Journal of Economic Dynamics & Control 53 (2015) 15–27

Contents lists available at ScienceDirect

Journal of Economic Dynamics & Control

journal homepage: www.elsevier.com/locate/jedc

Deforestation and optimal management

Adriana Piazza [a,1], Santanu Roy [b,*]

[a] Universidad Técnica Federico Santa María, Valparaíso, Chile
[b] Southern Methodist University, Dallas, TX, USA

ARTICLE INFO

Article history:
Received 28 May 2014
Received in revised form
12 January 2015
Accepted 14 January 2015
Available online 23 January 2015

JEL classification:
Q23
H23
Q58
C62

Keywords:
Deforestation
Optimal forest management
Conservation
Renewable resources
Extinction

ABSTRACT

In a general discrete time model of optimal forest management where land may be diverted to alternative use and stocks of standing trees may yield flow benefits, we investigate the economic and ecological conditions under which optimal paths lead to (total) deforestation i.e., complete long term removal of forest cover. We show that if deforestation occurs from some initial state, then it must occur in finite time along *every* optimal path so that zero forest cover is the globally stable optimal steady state. We develop a condition that is both necessary and sufficient for deforestation. Deforestation is less likely if the immediate profitability of timber harvest, the benefits from stocks of standing forests and the timber content of trees are higher. We characterize the minimum forest cover along optimal paths (when deforestation is not optimal). We design a simple linear subsidy on standing forest biomass that can motivate a private owner (who does not take into account the external benefits from standing trees) to conserve forests.

1. Introduction

Deforestation is an important environmental concern. Globally, around 13 million hectares of forests disappeared each year between 2000 and 2010 (Global Forest Resources Assessment, 2010).[2] Social scientists tend to focus on tropical deforestation[3] where weakness of property rights (leading to encroachment and illegal logging) and myopic management practices are some of the key human factors.[4] However, deforestation may occur even when property rights are well defined (and strongly enforced) and forest management is based on a long time horizon. Depending on the intertemporal costs and benefits that a forest manager takes into account, deforestation can be the consequence of an optimal strategy where trees

* Corresponding author. Department of Economics, Southern Methodist University, 3300 Dyer Street, Dallas, TX 75275-0496, USA. Tel.: +1 214 768 2714; fax: +1 214 768 1821.

E-mail addresses: adriana.piazza@usm.cl (A. Piazza), sroy@smu.edu (S. Roy).

[1] Departamento de Matemática, Universidad Técnica Federico Santa María. Avda. España 1680, Casilla 110-V, Valparaíso, Chile. Tel.: +56 22 3037 331; fax: +56 32 2654 563.

[2] According to FAO's Global Forest Resources Assessment 2010 (Global Forest Resources Assessment, 2010) forests currently cover about 4 billion hectares, comprising approximately 31 percent of the earth's land surface.

[3] There is a large literature on tropical deforestation (see, for instance, Angelsen and Kaimowitz, 1999) that focuses on somewhat different economic factors; this literature ignores the age structure of forests.

[4] For an excellent exposition of the literature, see Amacher et al. (2009).

http://dx.doi.org/10.1016/j.jedc.2015.01.004
0165-1889/

are cut down without replanting and forest land is diverted to alternative uses over time. This paper attempts to characterize the economic and ecological conditions under which optimal dynamic management leads to deforestation.

Besides timber and shelter, forests provide many environmental services such as biodiversity, water and soil conservation, water supply, climate regulation and carbon sequestration. Reduction in total forest area has led to a decrease in carbon stocks sequestered in the forest biomass by an estimated 0.5 Gt per year over the decade 2000–2010; this has contributed significantly to adverse climate change. Often, there are no markets for such non-timber products and services provided by forests.[5] We highlight the role that such benefits flowing from stocks of standing forests can play in determining whether or not deforestation is optimal, and the implications for design of optimal subsidies to incentivize private owners (that may not take into account these benefits) to conserve forests.

In the existing literature, conditions for optimal extinction and conservation have been characterized for renewable resources whose biological growth depends mainly (if not exclusively) on the size of the remaining population.[6] Forests are however somewhat different from many other renewable resources in that regeneration is largely dependent on the availability of land and on decisions regarding land use. Models of forest management also need to take into account the relatively long rotation, the multi-age structure and the age dependent timber content of trees. In managed forests, trees can always be planted and zero resource stock is not an absorbing state. As a result, the concept of "extinction of a forest" is somewhat different from that for other biological species.[7] In this paper, we focus on (total) deforestation which is said to occur when all available land is diverted to alternative (non-forest) use and the area under forest cover dwindles to zero over time.[8]

Optimal forest harvesting is a problem that dates back to the 19th century. In his seminal paper, Faustmann (1849) proposed an appropriate formula for valuation of an even-aged forest that allows one to determine the optimal rotation length. A wide literature has developed since then in numerous directions that range from more sophisticated growth models to models that allow for non-timber forest products such as tourism and environmental services. Optimal harvesting policies are frequently studied numerically with different types of even-flow constraints, or requiring convergence to a predetermined steady state (Johnson and Scheurman, 1977; Lyon and Sedjo, 1983, 1986).

