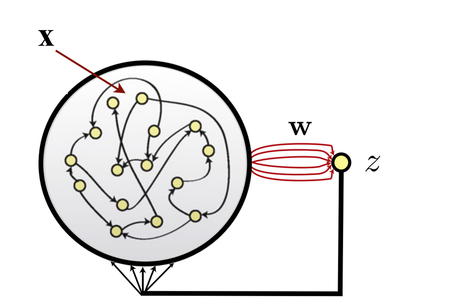
Symmetries constrain dynamics in a family of balanced neural networks

Andrea K. Barreiro[1], J. Nathan Kutz[2], Eli Shlizerman[2,3]
[1]Southern Methodist University, Department of Mathematics;
University of Washington; [2]Department of Applied Mathematics, [3]Department of Electrical Engineering

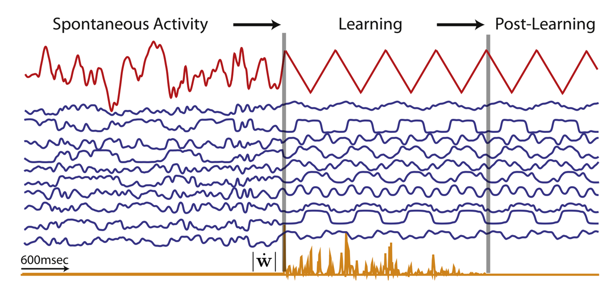
Abstract : Biological neural circuits display both spontaneous asynchronous activity, and complex, yet ordered activity while actively responding to input. Recently, researchers have demonstrated this capability in large, recurrently connected neural networks, or "echo-state" networks, with chaotic activity (Bertschinger and Natschläger 2004, Jaeger and Haas 2004, Sussillo and Abbott 2009). We study the transition to chaos in a family of such networks, and use principal orthogonal decomposition (POD) techniques to provide a lower-dimensional description of network activity. We find that key characteristics of this transition depend critically on whether a fundamental neurobiological constraint — that most neurons are either excitatory or inhibitory — is satisfied. Specifically, we find that constrained networks exhibit the transition to chaos at much higher coupling strengths than unconstrained networks. This property is the consequence of the fact that the constrained system may be described as a perturbation from a system with non-trivial symmetries. These symmetries imply the presence of both fixed points and periodic orbits that continue to act as an organizing center for solutions, even for large perturbations. In comparison, spectral characteristics of the network coupling matrix (Rajan and Abbott 2006, Wei 2012) are relatively uninformative about the behavior of the constrained system.

Background: echo-state networks

- Large recurrent neural networks (in simulations here, N = 200-1000)
- Network receives feedback from trained output units (Jaeger and Haass 2004).
- Network is spontaneously chaotic; chaos suppressed during training/testing
- ***Common finding:*** network dynamics are low-dimensional (Sussillo and Abbott 2009, 2012; Sussillo and Barak 2013)



$$\dot{\mathbf{x}} = -\mathbf{x} + \mathbf{G}\bullet(\tanh(g\mathbf{x})) + \mathbf{J}^{FB}z$$
$$z = \mathbf{w}^T(\tanh(g\mathbf{x}))$$
$$\mathbf{w}(t) \to \mathbf{w}(t+\Delta t)$$



Graphics: Sussillo and Abbott, Neuron 2009

We investigate behavior of RNNs (before and after training) with two choices of internal connectivity structures G

Network setup

$$\dot{\mathbf{x}} = -\mathbf{x} + \mathbf{G}\bullet(\tanh(g\mathbf{x})), \quad \mathbf{x}(0) = \mathbf{x}_0$$

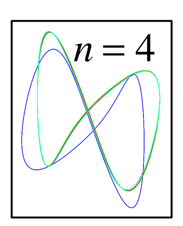
$\mathbf{x}$: activity variables $N \times 1$

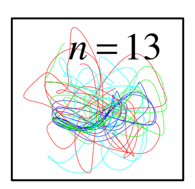
$\mathbf{G}$: coupling matrix $N \times N$

g : overall coupling parameter

$$\mathbf{G}_{ij} \sim P(\mu, \sigma^2, \ldots), \quad i \neq j$$

The elements of G are chosen from some probability distribution P; in general we choose $\mathbf{G}_{ii} = 0$ (no self-coupling)





