Data Assimilation in the Detection of Vortices

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Abstract We develop new algorithms for target detection in multi-sensor environments. These methods are applied to study point vortex motion based on Lagrangian tracer information. First we solve analytically the nonlinear filtering problem for the special case of equal strength vortices. Recently developed methods, the particle filters that are based on importance sampling Monte Carlo simulations, are used for the detection of vortices in the the general case. Unlike the well-known extended Kalman filter, it is applicable to highly nonlinear systems with non-Gaussian uncertainties.

1 Introduction

Nonlinear filtering methods are used to dynamically integrate the computational and measurement aspects of real-time applications. The range of subjects in which nonlinearity and noise play a significant role is enormous. In this paper, we are interested in the development and application of prediction techniques to a specific *fluid mechanics* problem – vortex-driven tracer dynamics. Because of low viscosity vortices are long-lived structures. We study the conditional law of the vortices on the basis of the tracer observations.

The main focus of filtering is to combine computational models with sensor data to predict the dynamics of large-scale evolving systems. Filtering deals with recursive estimation of a signal or state of a random dynamical system from noisy

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measurements. When the signal and the observation model is linear and Gaussian, the filtering equation is linear as well and it is given by the well-known Kalman-Bucy filter. Otherwise the filter has a more complicated nonlinear structure. The signal that is represented by a Markov process cannot be accessed or observed directly and is to be "filtered" from the trajectory of the observation process which is statistically related to the signal. Suppose we make a forecast about the behavior at a future time of a complex system with some uncertainties (randomness) and there is near-continuous data available from remote sensing instrumentation networks. As new information becomes available through observations, it is natural to ask how to best incorporate this new information into the estimation and prediction.

The optimal estimate is given by the conditional expectation and can be generated by a recursive equation, called filter, driven by the observation process. Sensor data usually contain noise, and mathematical models are limited in accuracy due to model uncertainties. But, when used together, the resulting prediction of the state of large-scale dynamical systems must be superior to using either models or data alone.

2 Modeling: Vortex-Driven Tracer Dynamics

To obtain the point-vortex model, one starts with the 2-D vorticity equations. If viscous and external forces are neglected, the vorticity-transport equation corresponding to the Euler equations is [9, 12]:

$$\frac{\partial \omega}{\partial t} + (\mathbf{u}\cdot\nabla)\,\omega = (\omega\cdot\nabla)\,\mathbf{u} \quad \text{with} \quad \nabla\cdot\omega = 0, \tag{1}$$

where the vorticity vector $\omega \equiv \nabla\times\mathbf{u}$. These equations simplify for two-dimensional flows in (x_1, x_2) plane to

$$\frac{\partial \omega}{\partial t} + (\mathbf{u}\cdot\nabla)\,\omega = 0 \quad \text{with} \quad \nabla^2\psi = \omega, \tag{2}$$

where $\omega \equiv \omega_{x_3}$ is the vorticity component normal to the (x_1, x_2) plane and ψ is the stream function defined by

$$u_{x_1} = -\frac{\partial \psi}{\partial x_2} \qquad u_{x_2} = \frac{\partial \psi}{\partial x_1}$$

If ω consists of isolated, well-separated vortices, then a reasonable approximation is to consider the vortices as singularities or "point" vortices. In this case we express the vorticity field as

$$\omega(\mathbf{x},t) = \sum_{i=1}^{n} \Gamma_i \delta\left(\mathbf{x}_t^i - \mathbf{x}\right) \quad \text{with} \quad \mathbf{x}_0^i = \mathbf{x}^i, \tag{3}$$

where $\Gamma_i \neq 0$ is the circulation of vortex i. By inserting (3) in the Euler equations (1), and using the divergence free constraint along with Biot-Savart law, one obtains

$$\dot{\mathbf{x}}_t^i = \sum_{j,j\neq i}^{n} \frac{\Gamma_j}{2\pi} \frac{\left(\mathbf{x}_t^j - \mathbf{x}_t^i\right)^{\perp}}{|\mathbf{x}_t^j - \mathbf{x}_t^i|^2} \quad \text{with} \quad \mathbf{x}_0^i = \mathbf{x}^i. \tag{4}$$