The optimal timber harvesting problem was reconsidered by Mitra and Wan (1985, 1986) in a discrete time dynamic theoretical framework that allows for more general analytical characterization. In this paper, we consider a variation of this well known model of optimal forest management. It assumes that the timber content per unit of forest area is related only to the age of the trees, so that the forest can be represented as a collection of age classes.[9] The focus of the Mitra–Wan papers (and indeed of much of the subsequent theoretical literature on optimal forest use, see for example Salo and Tahvonen, 2003; Tahvonen, 2004a) is on the dynamics of forest rotation and it is assumed that the total area under forests is fixed over time. In order to study the problem of depletion of forest area, we extend the model to allow for alternative use of land.

Such a model with alternative use was studied analytically by Salo and Tahvonen (2002, 2004). They focus on the existence and uniqueness of the steady state. They show the existence of optimal periodic cycles when the steady state is a pure forestry state (with no alternative use), and the impossibility of such cycles if the steady state is "mixed"; they also analyze the stability of the steady state in the latter case. While their analysis allows for the possibility of an optimal steady state where land is allocated exclusively to alternative use, they do not explicitly study the existence or stability of such a steady state and therefore, their analysis does not shed light on the specific issue of deforestation.

Apart from allowing for alternative use and focusing on deforestation, our model also allows for a flow of benefits from *stocks* of standing forests that may capture environmental externalities that could be taken into account by a social planner, as well as earnings from subsidies (for instance, based on forest area or timber content) and recreational use that add to the profits earned by a private owner.[10] This is in addition to the usual benefits from timber harvests. In the existing literature, a number of papers have studied optimal forest management with stock benefits of various kind; to the best of our knowledge, none of them allow for alternative simultaneous (non-forest) use of land. Bowes and Krutilla (1985, 1989) consider stock benefits from standing forests in an age class forestry model; they extend the Mitra–Wan framework to include a benefit that depends on the standing age class distribution of the forest. They characterize the steady states and the occurrence of periodic cycles.[11] Tahvonen (2004a,b) studies the optimal management of a forest with a somewhat different kind of stock benefit that depend on the positive environmental externality emanating from the forest area devoted to old trees which are never harvested for timber. Apart from the existence of optimal periodic cycles, it is shown that the introduction of such a stock benefit can lead to a continuum of steady states. Our paper extends the broad

[5] The non-market value of a large part of standing forests may exceed the value of timber extracted or even that of converting land to alternative uses (Pearce, 2001).

[6] See, for instance, Clark (1973).

[7] There is a literature on forest use using mining models that focuses on the effect of depletion of old-growth timber stands on prices of timber and allows for the possibility of full depletion. See, among others, Berck (1979), Lyon (1981) and Lyon and Sedjo (1983).

[8] Note that this concept differs from the usual sense of the term "deforestation" as referring to any decline in forest cover.

[9] This assumption may not be applicable to wild forests.

[10] For biological species and other renewable resources whose reproduction depends on existing population size, there is a fairly extensive analysis of optimal conservation and extinction in a framework that allows for stock effects in net benefits from harvesting. See, for instance, Olson and Roy (1996).

[11] Bowes and Krutilla compute examples where deforestation (in their framework, a situation where all land is eventually barren) is avoided only if the stock benefits are taken into account. In these examples, it is assumed that each age class is either fully harvested or left untouched.

framework in this literature by allowing for alternative economic use of land but our analysis is focused on characterization of one specific aspect of optimal paths namely, deforestation.

We show that if deforestation occurs, then it must occur in finite time (in fact, in our model, within the first n periods where n is the lifetime of a tree). We develop a tight condition for deforestation to be optimal; this is a verifiable condition on the exogenous elements of the model such as the benefit from timber harvesting and standing forest stocks, the cost of planting trees, the discount factor, the benefit from alternative use and the timber content of trees of various ages. The condition is both necessary and sufficient for deforestation and is, in fact, identical to the condition under which the state of deforestation is an optimal steady state, and also the condition under which such a state is globally stable. In other words, deforestation can occur along *some* optimal path if, and only if, zero forest cover is the unique optimal steady state and independent of the initial condition of the forest, this state is reached in finite time. One reason why this is interesting is because models of forest use are known to generate periodic cycles and optimal steady states that are not globally stable; our analysis indicates that such nonlinear dynamics and multiplicity of attractors do not characterize optimal paths in circumstances where deforestation is optimal.

Our general results are applied to the problem of a private owner who ignores the environmental services provided by the standing forest, and the problem of a social planner who takes them into account. We characterize situations under which deforestation is privately but not socially optimal and make policy suggestions about the design of subsidies based on standing forest stocks. Further, in situations where deforestation is not optimal for the forest manager, we derive lower bounds on the *forest cover* under the optimal harvesting policy; this is of particular relevance given the fact that actual forest cover may cycle over time along the optimal path. Finally, we derive some interesting comparative statics. For instance, we show that deforestation is less likely if the price of timber increases (for instance, because of an increase in demand for timber), the cost of planting new trees decreases (technology improvement), the timber content of trees increases and if the return on alternative use declines.