Principal orthogonal decomposition (POD)

Integrate f.r. equations, keeping N_T snapshots:

$$\{\mathbf{x}(t_0), \mathbf{x}(t_1), \ldots, \mathbf{x}(t_{N_T})\}, \quad N \times N_T$$

$$\mathbf{x}(t_0 : t_{N_T}) = \mathbf{V}\Sigma\mathbf{U}^T$$

$\mathbf{U}$: N_T x N projection onto modes at each time slice

Σ : N x N mode energy

$\mathbf{V}$: N x N principal orthogonal modes

Keep enough modes (n) to capture 99% of the energy

Choice of connectivity matrix ensemble

Unconstrained networks

$$\mathbf{G}_{ij} \sim N(0, \sigma^2/N), \quad i \neq j$$

- Eigenvalues: $\frac{p(\lambda)}{N} \sim \pi^{-1}, \quad |\lambda| \leq 1$
- Singular values:

$$\frac{p(\hat{\sigma})}{N} \sim \frac{2}{\pi}\sqrt{1 - \frac{\hat{\sigma}^2}{4}}, \quad 0 \leq \hat{\sigma} \leq 2$$

We expect a transition to chaotic activity for some g > 1 (Sompolinsky et al., 1988)

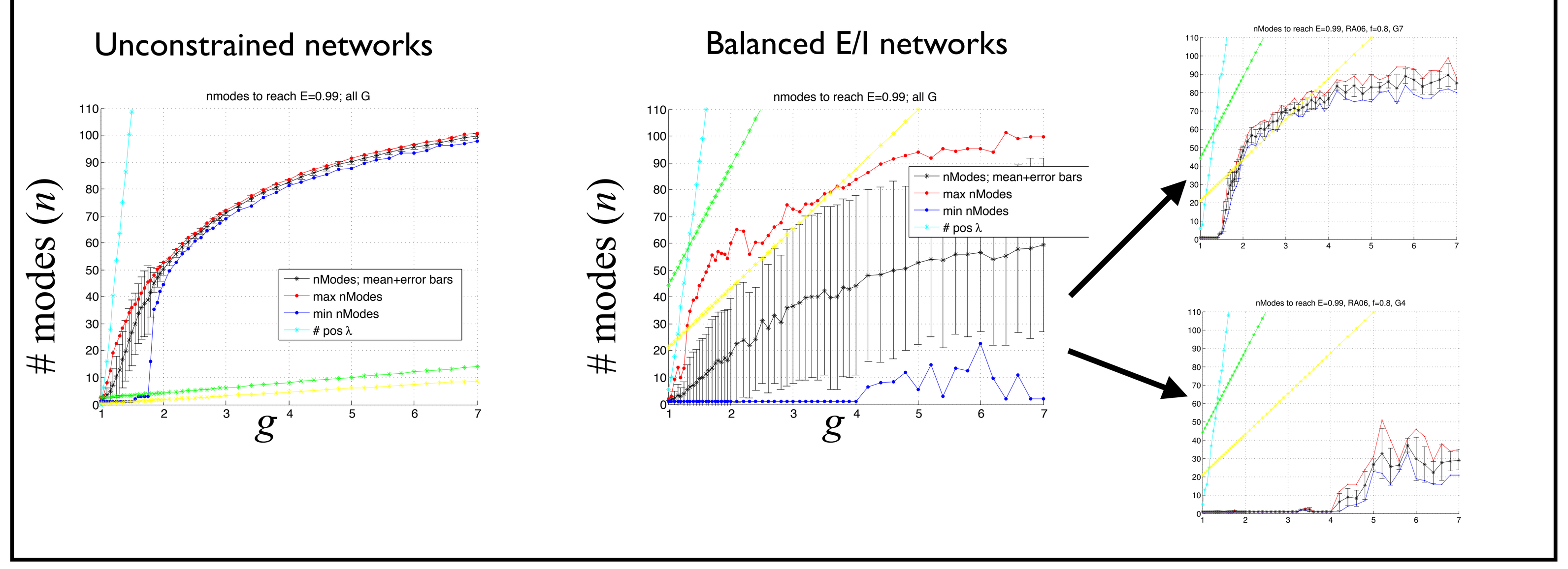
Balanced E/I networks

$$\mathbf{G}_{ij} \sim N(\mu_E/\sqrt{N}, \sigma_E^2/N), \quad i \neq j, \ 1 \leq j \leq fN$$
$$\sim N(\mu_I/\sqrt{N}, \sigma_I^2/N), \quad i \neq j, \ fN < j \leq N$$