Point vortex models that account for viscous effects also exist. Chorin [3] introduced the first random point vortex method to simulate viscous incompressible flows. Later Marchioro and Pulvirenti in [10] considered a continuous-time random vortex method with Gaussian random walks replaced by independent Brownian motions and proved a corresponding mean field type result. It was shown by Marchioro and Pulvirenti [10] and Agullo and Verga [1] that a stochastic vortex dynamics model approximates the evolution of vorticity with viscosity, in the same way in which the deterministic vortex dynamics simulates the Euler equations. Vortex dynamics with viscosity are governed by set of Langevin or stochastic differential equations:

$$\dot{\mathbf{x}}_t^i = \sum_{j,j\neq i}^{n} \frac{\Gamma_j}{2\pi} \frac{\left(\mathbf{x}_t^j - \mathbf{x}_t^i\right)^{\perp}}{|\mathbf{x}_t^j - \mathbf{x}_t^i|^2} + \sqrt{2\nu}\xi_t^i \quad \text{and} \quad \mathbf{x}_0^i = \mathbf{x}^i, \tag{5}$$

where $\xi^i(t) \equiv (\xi_1^i(t), \xi_2^i(t))$ are zero mean white noise processes and equations (5) show that the velocity of each vortex is the sum of two terms, namely, the fluid velocity at the vortex position and a diffusive (stochastic) perturbation proportional to the fluid viscosity.

Lagrangian meters, such as ocean drifters and floats, provide a substantial part of ocean data which are used to reconstruct mean large-scale currents, estimate the rate of relative dispersion and give insight into the formation, movement and interactions of coherent structures such as point vortices and eddies. Based on the near-continuous data available from these instrumentation networks and to lower computational costs, we would like to develop more practical techniques required to analyze and interpret the data for dispersion modeling.

Trajectories of a Lagrangian tracer contain quantitative information about the dynamics of the of the underlying flow [6]; a tracer is advected according to

$$\dot{\mathbf{y}}_t^i = \mathbf{J}\sum_{j}^{n} \frac{\Gamma_j}{2\pi} \frac{\mathbf{y}_t^i - \mathbf{x}_t^j}{|\mathbf{y}_t^i - \mathbf{x}_t^j|^2} + \sqrt{2\nu}\eta_t^i \quad \text{and} \quad \mathbf{y}_0^i = \mathbf{y}^i. \tag{6}$$

The coupling between the dynamical model of the vortices and the tracer allows us to extract maximal information about the vortices by tracking the tracer. We can also correct the model variables on the fly using data from the tracers.

The theory of nonlinear filtering forms the framework in which problems of data assimilation for the nonlinear models will be treated. This can also be achieved using the idea of reduced dimensional nonlinear filtering as explained in Park et al. [13], where dimensional reduction made it possible in principle to reduce the cost of such filtering algorithms by a considerable factor.

3 Analytical Results:Continuous Signal and Discrete Observations

The theoretical aspect of data assimilation will be accomplished by constructing nonlinear filter equations based on continuous dynamics and discrete observation. Consider, for example, a two-vortex problem. Its state variables are the positions and velocities of the vortices. In the deterministic two-vortex problem, the distance between the vortices r is a constant of motion and the pair of point vortices rotate rigidly about the center of vorticity with a constant angular frequency. We use (x_1, x_2) and (x_3, x_4) to represent the position coordinates of the first and second vortices respectively. The dynamics of the signal process $X_t \equiv \{x_t^1, x_t^2, x_t^3, x_t^4\}$ is governed by

$$dx_t^1 = -\frac{\Gamma_2}{2\pi}\frac{(x_t^2 - x_t^4)}{r^2}dt + \sqrt{2\nu}dW_t^1, \quad dx_t^2 = \frac{\Gamma_1}{2\pi}\frac{(x_t^1 - x_t^3)}{r^2}dt + \sqrt{2\nu}dW_t^2 \quad (7)$$

$$dx_t^3 = -\frac{\Gamma_2}{2\pi}\frac{(x_t^4 - x_t^2)}{r^2}dt + \sqrt{2\nu}dW_t^3, \quad dx_t^4 = \frac{\Gamma_1}{2\pi}\frac{(x_t^3 - x_t^1)}{r^2}dt + \sqrt{2\nu}dW_t^4, \quad (8)$$

where $r = \sqrt{(x^1 - x^3)^2 + (x^2 - x^4)^2}$. For various practical reasons, the initial value of the signal $X_0 = \bar{x}$ is unknown, but the distribution of the initial condition x is given by $p(x)$. Now we introduce *relative* and "center of mass" coordinates as

$$x^r = x^3 - x^1, \quad x^c = \frac{\Gamma_1 x^1 + \Gamma_2 x^3}{\Gamma_1 + \Gamma_2} \quad (9)$$

$$y^r = x^4 - x^2, \quad y^c = \frac{\Gamma_1 x^2 + \Gamma_2 x^4}{\Gamma_1 + \Gamma_2}. \quad (10)$$