The paper is organized as follows. Section 2 presents the model and derives a set of Euler inequalities satisfied by any optimal program. Section 3 contains the main contribution of this paper i.e., the necessary and sufficient conditions for deforestation and their implications for public policy. Section 4 discusses results on lower bounds on forest cover along the optimal path in situations where deforestation is not optimal. Section 5 illustrates our key results with a numerical example. Section 6 discusses possible extensions of the model and indicates how our results may be modified by some of these. Section 7 concludes. All proofs are contained in the appendix.

2. Preliminaries

Time is discrete and is indexed by $t = 0, 1, 2, \ldots$. Total area of land is constant over time and set equal to 1. Land is allocated between forest and an alternative use. The forest consists of trees whose age can vary from 1 to n where $n > 1$ represents the age at which a tree dies or loses its economic value. The timber content per unit of forest area is related only to the age of the trees. Hence, we can group the trees into n age classes and represent the state of the forest by specifying the area occupied by each one of them.[12] More precisely, let $x_{a,t}$ denote the area occupied by the output of trees of age a at the very beginning of period t *before any harvesting or planting of trees take place*. In each period t, the state of the forest can be represented by the vector $\boldsymbol{x}_t = (x_{1,t}, \ldots, x_{n,t})$,[13] where $\boldsymbol{x}_t$ belongs to the set $\mathcal{D}$ defined by

$$\mathcal{D} = \left\{ \boldsymbol{x} \in \mathbb{R}^n_+ : \sum_{a=1}^{n} x_a \le 1 \right\}.$$

We assume that all land not occupied by this standing forest has been dedicated to alternative use. The area under alternative use at the beginning of period t (or more precisely, during the full length of period $(t-1)$) is then given by

$$y_t = 1 - \sum_{a=1}^{n} x_{a,t}. \quad (1)$$

Given the current state $\boldsymbol{x}_t$ at the beginning of period t, the forest manager or owner (hereafter, agent) makes two sets of decisions; both are made at the beginning of period t. First, the agent decides on $c_{a,t} \in [0, x_{a,t}]$, the part of the land occupied by (the output of) trees of age-class a that is harvested in that period, $a = 1, \ldots, n$. The size of current harvest at the beginning of period t is then $\boldsymbol{c}_t = (c_{1,t}, \ldots, c_{n,t})$. Without loss of generality, we assume that $c_{n,t} = x_{n,t}$. After this harvest,

$$z_{a,t} = x_{a,t} - c_{a,t}$$

is the area occupied by trees of age class a that remain standing, $a = 1, \ldots n$. Second, given the harvesting decision $\boldsymbol{c}_t$, the total land available for replanting is $\sum_{a=1}^{n} c_{a,t} + y_t$; of this area, the agent plants an area $z_{0,t} \in [0, \sum_{a=1}^{n} c_{a,t} + y_t]$ with new seedlings and the rest (denoted by y_{t+1}) is diverted to alternative use during the full length of the period t. With the understanding

[12] In their papers, Mitra and Wan take n to be the age at which the biomass per unit of land is maximized. It was pointed out by Khan and Piazza (2012), that the concavity of the benefit function favors a homogeneously configured forest and that it may be optimal to postpone harvesting beyond age n in order to reshape the forest into a more homogeneous state. Following the approach used in Khan and Piazza (2012), we circumvent this by assuming n to be the age after which a tree dies.

[13] The expressions in bold print represent vectors.

that age 0 represents new seedlings and that $z_{n,t}=0$, the vector $\boldsymbol{z}_t=(z_{0,t}, z_{1,t}, \ldots z_{n-1,t})$ then gives us the land occupied by the *input* of trees of all age classes at the beginning of period *t after harvesting and replanting decisions are made*. This input generates the output of trees of various age classes at the beginning of period $t+1$ through the natural production process of aging. In particular, the area occupied by (the output of) trees of age $(a+1)$ at the very beginning of period $t+1$ is given by

$$x_{a+1,t+1} = z_{a,t}, a = 0, 1, \ldots n. \tag{2}$$

At the very beginning of period $t+1$ (prior to any harvesting and replanting), the agent faces a standing forest

$$\boldsymbol{x}_{t+1} = (x_{1,t+1}, \ldots x_{n,t+1}) = (z_{0,t}, x_{1,t} - c_{1,t}, \ldots, x_{n-1,t} - c_{n-1,t})$$

and an amount $y_{t+1} = 1 - \sum_{a=1}^{n} x_{a,t+1}$ of land that has been dedicated to alternative use (over the full length of the period t).

A sequence $\{\boldsymbol{x}_t\}_t$ is a *program* iff

$$\boldsymbol{x}_t \in \mathcal{D} \forall t \quad \text{and} \quad x_{a+1,t+1} \le x_{a,t} \text{ for } a = 1, \ldots, n-1 \ \forall t$$

Associated with any program, we have a sequence of land dedicated to the alternative use, $\{y_t\}$, and a sequence of harvests, $\{\boldsymbol{c}_t\}$, that are calculated using (1) and (2) respectively.