$$(f\mu_E + (1-f)\mu_I = 0)$$

- Mean input currents balanced: can sustain high variability
- Respects *Dale's Law*: most neurons are *either* excitatory or inhibitory
- no change in eigenvalue distribution (Rajan and Abbott 2006, Wei 2012)

Balanced E/I networks exhibit a delayed transition to chaos



Balanced E/I network has underlying symmetry

$$\mathbf{G}_{ij} \sim N(\mu_E/\sqrt{N}, 0), \quad i \neq j, \ 1 \leq j \leq fN$$
$$\sim N(\mu_I/\sqrt{N}, 0), \quad i \neq j, \ fN < j \leq N$$

(e.g.
$$\mathbf{G} = \frac{1}{\sqrt{N}}\begin{pmatrix} 0 & \mu_E & \mu_E & \mu_E & \mu_E & \mu_E & \mu_I & \mu_I \\ \mu_E & 0 & \mu_E & \mu_E & \mu_E & \mu_E & \mu_I & \mu_I \\ \mu_E & \mu_E & 0 & \mu_E & \mu_E & \mu_E & \mu_I & \mu_I \\ \mu_E & \mu_E & \mu_E & 0 & \mu_E & \mu_E & \mu_I & \mu_I \\ \mu_E & \mu_E & \mu_E & \mu_E & 0 & \mu_E & \mu_I & \mu_I \\ \mu_E & \mu_E & \mu_E & \mu_E & \mu_E & 0 & \mu_I & \mu_I \\ \mu_E & \mu_E & \mu_E & \mu_E & \mu_E & \mu_E & 0 & \mu_I \\ \mu_E & \mu_E & \mu_E & \mu_E & \mu_E & \mu_E & \mu_I & 0 \end{pmatrix}$$
)

Under these conditions the ODE has $S_{n_E} \times S_{n_I}$ symmetry

$$\dot{\mathbf{x}} = -\mathbf{x} + \mathbf{G}\bullet(\Phi(g\mathbf{x})), \quad \mathbf{x}(0) = \mathbf{x}_0$$

Left: example with $N = 8; \ f = 0.75$

Equivariant bifurcation theory characterizes the solutions we will see:

- Branch of fixed points; emerges at $g \approx \sqrt{N}/\alpha\mu_E$

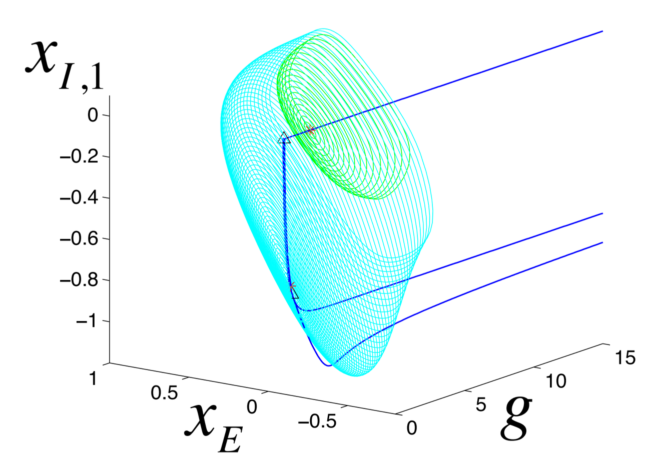
$$\mathbf{x} = (x_E(t), x_{I,1}(t), x_{I,2}(t))$$

