Spectral Gradient Flow and Equilibrium Configurations of Point Vortices

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Why seek equilibria of point vortex systems?

- Quasi-2D, vortex-dominated fluid flows
- Rotating superfluid ^4He
- Electron columns in a Malmberg-Penning trap

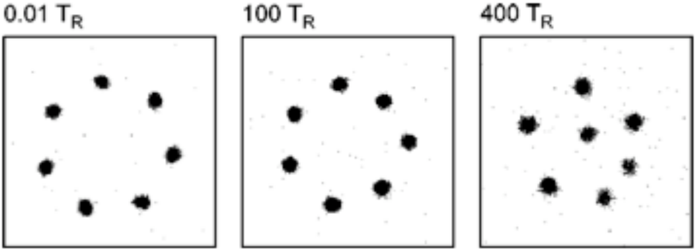


Figure: From Durbin and Fajan, *Physics of Fluids*, 2000

Some motivation for our work

- Many known techniques rely on symmetries (for an exception see Aref and Vainchtein, *Nature*, 1998)
- Complete classification is likely to be difficult (O'Neil 1987, Hampton and Moeckel 2006, *Trans. AMS*) particularly for large numbers of vortices.
- What is needed: a fast, reliable method for finding large N asymmetric equilibria.

N-point vortex problem

- $\vec{x}_\alpha \in R^2, \; \Gamma_\alpha \in R, \qquad 1 \leq \alpha \leq N$
- $$\begin{aligned} \Gamma_\alpha \dot{\vec{x}}_\alpha &= J \nabla_\alpha H \\ H &= -\frac{1}{2\pi} \sum_{\alpha<\beta}^{N} \Gamma_\alpha \Gamma_\beta \ln(r_{\alpha\beta}) \\ J &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{aligned}$$
- Conserved quantities: H, $I = \sum_{\alpha=1}^{N} \Gamma_\alpha |\vec{x}_\alpha|^2$, $V = \sum_{\alpha=1}^{N} \vec{x}_\alpha$

N-point vortex problem

- N point vortices at position $z_\alpha = x_\alpha + i y_\alpha$, circulations Γ_α
- Dynamics given by

$$\Gamma_\alpha \dot{z}_\alpha = -2i \frac{\partial H}{\partial z_\alpha^*}$$

$$H = -\frac{1}{2\pi} \sum_{\alpha<\beta}^{N} \Gamma_\alpha \Gamma_\beta \ln |z_\alpha - z_\beta|$$

- Or

$$\frac{dz_\alpha}{dt} = -\frac{1}{2\pi i} \sum_{\beta \neq \alpha} \frac{\Gamma_\beta}{z_\alpha^* - z_\beta^*}$$

N-point vortex problem

- How to find relative/stationary/translating equilibria?
- Rotating

$$\frac{dz_\alpha}{dt} = i\omega z_\alpha$$

for some ω

- Stationary

$$\frac{dz_\alpha}{dt} = 0$$

- Translating

$$\frac{dz_\alpha}{dt} = 1$$

The essential observation is that $\frac{d\vec{z}}{dt}$ is linear in $\vec{\Gamma}$ (Newton and Chamoun, *Proc. Roy. Soc. A*, 2007). If I define

$$M(\vec{z}) = \begin{pmatrix} 0 & \frac{i}{z_1^*-z_2^*} & \frac{i}{z_1^*-z_3^*} & \frac{i}{z_1^*-z_4^*} & \cdots & \frac{i}{z_1^*-z_N^*} & iz_1 \\ \frac{i}{z_2^*-z_1^*} & 0 & \frac{i}{z_2^*-z_3^*} & \frac{i}{z_2^*-z_4^*} & \cdots & \frac{i}{z_2^*-z_N^*} & iz_2 \\ \vdots & \vdots & \vdots & & \ddots & & \\ \frac{i}{z_N^*-z_1^*} & \frac{i}{z_N^*-z_2^*} & \frac{i}{z_N^*-z_3^*} & \cdots & \frac{i}{z_N^*-z_{N-1}^*} & 0 & iz_N \\ -iz_1 & -iz_2 & -iz_3 & \cdots & -iz_{N-1} & -iz_N & 0 \end{pmatrix}$$

then we must find $\vec{z}$, $\vec{\Gamma}$ such that

$$M(\vec{z})\vec{\Gamma} = M(\vec{z}) \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \\ \vdots \\ \omega \end{pmatrix} = 0$$

$$M(\vec{z}) = M_R(\vec{z}) + iM_I(\vec{z})$$

So we need to find vortex positions such that M_R, M_I share a real nullspace. $\vec{\Gamma}$ can then take any value in the nullspace.