As in the original model analyzed by Mitra and Wan (1985, 1986), the timber content of one unit of area covered by trees of age a is given by the production function $f(a) \ge 0$. For notational convenience, we define the biomass coefficients as $f_a = f(a)$ for $a = 1, \ldots, n$.[14] The total timber consumption in period t is given by c_t:

$$c_t = \sum_{a=1}^{n} f_a c_{a,t}. \tag{3}$$

After tree cutting at the beginning of period t, the agent receives net benefit from timber harvesting, $U(c_t)$. In addition, at the beginning of period t, the agent receives return $W(y_t)$ from the dedication of land y_t to alternative use over the length of the previous period $(t-1)$. Immediately after the cut, the seedlings of what will constitute the following period's young forest are planted; planting of new trees has an associated cost of $p>0$ per unit of area and the agent incurs a total planting cost of $pz_{0,t} = px_{1,t+1}$.

In addition to the benefits and costs described above, the agent also derives "stock" benefit $S(\boldsymbol{x}_t)$ that summarizes the flow of benefits from standing trees at the beginning of period t. Here, $S(\boldsymbol{x})$ may reflect amenity value, biodiversity preservation and carbon sequestration externalities for a social planner, private owner's earnings from subsidies on timber content or forest area, private and social benefits from tourism etc.

We impose the following assumptions on the benefit functions

A_1: U and W are concave, non-decreasing, continuous on $\mathbb{R}_+$ and differentiable on $\mathbb{R}_{++}$.
A_2: S is continuous on $\mathcal{D}$ and differentiable on the interior of $\mathcal{D}$. The partial derivatives (S_a) are non-negative and non-increasing with respect to every coordinate x_j.
A_3: U is strictly concave in a neighborhood of zero.

Assumptions A_1 and A_2 are retained throughout the paper and will not be specified in the statements of the results, while A_3 is only used in Theorem 1.[15] We follow the usual convention that $U'(0)$ is the right hand side derivative $U'_+(0) = \lim_{h \to 0^+} (U(h) - U(0))/h$. Similarly, the partial derivatives of S evaluated at a point in the boundary of $\mathcal{D}$ refer to the corresponding one-sided partial derivatives.

Note that A_1 and A_3 imply that U is strictly increasing in a neighborhood of zero. We point out that we do not require any of the functions to be increasing on their domains. Furthermore, we allow for $S \equiv 0$ and $W \equiv 0$, which corresponds to the classical forest management model without alternative use presented by Mitra and Wan (1985); with the difference that in our case land may be left fallow.

Given an initial state $\mathbf{x} \in \mathcal{D}$, the dynamic optimization problem is

$$\begin{cases} \text{maximize} & \sum_{t=0}^{\infty} b^t [U(c_t) + W(y_t) + S(\boldsymbol{x}_t) - px_{1,t+1}] \\ \text{subject to} & (1)\text{–}(3) \\ & \{\boldsymbol{x}_t\} \text{ is a program and } \boldsymbol{x}_0 = \mathbf{x} \end{cases} \tag{4}$$

where $0 < b < 1$ is the discount factor, $\boldsymbol{c}_t$ and y_t are the control variables and $\boldsymbol{x}_t$ is the state variable.

In the following lemma we state some useful Euler inequalities that any optimal program must satisfy.

Lemma 1. *Let $\{\boldsymbol{x}_t\}_{t=0}^{\infty}$ be an optimal program.*

[14] As in Khan and Piazza (2012), we dispense with most of the restrictions on the biomass coefficients that are found in the literature (for instance, in Mitra and Wan, 1985, 1986; Salo and Tahvonen, 2003, 2004; Tahvonen, 2004a).

[15] A weaker version of Theorem 1 can be obtained without A_3, see Footnote 16.

If $\min_{j=1\ldots a}\{y_{t+j}\}>0$ *for some* $t \geq 0$, *then*

$$p+\sum_{j=1}^{a} b^j W'(y_{t+j}) \geq b^a f_a U'(c_{t+a})+\sum_{j=1}^{a} b^j S_j(\boldsymbol{x}_{t+j}). \tag{5}$$

If $c_{a,t+a}>0$ *for some* t, *then*

$$p+\sum_{j=1}^{a} b^j W'(y_{t+j}) \leq b^a f_a U'(c_{t+a})+\sum_{j=1}^{a} b^j S_j(\boldsymbol{x}_{t+j}). \tag{6}$$

3. Deforestation

3.1. Necessary and sufficient conditions for deforestation

As mentioned in the introduction, the concept of deforestation is related to depletion of area under forest and diversion of such land to alternative (non-forest use). We begin by formally defining the concept of deforestation in the context of our model.