  - Limit cycle from Hopf bifurcation:

$$\mathbf{x} = (x_E(t), x_{I,1}(t), x_{I,2}(t))$$

- Limit cycle from Hopf at origin: $g \approx \sqrt{N}/\mu_E$

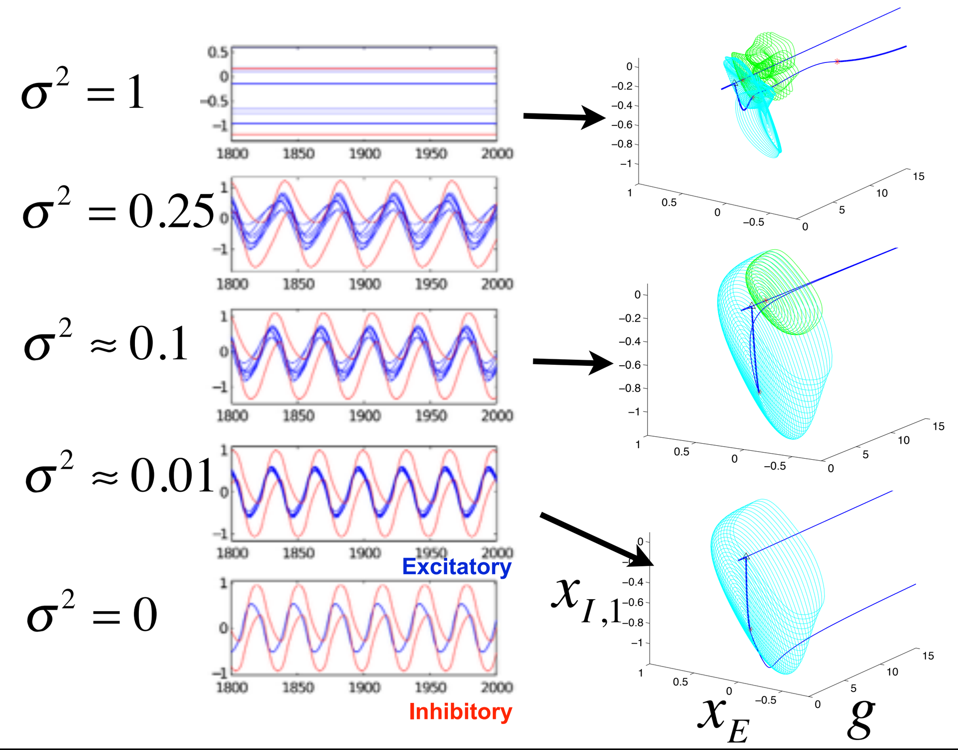
$$\mathbf{x} = (x_E(t), x_I(t))$$



Above: bifurcation diagram for $N = 10; \ f = 0.8$

Barreiro, Kutz and Shlizerman, arXiv:1602.05092, 2016

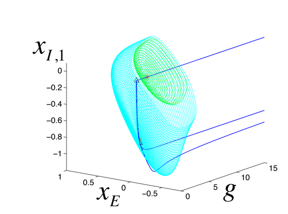
Solutions persist as randomness is added



Here we pick a specific realization of a random network and compute bifurcation diagrams as σ varies ($N = 10$).

