Discrete Function KL Spectrum Computation over Symmetry Groups of Arbitrary Size

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Abstract—*The Karhunen-Loève spectrum of a discrete function can vary depending on how symmetry group objects are mapped to function minterms. For completely specified functions with a range space smaller than the cardinality of a symmetry group, we propose a technique to assign the symmetry group don't cares to maximize the number of zero-valued KL spectral coefficients of the function. To evaluate these approaches, we compare the average number of zero-valued KL spectral coefficients with those occurring in the Chrestenson spectrum.*

Index Terms— Karhunen-Loève Transform, Symmetry group.

I. INTRODUCTION

The Karhunen-Loève (KL) transform of a discrete multiple-valued logic function has been studied recently in [1] where a new way of computing the KL spectrum is described. One important observation in [1] is that the spectrum of a Cayley graph defined over the symmetry group is equivalent to the KL spectrum of a discrete function when the Cayley graph is generated using that function. It is also observed that the autocorrelation of the discrete function using the symmetry group operator is equivalent to the adjacency matrix of the Cayley graph. Based on the observations described above, the KL spectrum of a discrete multiple-valued logic function can be calculated efficiently.

The KL transform is developed as a series expansion method for continuous random processes by Kari Karhunen and Michel Loève [2, 3]. The KL transform is heavily utilized for performance evaluation of compression algorithms in the digital signal processing community since it has been proven to be the optimum transform for the compression of a sampled sequence in the sense that the KL spectrum contains the largest number of zero-valued coefficients [4]. Because the basis functions of the KL transform are data dependent, the KL spectrum is generally used as a benchmark to judge the effectiveness of the data compression capability of other more easily computed transforms [1].

The KL transform is a unitary transform with basis functions that are orthogonal eigenvectors of the covariance matrix of a data set or measurement vector. In this paper, we are interested in discrete binary and *Multiple-valued Logic* (MVL) functions, thus the eigenvectors of the autocorrelation matrix of the function form the basis. The KL spectrum is defined as the set of eigenvalues associated with the basis functions. It is the fact that the basis functions depend on the actual function to be transformed that discourages widespread use of the KL transform as compared to other transforms that have a common, well-known set of basis functions independent of the function to be transformed and often with other desirable features such as a recursive construction characteristic.

Since the KL spectrum contains the maximum number of zero-valued coefficients as compared to all other transforms, a possible application of this work is in the evaluation of different decision diagrams. By representing a function in the KL spectral domain, an extension of the use of *zero suppressed binary decision diagrams* (ZBDDs) [5] to the KL spectrum of a multiple-valued function will lead to compact representations. ZBDDs are very efficient representations for sparse functions meaning that a large number of zeros exist in truth table of the function.

The paper is organized as follows. In Section II, we review the computation of the KL spectrum based on Cayley graphs which is a condensed version of the methods described in [1]. In Section III we propose an algorithm to maximize the number of zero-valued spectral coefficients by efficient mapping and assigning don't cares. Section IV describes experimental results and conclusions are presented in Section V.

II. PRELIMINARY INFORMATION

The computation of the KL spectrum of a discrete MVL function, f, is described in [1]. Because the KL spectrum is defined as the set of eigenvalues of the autocorrelation matrix of a discrete function, and by the observation in [1], this matrix is equivalent to the adjacency matrix of a Cayley graph defined over the symmetric group S using the generator given in the following equation.

$$e = f(s_i \circ s_j) \qquad (1)$$

where $s_i, s_j \in S$ with each s_i, s_j uniquely corresponding to a minterm of f, and e corresponds to the color of an edge in a Cayley graph.. The group S consists of n! members with a product operator that is denoted as $\circ$ representing the permutation operation. Thus, the KL spectrum of the discrete function f is also the Cayley graph spectrum.