Then by Itô lemma

$$\begin{pmatrix} dx_t^r \\ dy_t^r \\ dx_t^c \\ dy_t^c \end{pmatrix} = \begin{pmatrix} \frac{\Gamma_1+\Gamma_2}{2\pi}\frac{y^r}{r^2} \\ -\frac{\Gamma_1+\Gamma_2}{2\pi}\frac{x^r}{r^2} \\ 0 \\ 0 \end{pmatrix} dt + \sqrt{2\nu} \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ \frac{\Gamma_1}{\Gamma_1+\Gamma_2} & \frac{\Gamma_2}{\Gamma_1+\Gamma_2} & 0 & 0 \\ 0 & 0 & \frac{\Gamma_1}{\Gamma_1+\Gamma_2} & \frac{\Gamma_2}{\Gamma_1+\Gamma_2} \end{pmatrix} \begin{pmatrix} dW_t^1 \\ dW_t^2 \\ dW_t^3 \\ dW_t^4 \end{pmatrix} \quad (11)$$

Defining

$$z = \{x^r, y^r, x^c, y^c\}, \quad \tau = \Gamma_1 + \Gamma_2, \quad \text{and} \quad \kappa_i = \frac{\Gamma_i}{\Gamma_1 + \Gamma_2}$$

the generator of the Markov process is given by

$$(\mathscr{L} f) = \nu\left(\kappa_1^2 + \kappa_2^2\right)\left(\frac{\partial^2 f}{\partial z_3^2}(z) + \frac{\partial^2 f}{\partial z_4^2}(z)\right) + 2\nu\left(\kappa_2 - \kappa_1\right)\left(\frac{\partial^2 f}{\partial z_1 \partial z_3}(z) + \frac{\partial^2 f}{\partial z_2 \partial z_4}(z)\right)$$

$$+\tfrac{\tau}{2\pi}\tfrac{1}{z_1^2+z_2^2}\left(-z_2\tfrac{\partial f}{\partial z_1}+z_1\tfrac{\partial f}{\partial z_2}\right)+2\nu\left(\tfrac{\partial^2 f}{\partial z_1^2}(z)+\tfrac{\partial^2 f}{\partial z_2^2}(z)\right) \tag{12}$$

for $f \in C^2(\mathbb{R}^4)$. The probability density is governed by the forward Kolmogorov equation

$$\frac{\partial P}{\partial t} = \mathscr{L}^* P \tag{13}$$

where $\mathscr{L}^*$ is the adjoint operator.

3.1 Continuous Signal : Two-vortices with Equal Strengths

In this section, we consider a special case of the signal process. Our signal process is given by the two-vortex dynamics given in (8) with equal vortex strengths, that is, $\kappa_1 = \kappa_2 = 1/2$. Then the generator (12) of the signal process is separable into the "relative" and "center of mass" coordinates. The center of mass (x^c, y^c) and relative distance (x^r, y^r) evolve independently; if one solves the forward equation (13) with initial conditions such that $p(z,t_0) = p_c(x^c, y^c, t_0)p_r(x^r, y^r, t_0)$, then $p(z,t) = p_c(x^c, y^c, t)p_r(x^r, y^r, t)$ for all time $t > t_0$. This suggests that the evolution equation - here a PDE with four spatial dimensions - can be simplified by considering the evolution of (x^c, y^c) and (x^r, y^r) separately.

The evolution equation for the density of (x^c, y^c) is simply the heat equation. So we are left with the evolution of the density of (x^r, y^r), which in polar coordinates is given by the generator

$$(\mathscr{L}^{r,\theta} f) = \frac{\tau}{2\pi}\frac{1}{r^2}\frac{\partial f}{\partial \theta} + \nu\left(\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}\right). \tag{14}$$

We can rescale time and redefine ν so that the constant $\tau/2\pi$ is removed from the equations.