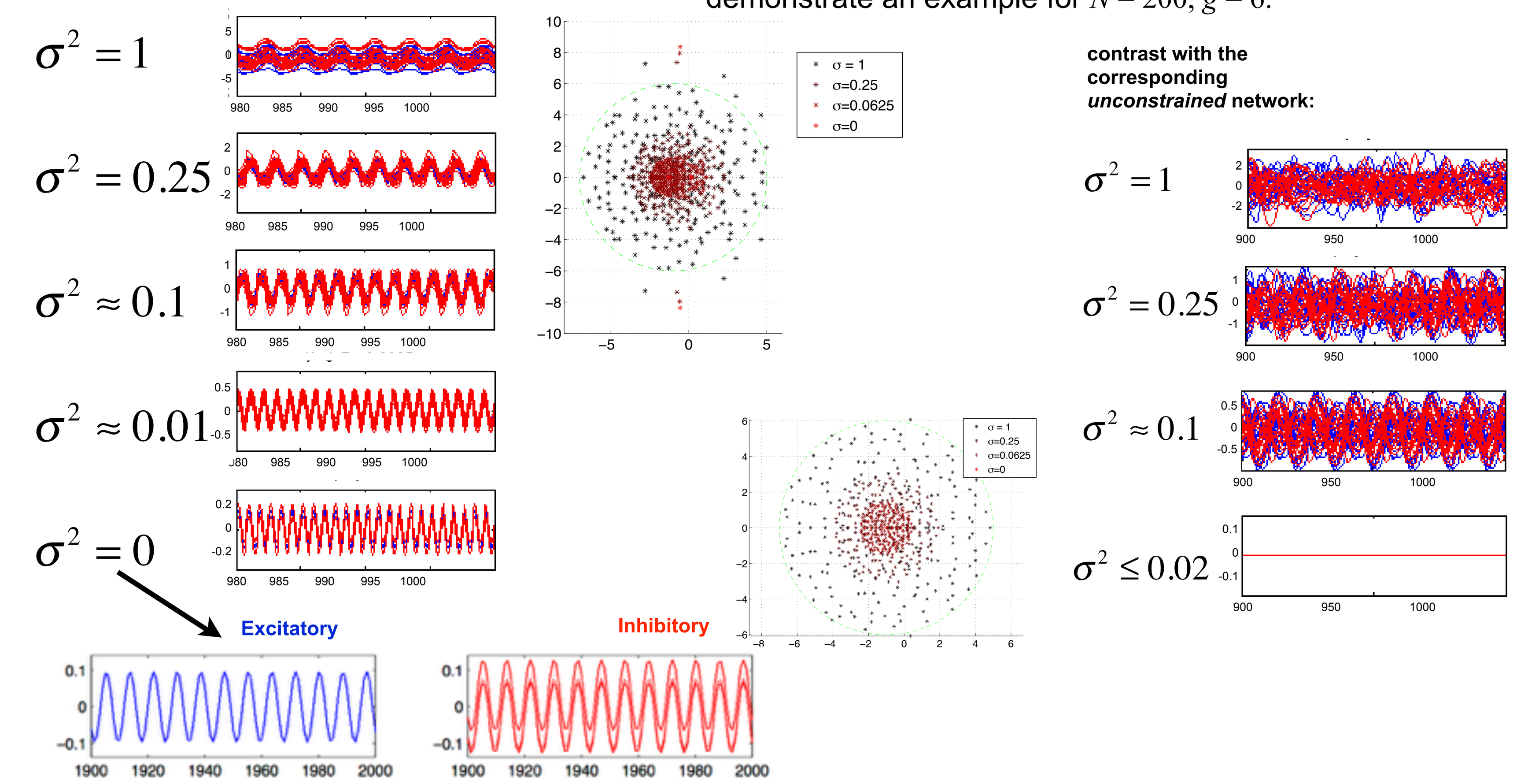
$$\mathbf{G}_{ij} = \begin{cases} \mu_E/\sqrt{N} + \sigma A_{ij}, & i \neq j, 1 \leq j \leq n_E \\ \mu_I/\sqrt{N} + \sigma A_{ij}, & i \neq j, n_E < j \leq N \end{cases}$$

(compare with $\sigma = 0$ below:)



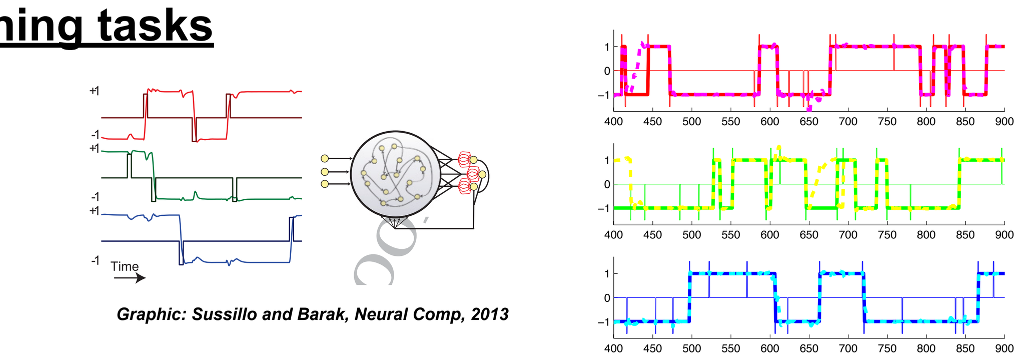
Large networks

This bifurcation structure does not depend on N; here we demonstrate an example for $N = 200, g = 6$.