Definition 1. A program $\{\boldsymbol{x}_t\}_{t=0}^{\infty}$ is said to be characterized by *eventual deforestation* if

$$\lim_{t\to\infty} y_t = 1.$$

Definition 2. A program $\{\boldsymbol{x}_t\}_{t=0}^{\infty}$ is said to be characterized by *immediate deforestation* if y_t is non-decreasing and

$$y_t = 1 \quad \text{for all } t \geq n.$$

Obviously, immediate deforestation implies eventual deforestation. We will show that under assumption A_3, immediate deforestation is optimal from any initial state if, and only if, eventual deforestation is optimal from some initial state. Further, they are also equivalent to the zero forest cover state, $\mathbf{0}=(0,\ldots,0)$ being a (optimal) steady state. These results are part of the next proposition, the main theoretical result of this paper, that develops a condition for deforestation that is both necessary and sufficient.

Proposition 1. *Under* A_3 *the following assertions are equivalent*[16]

1. *Every optimal program is characterized by immediate deforestation.*
2. *There exists an optimal program that is characterized by eventual deforestation.*
3. *The following conditions hold*:

$$\frac{b}{1-b}W'(1) \geq \frac{b^a}{1-b^a} f_a U'(0)+\frac{1}{1-b^a}\sum_{j=1}^{a} b^j S_{x_j}(\mathbf{0})-\frac{p}{1-b^a} \quad \forall a=1,\ldots,n \tag{7}$$

4. *The optimal policy is one where no new trees are planted in any period.*
5. *The state of zero forest cover* ($\boldsymbol{x}=\mathbf{0}$) *is a globally stable steady state.*
6. *The state of zero forest cover* ($\boldsymbol{x}=\mathbf{0}$) *is a steady state.*

Proposition 1 indicates that the economic and ecological conditions under which optimal forest management leads to immediate deforestation and those that lead to eventual deforestation are identical and in fact, they are the very same conditions under which starting from a state where there is zero forest cover on land, it is optimal to not plant any new trees and to remain in that state of zero forest cover forever. Under these conditions, the state of zero forest cover becomes a globally stable steady state. Eq. (7) provides a very tight characterization of when all of these are optimal; it is a condition on the exogenous elements of the model and is verifiable from its economic and ecological primitives.

The left hand side of (7) reflects the marginal (opportunity) cost of bringing some land under forests once and forever; i.e., the foregone benefit from alternative use. The right hand side of (7) captures the marginal benefit from planting trees; this includes the marginal benefit from timber harvesting (realized at the points of time the planted trees are harvested) and the flow of marginal stock benefits from the standing trees until their harvest minus the cost of planting trees. Eq. (7) requires that starting from a state of zero forest cover, the dynamic marginal cost of planting trees exceeds the dynamic marginal benefit of doing so. This suggests that it is not optimal to move from a state of zero forest cover i.e., that the latter is an optimal steady

[16] Without A_3 we have that $1 \Rightarrow 2 \Rightarrow 3$ and that the strict inequality version of (7) implies 1. Hence, the equivalence of the assertions is lost.

state. The fact that this condition also implies that it is optimal to converge to this steady state in finite time from any state (i.e., independent of the extent of forest cover and the age structure of the forest) is the interesting part of Proposition 1.

It is easy to see from (7) that low timber content of trees, low price for timber (low marginal benefit from timber harvesting), high planting cost and high marginal return on alternative use are likely to create greater incentives for deforestation. On the other hand, technological changes that increase the timber content (or forest growth) or reduce harvesting costs are likely to encourage conservation of forests. An increase in the stock benefits from standing forests internalized by the forest manager also encourages conservation.

Observe that as $b>0$, if $U'(0)=\infty$, then deforestation can never occur (no matter how much the future is discounted). It is already known that in this case, timber consumption must be positive in every time period, but the contribution of Proposition 1 is stronger: if $U'(0)=\infty$, there is *no* optimal program along which the forest cover converges to zero. However, the area of forest cover might be very small.

Though we assume $b>0$, it is easy to check that in our framework deforestation is always optimal for a myopic decision maker with $b=0$. Further, if $U'(0)<\infty$ then (7) is always satisfied and deforestation is optimal if $b>0$ is small enough. However, the effect of an increase in the discount factor b is ambiguous as both sides of (7) are increasing in b. More assumptions are necessary to determine the effect of the variations of b. We will elaborate more in this effect in the next subsection, when the function $S(\boldsymbol{x})$ depends only on the total timber content of the standing forest.

3.2. Implications for public policy

Much of the stock benefits from standing forests (such as carbon sequestration) are in the nature of environmental externalities that are typically not part of the cost-benefit calculations of a profit maximizing private owner (unless there is a regulatory structure to ensure that these externalities are internalized). In this subsection, we use the general characterization in Proposition 1 to focus on the difference between the private and social incentives to engage in deforestation. In particular, we compare the condition for deforestation when the forest is managed by a private owner that ignores all stock benefits to that of a social planner who takes into account these stock benefits. When deforestation is privately but not socially optimal, we examine how simple public subsidy schemes can be designed to provide incentives to a private owner to avoid deforestation.

For simplicity, we assume that the direct social benefit from timber harvest is identical to the private owner's benefit or profit; this holds for instance, if the timber harvest is entirely exported. As a result, the only difference between the private and social optimization problems is that the former sets stock benefits equal to zero.