One equivalent condition is that

$$f(\vec{z}) = \det(M_R^T M_R + M_I^T M_I) = 0$$

We propose driving $f(\vec{z})$ to zero via a gradient flow. Define the gradient of f with respect to a $n \times n$ real matrix M as

$$(\nabla_M f)_{j,k} = \frac{\partial f}{\partial M_{j,k}}.$$

Using the chain rule, we evolve coordinates according the equation

$$\begin{aligned} \frac{dx_i}{ds} &= -\frac{\partial}{\partial x_i} f(M) \\ &= -\sum_{j,k} \frac{dM_{j,k}}{dx_i} (\nabla_M f)_{j,k} \end{aligned}$$

The flow on $\det(M_R^T M_R + M_I^T M_I)$ has some nice properties. Because

$$M_s = det(M)M^{-1}$$

the singular value basis is unchanged under the flow. Therefore we can write it in terms of the singular values of M:

$$\frac{d\sigma_j}{ds} = -2\sigma_j \prod_{k \neq j} \sigma_k^2$$

and show that the difference in squares of singular values is conserved.

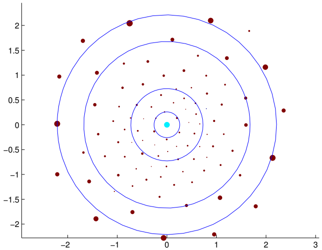
$$\sigma_j^2 - \sigma_k^2 = C_{jk}$$

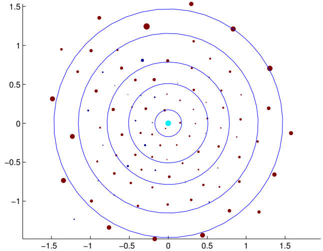
Practical problem: gradient descent is slow.

To speed convergence, we use a subspace trust region minimization algorithm (suitable for large-scale problems), Implemented in the Matlab Optimization Toolbox (Branch et al., *SIAM J. Sci. Comp*, 1999).

(So I just threw my nice properties out the window, for now)

Here are some examples found by minimizing $\det(M_R^T M_R + M_I^T M_I)$: rotating equilibria, with *free* vortex strengths

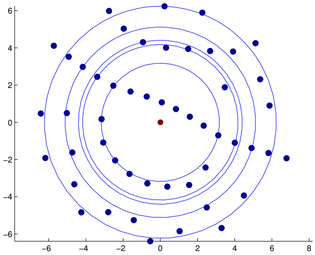


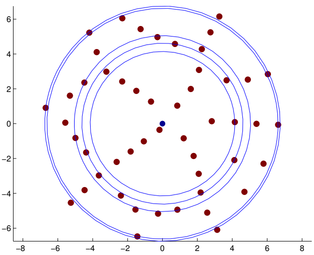


- We can choose other functions to minimize which (if they can be driven to zero) would imply an equilibrium.
- Suppose we specify vortex strengths $\vec{\Gamma} \equiv \vec{v}$. Then we could minimize

$$\| M_R \vec{v} \|^2 + \| M_I \vec{v} \|^2$$

These are rotating equilibria, with *fixed* vortex strengths. Vortex strengths have been chosen to be identical.