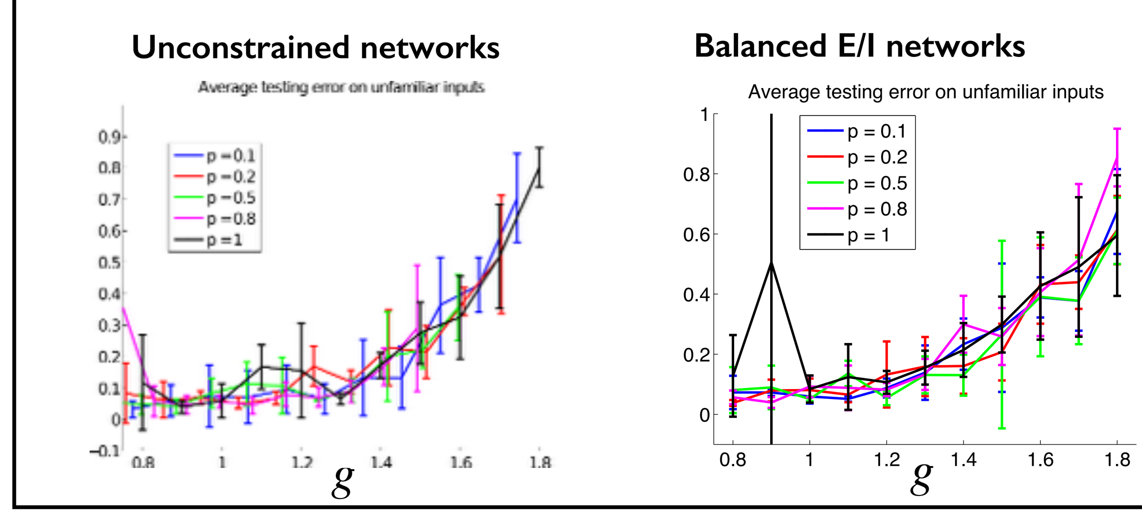


On-going work: impact on learning tasks

$$\dot{\mathbf{x}} = -\mathbf{x} + \mathbf{G}\bullet(\tanh(g\mathbf{x})) + \mathbf{W}^{FB}\mathbf{z} + \mathbf{B}\mathbf{u}$$
$$\mathbf{z} = (\mathbf{W}^{FF})^T(\tanh(g\mathbf{x}))$$
$$\mathbf{W}^{FF}(t) \to \mathbf{W}^{FF}(t + \Delta t)$$