In the absence of any public policy or regulation, a private owner solves Problem (4) taking $S(\boldsymbol{x})\equiv 0$. The necessary and sufficient condition for deforestation (7) then reduces to

$$\frac{b}{1-b}W'(1)\geq\frac{b^a}{1-b^a}f_aU'(0)-\frac{p}{1-b^a}\quad \forall a=1,\dots,n \tag{8}$$

The social planner however does take into account the stock externalities from the standing forest. For simplicity, we impose some more structure on the nature of stock benefits or externalities by assuming that it depends only on the total timber content of the standing forest[17] i.e.,

$$S(\boldsymbol{x}_t)=A\left(\sum_{a=1}^{n}f_a x_{a,t}\right). \tag{9}$$

In addition, we impose the following assumption on $A(\cdot)$,

A_2': A is concave, non-decreasing, continuous in $\mathbb{R}_+$ and differentiable in $\mathbb{R}_{++}$.

It is easy to see that if $A(\cdot)$ satisfies A_2' then $S(\boldsymbol{x})=A(\sum_{a=1}^{n}f_a x_{a,t})$ satisfies A_2. Once again, using (7) we have that deforestation is socially optimal if and only if

$$\frac{b}{1-b}W'(1)\geq\frac{b^a}{1-b^a}f_aU'(0)+\frac{1}{1-b^a}A'(0)\sum_{j=1}^{a}b^jf_j-\frac{p}{1-b^a}\quad \forall a=1,\dots,n \tag{10}$$

It is easy to check that the right hand side of (8) is smaller than or equal to that of (10) i.e., deforestation is privately optimal if it is socially optimal, but for a subset of the parameter space deforestation is privately optimal but not socially optimal.

While it is always possible to fully align the private owner's incentives with the social planners by using a nonlinear subsidy, we want to focus on a simple subsidy scheme. We will consider subsidies that are linear in total timber content of the standing forest. In particular, in each period t, the total subsidy received by the private owner is $\lambda\sum_a f_a x_{a,t}$ where $\lambda\geq 0$.

[17] A very similar option is to take subsidies that are a function of the total forestry area, $(\sum_a x_{a,t})$, which can also be viewed as a tax on the alternative use, as $\sum_a x_{a,t}=(1-y_t)$. As the analysis for these two options is analogous we will only present our results when the subsidy depends on the total standing biomass leaving the details of the latter case to the reader.

With such a subsidy, the private owner now solves the optimization (4) where

$$S(\boldsymbol{x}) = \lambda \sum_a f_a x_{a,t}. \tag{11}$$

Note that the specific form (11) of stock benefits is a special case of (9) and satisfies assumption A_2. Using (7), a private owner receiving such a subsidy will engage in deforestation if and only if

$$\frac{b}{1-b}W'(1) \geq \frac{b^a}{1-b^a} f_a U'(0) + \frac{\lambda}{1-b^a}\sum_{j=1}^{a} b^j f_j - \frac{p}{1-b^a} \quad \forall a = 1, \ldots, n$$

Deforestation is avoided if λ satisfies

$$\lambda > \min_a \left\{ \frac{1}{\sum_{j=1}^{a} b^j f_j} \left[p + b\frac{1-b^a}{1-b}W'(1) - b^a f_a U'(0) \right] \right\}. \tag{12}$$

This gives us an explicit lower bound on the subsidy needed to avoid deforestation and indicates that forest stock subsidies that are lower than a critical level are unlikely to have any effect on deforestation (small interventions are ineffective).

If (8) holds but (10) does not hold, i.e., deforestation is privately but not socially optimal, $A'(0)$ is greater than the right hand side of (12). While a subsidy equal to $A'(0)$, the marginal stock benefit at zero, can definitely avoid deforestation, we can achieve the same by a smaller subsidy that lies between the right hand side of (12) and $A'(0)$.

Note that improvement in timber content, the timber planting technology and an increase in the price at which timber can be sold (an increase in demand for timber) reduce the minimal subsidy needed to avoid deforestation. As long as deforestation is not *socially* optimal, any further increase in stock benefit from forests does not affect the subsidy needed to avoid deforestation. On the other hand, the minimal subsidy is increasing in p, the cost of planting trees, and in $W'(1)$, the marginal benefit from alternative use of land (at zero forest cover); both discourage forestry. Finally, if we assume that the biomass coefficients are increasing in age, then it can be shown that minimal subsidy decreases with an increase in the discount factor i.e., milder discounting promotes forest conservation. We prove this in Remark 1 in the Appendix.

Of course, even if deforestation is avoided, the forest cover ensured by this subsidy can be arbitrarily small. The next section discusses how a subsidy can be designed to ensure that the forest cover is at a socially optimal level.