We use a Green's function approach to find a solution from an arbitrary initial condition. Let $\rho(r,\theta,t)$ be a probability distribution evolving according to this law. Then

$$\rho(r,\theta,t) = \int ds\, s \int_0^{2\pi} d\phi\, P(r,\theta,t;s,\phi)\rho(s,\phi,0), \tag{15}$$

where $P(r,\theta,t;s,\phi)$ is the solution to

$$\frac{\partial P}{\partial t} = (\mathscr{L}^{r,\theta})^* P \tag{16}$$

with initial condition $P(r,\theta,0;s,\phi) = \delta(r-s)\delta(\theta-\phi)$. Without loss of generality we can set $\phi = 0$.

An exact solution is available for this initial value problem, due to Agullo and Verga[1]; we state the final result here.

$$P(r,\theta,t;s,0) = \frac{1}{4\pi\nu t}\sum_{p\in Z} e^{ip\theta} e^{-(r^2+s^2)/(4\nu t)} I_{\mu_p}(\frac{rs}{2\nu t}) \tag{17}$$

where $I_m(z)$ is the modified Bessel function of the first kind with order m and argument z and $\mu_p^2 = p^2 + ip/\nu$, and the root should be chosen so that $Re(\mu_p) \leq 0$. In terms of the variables (x^c, y^c, x^r, y^r) our probability density is

$$P(x^r, y^r, x^c, y^c, t) = p_r(x^r, y^r, t) p_c(x^c, y^c, t) \tag{18}$$

where

$$p_r(x^r, y^r, t) = \frac{1}{4\pi\nu t}\int\int d\xi d\eta\, e^{-(|\bar{x}^r|^2+|\bar{\xi}|^2)/(4\nu t)} \times \left[\sum_{p\in Z} e^{ip\tan^{-1}(y^r/x^r) - ip\tan^{-1}(\eta/\xi)} I_{\mu_p}(\frac{|\bar{x}^r||\bar{\xi}|}{2\nu t})\right] p_r(\xi,\eta,0), \tag{19}$$

$$p_c(x^c, y^c, t) = \frac{1}{\pi\nu t} e^{-((x^c)^2+(y^c)^2)/(\nu t)} \tag{20}$$

and $\bar{x}^r \equiv (x^r, y^r)$ and $\bar{\xi} \equiv (\xi, \eta)$.

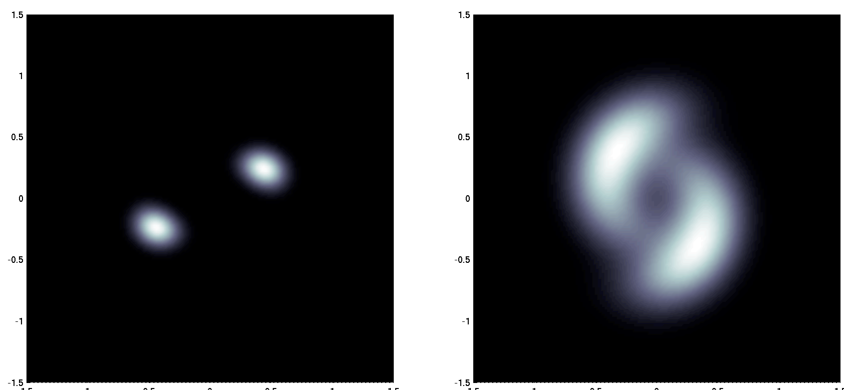

Fig. 1 The left figure shows the superimposed distributions of (x^1, x^2) and (x^3, x^4) at $t = 1$. The right figure shows $t = 5$.

We use this formula to observe the evolution of a pair of vortices, initially at a distance $r = 1$ and $\theta = 0$. We evaluate the modified Bessel functions using a freely available code developed for Coulomb functions [14], of which the Bessel functions are a subset. We show the distribution of the vortex positions at $t = 1$ and $t = 5$. These can be compared to the histograms presented in Fig. 3 of [1].