As an example, consider the function as shown in Figure 1 where $g = f(x, y)$ $x, g \in \{0,1\}$ and $y \in \{0,1,2\}$.

x_1 x_2	X	Mapping	g
0 0	0	0	0
0 1	1	1	1
0 2	2	2	1
1 0	3	3	0
1 1	4	4	1
1 2	5	5	0

Figure 1: Example Function

The members of the group S_3 may be considered to represent permutation operations of objects labeled *a*, *b*, and *c*. For convenience, these are assigned numerical values in an arbitrary manner. Consider the following assignment (or mapping):

$$\begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix} = 0 \quad \begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix} = 1 \quad \begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix} = 2$$
$$\begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix} = 3 \quad \begin{pmatrix} a & b & c \\ c & a & b \end{pmatrix} = 4 \quad \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix} = 5$$

Figure 2: Example Mapping for S_3 Elements

For the mapping defined in Figure 2, we use the group permutation operation and formulate the adjacency matrix of the Cayley graph with Equation (1). For this mapping, matrix A is given in general as:

$$A = \begin{bmatrix} f(0) & f(1) & f(2) & f(3) & f(4) & f(5) \\ f(1) & f(0) & f(3) & f(2) & f(5) & f(4) \\ f(2) & f(4) & f(0) & f(5) & f(1) & f(3) \\ f(3) & f(5) & f(1) & f(4) & f(0) & f(2) \\ f(4) & f(2) & f(5) & f(0) & f(3) & f(1) \\ f(5) & f(3) & f(4) & f(1) & f(2) & f(0) \end{bmatrix}$$

Figure 3: General Cayley Graph Adjacency Matrix

Using the example function in Figure 1, and the group object mapping in Figure 2, the Cayley graph adjacency matrix becomes:

$$A_g = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Solving for the roots of the characteristic equation of A_g yields the KL spectrum of g:

$$\Lambda=(0,0,0,-1,-1,3)$$

III. GROUP OBJECT MAPPING

As indicated in [1], the assignment of an object of a symmetry group to each minterm of a function is referred to as mapping. There are many different mappings available, $(n!)!$, and different mappings will produce different KL spectra. Here we investigate the relationship between alternative mappings and the resultant KL spectra. Next, we investigate the algorithm to maximize number of zero coefficients in the resultant spectrum.

For the function shown in Figure 1, there are 6 minterms for the function and therefore the 3-element symmetry group, S_3, is appropriate for mapping purposes. The 3-element symmetry group contains $3!=6$ members. In this case, the number of minterms is equal to the number of members in symmetry group, we refer to this case as *fully-specified mapping*. Alternatively, if the number of minterms is less than the number of members in symmetry group, it is referred to as *incompletely-specified mapping.*

For the 3-element symmetry group with 6 objects, the number of possible mapping is:

$$P = 6! = 720$$

Of interest is the fact that, for all 720 possible mappings for the example function, there are only 3 equivalence classes of differing KL spectra. These are (0,0,0,±3,±3), (0,0,±1,±2,±3), and (0,0,±1,±1,±3). Table 1 shows the distribution of the spectra over different mappings for the function g.

Table 1: Distribution of Spectra of *g* over All Possible Mappings

Spectrum	Percentage
$\{0,0,0,\pm3,\pm3\}$	10%
$\{0,0,\pm1,\pm2,\pm3\}$	30%
$\{0,0,\pm1,\pm1,\pm3\}$	60%

For incompletely-specified mappings, it is important to address don't care assignments.

An example function is shown in Figure 4, $h = f(x,y)$ with $x, y \in \{0,1\}$ and $g \in \{0,1\}$.

x_1 x_2	X	Mapping	h
0 0	0	0	0
0 1	1	1	1
1 0	2	2	1
1 1	3	3	0

Figure 4: Function with Incompletely-Specified Mapping

There are a total of 4 minterms for the fully-specified function and therefore the 3-element symmetry group S_3 is sufficient for mapping purposes. The 3-element symmetry group contains $3!=6$ group elements. Possible mappings number $\binom{3!}{4}4!=360$. Using the arbitrary mapping shown in Figure 4, the adjacency matrix A_h as shown in Figure 5 results.

$$A_h = \begin{bmatrix} f(0) & f(1) & f(2) & f(3) \\ f(1) & f(0) & f(3) & f(2) \\ f(2) & f(4) & f(0) & f(5) \\ f(3) & f(5) & f(1) & f(4) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & d_4 & 0 & d_5 \\ 0 & d_5 & 1 & d_4 \end{bmatrix}$$

Figure 5: Matrix for Function *h*

Notice that in the above matrix, we do not have specified values for $f(4)$ and $f(5)$. These can be interpreted as don't cares for the sake of the spectral computation. We point out that although these are don't cares for the KL spectral computation, the function being transformed is fully specified. We compare the two adjacency matrices with different don't care assignments, a) $d_4 = d_5 = 0$ and b) $d_4 = 0$ and $d_5 = 1$

$$A_{h_a} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad A_{h_b} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Figure 6: Two Possible Adjacency Matrices

The corresponding KL spectra for each matrix are:

$$\Lambda_a = [1.554, -1.554, 0.644j, -0.644j]$$

$$\Lambda_b = [2, -2, 0, 0]$$

From the above example, we note the importance of adjacency matrix don't care assignment for maximizing the number of zero-valued KL spectral coefficients. The following lemma provides useful guidelines for formulating a method for don't care assignment.