4. Minimum forest cover

When economic and ecological conditions rule out deforestation, it is of interest to understand the amount forest cover that is sustained over time. Now, if the (necessary and sufficient) condition for deforestation (7) in Proposition 1 does not hold, then all we know is that

$$\liminf_{t\to\infty} y_t < 1,$$

so that not only can the amount of forest cover be arbitrarily small but in fact, the forest cover may actually be zero along a subsequence of time periods. In this paper, we do not focus on characterizing these possibilities (though an optimal path that cycles between zero and positive forest cover would be qualitatively consistent with existing results on periodic cycles in models of forest management). Instead, we will characterize the *minimum forest cover that must be sustained along the optimal path*.

In what follows, we assume that (7) does not hold for at least one value of a so that deforestation is not optimal. Further, we impose an additional assumptions on the function S. Let m be the index of the maximal biomass coefficient (i.e., $f_m \geq f_a$ for all $a = 1, \ldots, n$). Let $\boldsymbol{e}_m$ be the unit vector such that $e_m = 1$ and $e_a = 0$ for all $a \neq m$. We assume

A_4: $S_a(\boldsymbol{x}) \geq S_a(\boldsymbol{e}_m \sum_j x_j)$ for all $a = 1, \ldots, n$ and for all $\boldsymbol{x} \in \mathcal{D}$.

Although A_4 seems difficult to verify in general, it is satisfied if S depends only the total standing biomass (as in (9)) or the total area under forest, and it is a concave function.

We assume A_1, A_2 and A_4 for all the results in this section.

4.1. Deriving the minimum forest cover

We start defining the auxiliary functions $g^a(y), a = 1, \ldots, n$, where

$$g^a(y) = p + b\frac{1-b^a}{1-b}W'(y) - b^a f_a U'\left(f_m(1-y)\right) - \sum_{j=1}^{a} b^j S_j((1-y)\boldsymbol{e}_m). \tag{13}$$

Using A_1 and A_2, one can check that g^a is continuous and decreasing for all $a = 1, \ldots, n$.[18]

[18] They are strictly decreasing if either U or W is strictly concave or S_1 is strictly decreasing with respect to x_m.

Let $\tilde{y}^a$ be defined by

$$\tilde{y}^a = \sup\{y \in [0,1]: g^a(y) \geq 0\} \tag{14}$$

where $\tilde{y}^a = 0$ if $g^a(0) < 0$. Further, if $g^a(0) \geq 0$ and $g^a(1) \leq 0$ then $\tilde{y}^a$ is the largest (unique, if g^a is strictly decreasing) solution to $g^a(y) = 0$ in $[0,1]$.

The next result identifies upper bounds on the area under alternative use.

Lemma 2. *Let $\{\boldsymbol{x}_t\}_{t=0}^{\infty}$ be an optimal program. Then,*

$$\min_{j=1,\ldots,a} \{y_{t+j}\} \leq \tilde{y}^a \quad \forall a, \ \forall t \geq 0.$$

The next proposition, which follows immediately from the above lemma, outlines lower bounds on the forest cover.

Proposition 2. *Let $\{\boldsymbol{x}_t\}_{t=0}^{\infty}$ be an optimal program and let $\tilde{y}^a$ be as defined in (14). Then the following holds:*

$$\max_{j=1,\ldots,a} \{1 - y_{t+j}\} \geq 1 - \tilde{y}^a \quad \forall a, \ \forall t > 0,$$

i.e., the forest cover is above $1 - \tilde{y}^a$ at least once every a periods.

We single out the case $a=1$ for its importance in terms of the conclusions that can be obtained. The proposition above implies that, in particular,

Corollary 1. $1 - y_t \geq 1 - \tilde{y}^1 \, \forall t > 0$, *i.e., the forest cover is at least as large as $1 - \tilde{y}^1$ at every time period.*

Observe that $g^a(1) \geq 0 \forall a$ if, and only if, (7) holds. Hence, in the absence of deforestation, we have $g^a(1) < 0$ for at least one value of a. Whenever $g^1(1) < 0$, we have $\tilde{y}^1 < 1$, i.e., the forest cover is bounded away from zero which can be thought of as strong avoidance of deforestation. Whenever $g^a(1) < 0$ for some value of $a > 1$, we have $\tilde{y}^a < 1$, i.e., there is partial forest cover at least once every a periods – a somewhere weaker form of avoidance of deforestation. If $U'(0) = \infty$, $1 - \tilde{y}^a > 0$ for all a.

Corollary 2. *The following hold*

- *If $g^1(0) < 0$, i.e., if*

$$p + bW'(0) - bf_1 U'(f_m) - bS_1(\boldsymbol{e}_m) < 0 \tag{15}$$

  then $\tilde{y}^1 = 0$ and there is full forest cover every period.
- *If $g^a(0) < 0$, i.e., if*

$$p + b\frac{1-b^a}{1-b} W'(y) - b^a f_a U'(f_m) - \sum_{j=1}^{a} b^j S_j(\boldsymbol{e}_m) < 0 \tag{16}$$

  then $\tilde{y}^a = 0$ and there is full forest cover at least once every a periods.