3.1.1 Nonseparable probability densities

In general, particularly after the incorporation of an observation, the probability density will no longer be separable into radial and center coordinates as in (18). In this general case, the forward Kolmogorv equation (13) must be solved by first dividing the density into separable components with a spectral transform. We apply the cosine tranform to P, that is

$$P(x^r,y^r,x^c,y^c,0) = \sum_{k,l} \hat{P}_{k,l}(x^r,y^r,0)\cos(k\pi x^c/L)\cos(l\pi y^c/L) \tag{21}$$

if x^c,y^c are defined on $(0,L)$ (if x^c and y^c are defined on another interval, such as $(-L/2,L/2)$, then the argument of the cosine functions would be shifted appropriately).

We choose the cosine tranform as most natural to a probability distribution, where we wish to have "no-flux" boundaries (ideally, we would resolve the probability distribution on a large enough grid that the probability would be negligible near the boundaries).

For each pair k,l we compute $\hat{P}_{k,l}(x^r,y^r,t)$ to be the solution of (16), with initial condition $\hat{P}_{k,l}(x^r,y^r,0)$. This solution is given explicitly by equation (19).

The final answer is given by applying the diffusion operator to the (x^c,y^c) coordinates that is appropriate to the Fourier mode, and taking the inverse transform to yield

$$P(x^r,y^r,x^c,y^c,t) = \sum_{k,l} e^{-\nu(k^2+l^2)\pi^2 t/2L^2}\hat{P}_{k,l}(x^r,y^r,t)\cos(k\pi x^c/L)\cos(l\pi y^c/L). \tag{22}$$

3.2 *Discrete Observations: Tracer Advection*

The observations are defined by the tracers and are taken at discrete time instants t_k. The model that we will use for y is the following:

$$y_k^i = h_i(z_k,y_{k-1}) + v_k^i, \qquad z_k = z_{t_k}, \quad y_k^i = y_{t_k}^i, \qquad i = 1\dots 2n \tag{23}$$

The sensor functions are given by first-order approximation to the tracers' equations of evolution (6); in the case of one tracer, we have

$$h_1(z_k,y_{k-1}) = y_{k-1}^1 + \Delta t\left(\frac{-(y_{k-1}^2-(y^c-y^r/2))}{r_{1,2}^2} + \frac{-(y_{k-1}^2-(y^c+y^r/2)}{r_{3,4}^2}\right) \tag{24}$$

$$h_2(z_k,y_{k-1}) = y_{k-1}^2 + \Delta t\left(\frac{y_{k-1}^1-(x^c-x^r/2)}{r_{1,2}^2} + \frac{y_{k-1}^1-(x^c+x^r/2)}{r_{3,4}^2}\right) \tag{25}$$

where

$$r_{1,2} \equiv |(y^1,y^2)-(x^1,x^2)|^2 = |(y^1,y^2)-(x^c-x^r/2, y^c-y^r/2)|,$$

$$r_{3,4} \equiv |(y^1,y^2)-(x^3,x^4)|^2 = |(y^1,y^2)-(x^c+x^r/2, y^c+y^r/2)|$$

give the distances of the tracer from the first and second vortices respectively. $v_k = v_{t_k}$ is a $\mathbb{R}^2$-valued white Gaussian noise process independent of z_k, that is $v_k \sim N(0, R_{v_k})$. Here $z = (x^1,x^2)$, (x^3,x^4) are the signal variables, and (y^1,y^2) are the observation variables; corresponding formulas would be used for additional tracers.

Once again the observation σ-field

$$\mathscr{F}_t^y = \sigma\{y_l \sup_{0\le t\le T} 0 \le t_l \le t\} = \{y_l \sup_{0\le t\le T} l = 1,2,\cdots,n; n\tau \le t\}$$

where τ is the sampling intervals. $\mathscr{F}_t^Y$ contains all the information available upto time instant t. To solve the filtering problem, for each $t \ge 0$, we would like to find the conditional **pdf** called the *posterior density*. Assume that the conditional probability distribution of the state z_t, given the observation up time t, denoted by

$$\pi_t(dz) = \mathbb{P}\left(z_t \in dz|\mathscr{F}_t^y\right)$$

has a conditional **pdf** $p(z,t|\mathscr{F}_t^y)$.