Lemma 1: *The number of zero-valued eigenvalues is minimally bounded by d-r. Where d is the dimension of the matrix and r is the rank.*

Based on above Lemma 1, we formulate a method that minimizes the rank of the Cayley graph adjacency matrix by assigning don't care value. This method is presented in algorithmic form in Figure 7.

```
dontcare_assignment (matrix[n, n]) {
  for (j=0; j<n; j++) {        // for each row
     if(row[j] contains don't care){
         //compare with all previous rows
         for(i=0; i<j; i++)){
           //check if row i and j could equal by
           //assigning don't cares
           possible_equal=check_equal(i, row);
         }
         if (possible_equal)
             assign_values();
       }
    }
}
```

Figure 7: Algorithm for Don't Care Assignment

The algorithm consists of two nested loops. If a row contains one or more don't care values, it will be compared with all previous rows to check if it is possible to make any two rows equal by assigning don't cares, i.e, for the previous example shown in Figure 5, row 3 contains two don't cares, comparing row 3 with row 1, it is impossible to make these two rows equal; comparing row 3 with row 1, it is possible to make these two rows equal through assigning $d_4 = 0$ and $d_5 = 1$.

IV. EXPERIMENTAL RESULTS

To test our algorithm and also to compare the KL spectrum with Chrestenson Spectrum [6], the following experiment is performed. We compare the average number of zero-valued spectral coefficients for the Chrestenson transform and the KL transform for some example functions. The results are shown in Table 2.

The first column shows 5 classes of functions that were chosen. The second column contains the number of functions for each class. Column 3 is the average number of zero-value Chrestenson Spectral coefficients. Columns 4 and 5 are the average number of zero-value KL spectral coefficients with an arbitrary don't care assignment and with the don't care assignment method shown in Figure 7 respectively. An arbitrary don't care assignment here means that all don't cares are assigned to zero as a default value.

From Table 2, we can see that the KL spectrum always has more zero-valued spectral coefficients than the corresponding Chrestenson spectra. Also, with our don't care assignment method, we improve the number of zero-value spectral coefficients except for the second class of functions where our results are equal to the arbitrary case.

V. CONCLUSIONS AND FUTURE WORK

A brief overview of the relationship between the KL spectrum and a Cayley graph defined of the symmetry group for discrete valued functions was given. The KL spectrum is of interest because it provides the theoretical lower bound on the number of zero-valued coefficients possible for any spectrum. In this paper, we proposed an algorithm for adjacency matrix don't care assignment to maximize the number of zero-valued KL spectral coefficients when the function of interest has fewer minterms than the corresponding symmetry group cardinality. Experimental results show the effectiveness of this method.

In the future, we plan to generalize the don't care assignment method described here to generate effective mappings for fully-specified mappings. We also plan to develop software to generate decision diagrams (DDs) of the KL spectrum of functions, and, in particular to develop multiple-valued decision diagrams that utilize zero-suppression as described in [5]. Such zero-suppressed KL-DDs should provide a lower bound in the number of DD vertices and could be useful for evaluating the effectiveness of other types of DDs.

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Table 2 Number of zeros in average for functions

Function Class	***# of functions***	***# of zeros in average***		
		Chrestenson Spectrum	***KL Spectrum***	
			arbitrary don't care assignment	***with don't care assignment***
$g = f(x,y)\ x,y,g \in \{0,1\}$	8	1.187	1.375	1.6875
$g = f(x,y)\ x,g \in \{0,1\}$ and $y \in \{0,1,2\}$	64	0.953	2.266	2.266
$g = f(x,y)\ x,y \in \{0,1\}$ and $g \in \{0,1,2\}$	81	0.716	0.741	1.136
$g = f(x,y)\ g \in \{0,1\}$ and $x,y \in \{0,1,2\}$	512	0.877	1.688	1.973
$g = f(x,y)\ x,y,g \in \{0,1,2\}$	19683	0.338	0.748	1.029