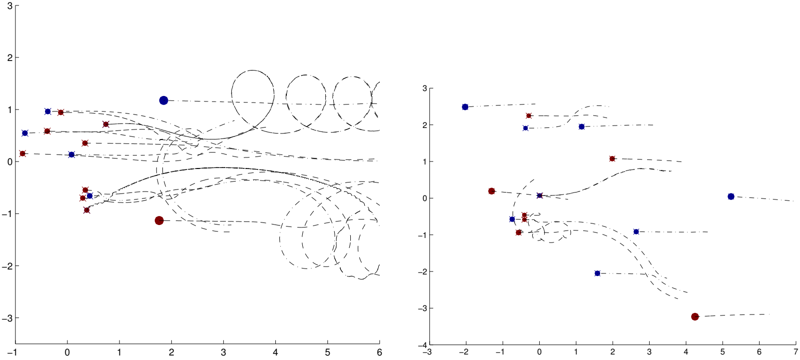
Or, we can seek translating vortex patterns, by finding $\vec{z}$,$\vec{\Gamma}$ s.t.

$$\begin{aligned} \tilde{M}_R(\vec{z})\vec{\Gamma} &= \vec{1} \\ \tilde{M}_I(\vec{z}) &= \vec{0} \end{aligned}$$

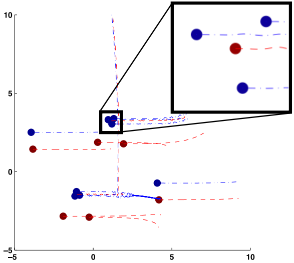
where $\tilde{M}$ is obtained from the previous M by the last (the $(N+1)$th) row and column.
So we minimize $\| \tilde{M}_I^{-1}\tilde{M}_R \|^2$ and recover $\vec{\Gamma} = \tilde{M}_R^{-1}\vec{1}$.
We find *fixed* strength patterns by minimizing

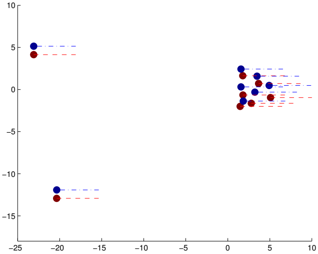
$$\| \tilde{M}_R - \vec{1} \|^2 + \| \tilde{M}_I \|^2$$

Some examples of translating solutions, for free vortex strengths...



...and for fixed vortex strengths.

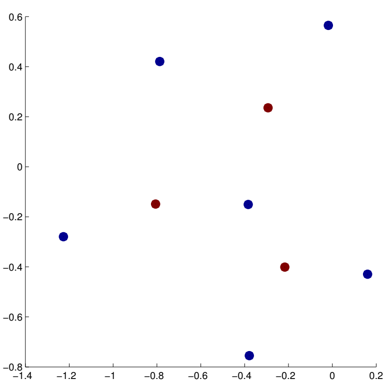


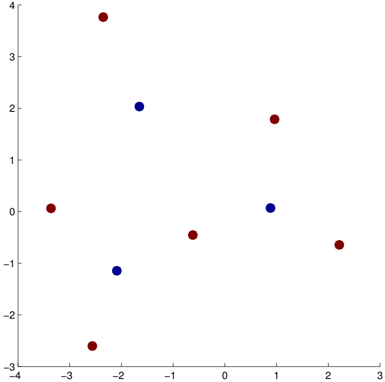


Here we begin to notice something interesting: clustering into locally rigid configurations.

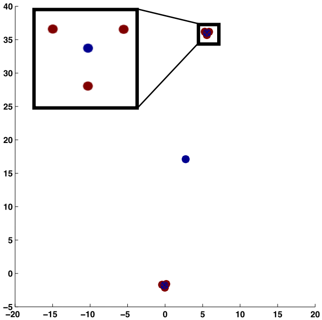
We seek stationary solutions by minimizing

$$\| \tilde{M}_R \vec{v} \|^2 + \| \tilde{M}_I \vec{v} \|^2, \qquad \vec{v} \textit{ fixed}$$





We can also find solutions with clustering into locally rigid or stationary configurations.



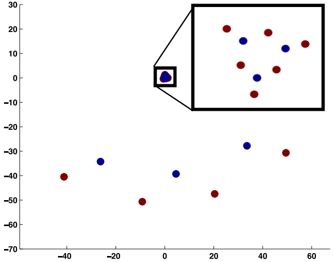


Figure: (right inset: see, for example, Aref, *Fluid Dynamics Research*, 2007

Future Work

- We have little analytical understanding of equilibria.
- Can we make gradient descent viable?
- Can we drive equilibria to stability?