3.3 Nonlinear Filters

Hence, *between observarions*, the conditional **pdf** $p(z,t|\mathscr{F}_t^y)$ is governed by the Kolmogrov's forward equation [7, 8], that is,

$$\begin{aligned} \frac{\partial}{\partial t} p(z,t|\mathscr{F}_{t_k}^y) &= \mathscr{L}^* p(z,t|\mathscr{F}_{t_k}^y), \quad t_k < t < t_{k+1}, \\ \text{with} \quad \lim_{t\to t_k} p(z,t|\mathscr{F}_{t_k}^y) &= p(z,t_k|\mathscr{F}_{t_k}^y). \end{aligned} \tag{26}$$

where $(\mathscr{L})^*$ is the adjoint of the operator given in (12). This implies that in the discrete observation case, once we know the initial condition at $t = t_k$ given by $p(z,t_k|\mathscr{F}_{t_k}^y)$, we can compute the conditional **pdf** $p(z,t|\mathscr{F}_{t_k}^y)$ using the explicit solution (18) at any time $t > t_k$. However, at time $t = t_{k+1}$, we get more information from the observation y_{k+1}, which has to be used to *update* this conditional **pdf** at $t = t_{k+1}$.

The natural question is how to determine the initial conditional **pdf** $p(z,t_k|\mathscr{F}_{t_k}^y)$ at $t = t_k$ given in (26), knowing the previous evolution $p(z,t_k|\mathscr{F}_{t_{k-1}}^y)$ evaluated at $t = t_k$, and the new information y_k. Then, by Bayes' rule we have

$$p(z,t_k|\mathscr{F}_{t_k}^y) = \frac{p\left(y_k\middle|\{z,t_k\},\mathscr{F}_{t_{k-1}}^y\right) p\left(z,t_k\middle|\mathscr{F}_{t_{k-1}}^y\right)}{p\left(y_k\middle|\mathscr{F}_{t_{k-1}}^y\right)} \tag{27}$$

The denominator in (27) is just the normalization of the numerator and can be calculated from

$$p\left(y_k\middle|\mathscr{F}^y_{t_{k-1}}\right) = \int p\left(y_k\middle|\{z,t_k\},\mathscr{F}^y_{t_{k-1}}\right) p\left(z,t_k\middle|\mathscr{F}^Y_{t_{k-1}}\right)dz. \tag{28}$$

The conditional **pdf** on the right-hand side of (27)

$$p\left(z,t_k\middle|\mathscr{F}^Y_{t_{k-1}}\right)$$

is given by (18). Since $\{v_k\}$ is a white noise and z_k is independent of v_k, the conditional **pdf**

$$p\left(y_k\middle|\{z,t_k\},\mathscr{F}^y_{t_{k-1}}\right),$$

from the observation y_k can be simplified somewhat. The observation y_k at time k conditioned on z_k, is independent of all other measurements but y_{k-1}.

$$p\left(y_k\middle|\{z,t_k\},\mathscr{F}^y_{t_{k-1}}\right) = p\left(y_k\middle|\{z,t_k\},y_{k-1}\right) = p\left(y_k\middle|z_k,y_{k-1}\right). \tag{29}$$

Once $p\left(y_k\middle|z,y_{k-1}\right)$ is determined, we can get the desired map for the conditional density at an instant of observation.

Consider the observation equation (23). Since v_k is Gaussian and y_k is linear in v_k, for a given value of $z_k = z$ and $y_{k-1} = y$

$$p_y\left(y_k\middle|z_k = z, y_{k-1} = y\right) = p_{v_k}\left(y_k - h(z,y,t_k)\right)\left|\frac{\partial v_k}{\partial y_k}\right| = p_{v_k}\left(y_k - h(z,y,t_k)\right)$$

Since $v_k \sim N(0,R_k)$ we can explicitly write

$$p\left(y_k\middle|z,y\right) = \frac{1}{(2\pi)^{\frac{m}{2}}|R_k|^{\frac{1}{2}}}\exp\left\{-\frac{1}{2}\left(y_k - h(z,y,t_k)\right)^T R_k^{-1}\left(y_k - h(z,y,t_k)\right)\right\} \tag{30}$$

We can summarise the results by combining the equations (18), (27), (29) and (30) as follows. The conditional **pdf** $p(z,t|\mathscr{F}^y_t)$ satisfies the following partial differential equation

$$\frac{\partial}{\partial t}p(z,t|\mathscr{F}^y_t) = \mathscr{L}^* p(z,t|\mathscr{F}^y_t), \quad t_k < t < t_{k+1}, \tag{31}$$

with the initial conditions at t_k given by the updating equations

$$p(z,t_k|\mathscr{F}^Y_{t_k}) = C_k\,\psi_k(z)\,p(z,t|\mathscr{F}^y_{t_{k-1}}), \tag{32}$$

where

$$\psi_k(z) = \exp\left\{-\frac{1}{2}\left(y_k - h(z,y_{k-1},t_k)\right)^T R_k^{-1}\left(y_k - h(z,y_{k-1},t_k)\right)\right\},$$