Graphic: Sussillo and Barak, Neur Comp, 2013

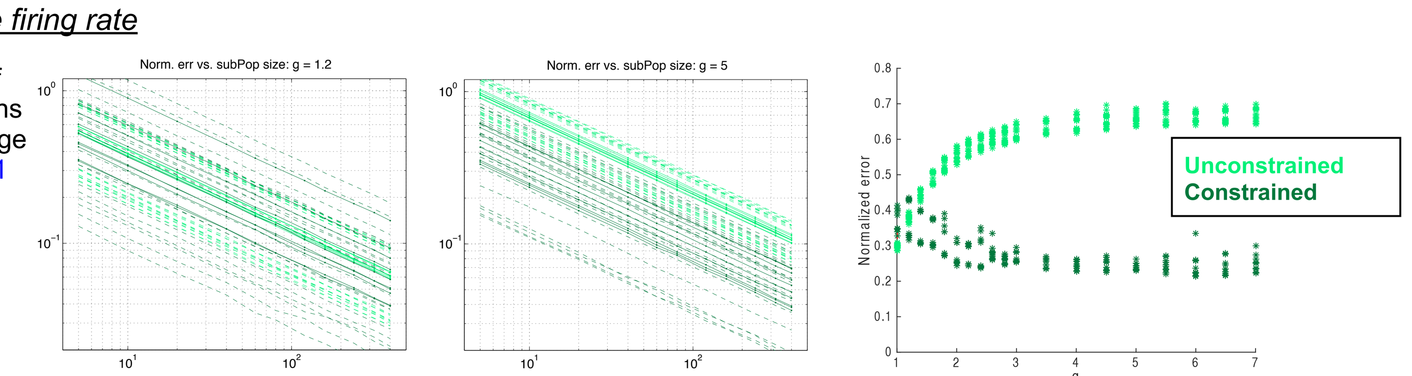


No obvious difference in performance between unconstrained and balanced networks

On-going work: impact on coding tasks

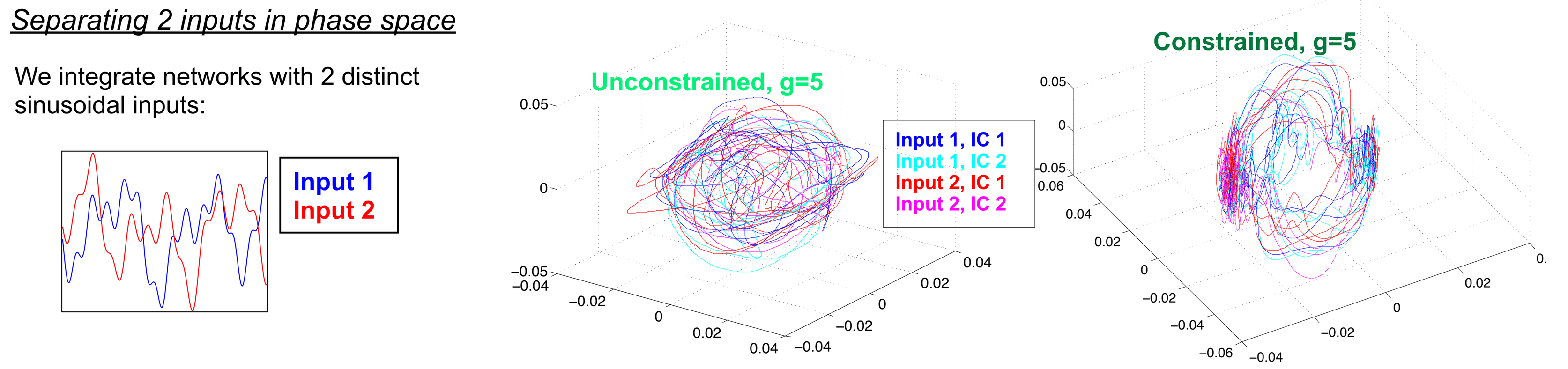
Encoding average firing rate

We test the ability of neural subpopulations to encode the average firing rate (on Input 1 below):



Separating 2 inputs in phase space

We integrate networks with 2 distinct sinusoidal inputs:



References

Barreiro, A.K., J.N. Kutz and E. Shlizerman, submitted, 2016 (**arXiv:1602.05092**)
Bertschinger, N. and T. Natschläger, *Neur. Comp.*, 2004.
Buonomano, D.V. and W. Maass, *Nat Rev Neurosci*, 2009.
Jaeger, H. and H. Haas, *Science*, 2004.
Garcia del Molina, L.C. et al., arXiv:1306.2576v3, 2013.
Golubitsky, M., I. Stewart and D.G. Schaeffer, *Singularities and Groups in Bifurcation Theory*, 1988.
Ostojic, S., *Nat Neurosci*, 2014.
Rajan, K. and L.F. Abbott, *PRL*, 2006.
Sompolinsky, H., A. Crisanti and H.J. Sommers, *PRL*, 1988.
Sussillo, D. and L.F. Abbott, *Neuron*, 2009.
Sussillo, D. and L.F. Abbott, *PLOS One*, 2012.
Sussillo, D. and O. Barak, *Neur. Comp.*, 2013.
van Vreeswijk, C. and H. Sompolinsky, *Neur. Comp.*, 1998.
Wei, Y., *Physical Review E*, 2012.

Acknowledgements: *NSF/NIGMS DMS-1361145 (ES), Washington Research Foundation Fund (ES), MBI Early Career Award (AKB).*