C_k is a normalizing factor and $\mathscr{F}^Y_{t_{k-1}}$ the information on Y up to the instant in time right before t_k. The equation (31) has an explicit solution given by (18). The first equation (31) is the Kolmogorov's forward equation which is used to compute predictions between measurements, while the second equation (32) is used to update the information about the state via Bayes' rule.

4 Numerical Results: Particle Filters

One of the recent, more efficient and most popular classes of filtering methods is called particle methods. Importance sampling Monte Carlo offers powerful approaches to approximating Bayesian updating in sequential problems. Specific classes of such approaches are known as particle filters. The particles in these methods refer to independent samples generated with the Monte Carlo method, and they include sequential Monte Carlo, ensemble Kalman filter and interacting particle filters. The popularity of particle methods is attested by the recent surge of papers in this area. Particle algorithms are techniques for implementing a recursive Bayesian filter by MC simulations [5] (see for example, Arulampalam et al [2]). In all particle methods, we evolve the particles between measurements by a set of random samples with associated weights and update the ensemble using Bayes' rule at the measurement time based on these samples and weights. Particle methods are very flexible and easy to implement; also they are ideally suited for a parallel computing architecture. This method has recently given rise to extensive mathematically rigorous studies, see for instance [11, 4] for the nonlinear filtering problem.

The idea is based on the Importance Sampling technique, that is, we can calculate an expected value

$$\mathbb{E}[f(z_k)] = \int f(z_k) p(z_k|y_{1:k}) dz_k = \int f(z_k) \frac{p(y_{1:k}|z_k)\, p(z_k)}{p(y_{1:k})\, q(z_k|y_{1:k})} q(z_k|y_{1:k}) dz_k \tag{33}$$

by using a known and simple proposal distribution $q(\cdot)$. This can be further simplified to

$$\mathbb{E}[f(z_k)] = \int f(z_k) \frac{w_k(z_k)}{p(y_{1:k})} q(z_k|y_{1:k}) dz_k, \text{ where } w_k(z_k) = \frac{p(y_{1:k}|z_k)\, p(z_k)}{q(z_k|y_{1:k})} \tag{34}$$

is defined as the filtering non-normalized weight at step k. Hence,

$$\mathbb{E}[f(z_k)] = \frac{\mathbb{E}_q[w_k(z_k)\, f(z_k)]}{\mathbb{E}_q[w_k(z_k)]} = \mathbb{E}_q[\hat{w}_k(z_k)\, f(z_k)], \quad \text{where} \quad \hat{w}_k(z_k) = \frac{w_k(z_k)}{\mathbb{E}_q[w_k(z_k)]} \tag{35}$$

These procedures rely on the simulation of samples or ensembles of the unknown quantities and the calculation of associated weights for the ensemble members. Hence, using Monte-Carlo sampling from the distribution $q(z_k|y_{1:k})$ we can write

$$\mathbb{E}[f(z_k)] \approx \sum_{i=1}^{N} \hat{w}_k(z^i_k)\, f(z^i_k), \quad \text{where} \quad \hat{w}_k(z^i_k) = \frac{w_k(z^i_k)}{\sum_{i=1}^{N} w_k(z^i_k)} \tag{36}$$

In addition if our proposal distribution $q(\cdot)$ satisfies the Markov property, it can be shown that $\hat{w}_k(z_k^i)$ satisfies a cursive relationship. The basic ideas of particle filters are: 1) represent the required posterior density function by a set of random samples with associated weights; 2) compute estimates based on these samples and weights.

In principle, armed with these algorithms, we should be able to handle a large class of nonlinear filtering problems. The problem of this method is that for high dimensional systems, these stochastic algorithms are usually slow and computational complexity grows too quickly with dimension. In extreme cases, after a sequence of updates the particle system can collapse to a single point or to several particles with so much internal correlation that summary statistics behave as if they are derived from a substantially smaller sample. To compensate, large numbers of particles are required in realistic problems. Hence, the method is not always implementable in real time nonlinear applications when the state space is too large.

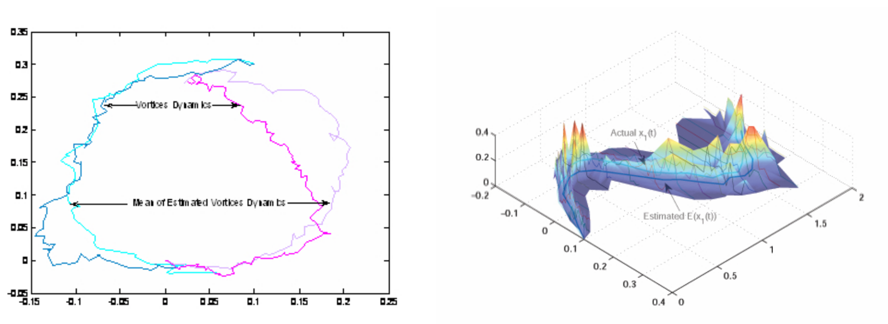

Fig. 2 The left figure shows the mean value of estimated position of the vortices by tracking the single tracer. The right figure shows the conditioned pdf of the position

4.1 Data Fusion

The results presented thus far are, in general, well understood in terms of single sensor filtering theory. However, when there are multiple sensors, then the problem of combining information from them arises. We consider some approaches generally proposed in the literature and discuss some criticisms associated with them.

To begin with, we assume that M sensors are available and the observations from the k^{th} are given by the vector $y^k \in \mathbb{R}^m$ (i.e., the number of observations m is the same for all sensors). What is now required is to compute the global posterior distribution $p(z|y^1, y^2, \cdots, y^M)$, given the information contributed by each sensor. We shall assume that each sensor provides either a local posterior distribution $p(z|y^k)$, or a likelihood function $p(y^k|z)$.

Since the information is received from different sensors, the natural question to ask in tackling the problem of fusion, is how relevant and how reliable is the infor-

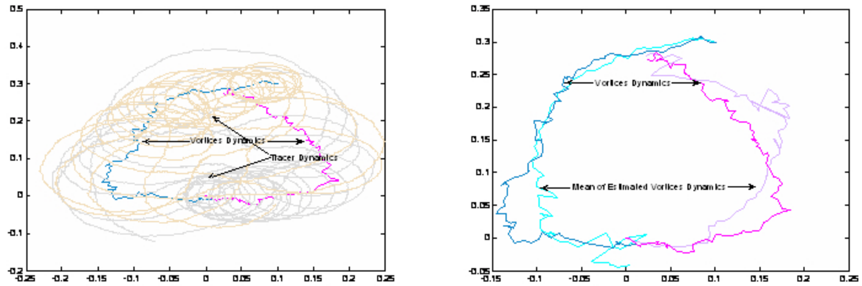

Fig. 3 The left figure shows the vortex-tracer dynamics.The right figure shows that with two or more tracers, the extraction results can be improved.

mation from each sensor. One of the ways to address this problem is by attaching a weight to the information provided by each sensor.

On the other hand when each information source has common prior information, i.e. information obtained from the same origin, the situation is better described by the *independent likelihood pool*, which is derived as follows. According to Bayes' theorem for the global posterior, we obtain

$$p(z|y^1,y^2,\cdots y^M) = \frac{p(y^1,y^2,\cdots y^M|z)p(x)}{p(y^1,y^2,\cdots y^M)}. \tag{37}$$

For a system of tracers it is reasonable to assume that the likelihoods from each tracer $p(y^m|z), m = 1,2,\cdots,M$ are independent since the only parameter they have in common is the state x of the vortices., that is,

$$p(y^1,y^2,\cdots y^M|z) = p(y^1|z)p(y^2|z)\cdots p(y^M|z).$$

Thus, the Independent Likelihood Pool is defined by the following equation

$$p(z|y^1,y^2,\cdots y^M) = p(z)\Pi_{m=1}^M p(y^m|z). \tag{38}$$

As may be seen from the above both the Independent Opinion Pool and the Independent Likelihood Pool more accurately describe the situation in multi-sensor systems where the conditional distribution of the observation can be shown to be independent. However, in most cases in sensing the Independent Likelihood Pool is the most appropriate way of combining information since the prior information tends to be from the same origin. If there are dependencies between information sources the Linear Opinion Pool should be used.

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