Note that high planting cost or high marginal benefit from alternative use increase the value of $\tilde{y}^a_A$ for $a = 1, \ldots, n$, implying that we have a smaller lower bound on forest cover. Further, a decrease in the discount factor b, is likely to increase the value of $\tilde{y}^a_A$ implying that the lower bound on forest cover is reduced. On the other hand, an increase on the price of timber or in the marginal benefit from standing forest stock is likely to decrease the value of $\tilde{y}^a_A$ implying that we have a larger lower bound on forest cover. Finally, the effect of a technological change that increases timber content or reduces harvesting cost is ambiguous; we need more specific structure to obtain definite predictions.

4.2. Implications for public policy

To see how Proposition 2 can provide specific guidelines for public policy, we revert to the specific framework in Section 3.2 where we consider a private owner that ignores all stock benefits (set $S(\cdot) = 0$), a social planner that takes into account all stock benefits (where $S(\cdot)$ depends only on the total timber content as specified in (9)) and finally, a public subsidy policy where the subsidy is linear in the total timber content (i.e., of the form $\lambda \sum_a f_a x_a$).[19]

Note that the specific form of the stock benefit function $S(\boldsymbol{x}) = A(\sum_a f_a x_a)$ used by the social planner satisfies A_2 and A_4 whenever A fulfills A_2'. Further, with the linear subsidy, the private owner sets $S(\boldsymbol{x}) = \lambda \sum_a f_a x_a$ and this satisfies A_4. The auxiliary functions in (13) can now be re-written for each of the above mentioned cases. In the social planner's problem, the

[19] This is just a specific version of Pigou tax for externalities (Pigou, 1932).

corresponding g^a function is

$$g_A^a(y) = p + b\frac{1-b^a}{1-b}W'(y) - b^a f_a U'\left(f_m(1-y)\right) - A'\left(f_m(1-y)\right)\sum_{j=1}^{a} b^j f_j$$

and in the subsidized private owner facing linear subsidy $\lambda\sum_a f_a x_a$ it is given by

$$g_\lambda^a(y) = p + b\frac{1-b^a}{1-b}W'(y) - b^a f_a U'\left(f_m(1-y)\right) - \lambda\sum_{j=1}^{a} b^j f_j$$

The definitions of $\tilde{y}_A^a$ and $\tilde{y}_\lambda^a$ are analogous to (14).

To ensure that in the problem faced by the subsidized private owner the lower bound on the forest cover $\tilde{y}_\lambda^1$ coincides with the socially optimal lower bound $\tilde{y}_A^1$ it is sufficient to choose subsidy rate λ equal to

$$\lambda^1 = A'(f_m(1-\tilde{y}_A^1)) \tag{17}$$

If, on the other hand, to ensure that the "at least once every a periods" – lower bound on the forest cover $\tilde{y}_A^a$ is also met by the private owner, the subsidy rate λ can be set equal to

$$\lambda^a = A'(f_m(1-\tilde{y}_A^a)). \tag{18}$$

Under assumption A_2', these values of λ are non-negative.

5. Example

To illustrate the conditions presented in the previous sections, we provide a numerical example. We consider a dual aged forest ($n=2$) where the biomass coefficients are $f_1 = 0.45$ and $f_2 = 1$.[20] We choose a discount factor $b=0.9$. We assume that the benefit functions U and S are quadratic and W is linear; in particular,

$$U(c) = \alpha_1 c - \frac{\alpha_2}{2}c^2 \quad \text{with } \alpha_1 = 2,\ \alpha_2 = 0.8$$

$$A(x) = \beta_1 x - \frac{\beta_2}{2}x^2 \quad \text{with } \beta_1 = 1.5,\ \beta_2 = 0.4$$

$$W(y) = \omega y \quad \text{with } \omega = 1.25$$

With these parameters' values, it is easy to check that while (10) is not satisfied, (8) holds. We are then in a situation where deforestation is privately but not socially optimal in the absence of subsidies. If we design a simple subsidy that is linear in total timber content i.e., of the form $\lambda\sum_a f_a x_a$, then (12) implies that to avoid total deforestation the parameter λ must be larger than

$$\min\left\{\frac{1}{bf_1}(p+b\omega) - \alpha_1, \frac{1}{bf_1 + b^2 f_2}\left[p + \left(b+b^2\right)\omega - b\alpha_1\right]\right\} \approx 0.59$$

Now suppose the objective of public policy is to go beyond the simple avoidance of total deforestation and try to ensure certain lower bounds on the forest cover (in the privately optimal path). With the parameters' values defined above we find that $\tilde{y}_A^1 \approx 0.81$ and $\tilde{y}_A^2 \approx 0.03$. From (17), we see that if the social planner wants to assure a minimal forest cover (derived from the social planner's problem) of $1-\tilde{y}_A^1 \approx 0.19$ every time period, the subsidy rate λ_1 must be at least $A'(f_2(1-\tilde{y}_A^1)) \approx 1.42$. If, on the other hand, if the social planner is satisfied with assuring $1-\tilde{y}_A^2 \approx 0.97$ at least once every two periods, then the subsidy can be smaller, from (18) we see that it is sufficient to choose the subsidy rate $\lambda_2 \geq A'(f_2(1-\tilde{y}_A^2)) \approx 1.11$.

It is somehow surprising that λ_2 is smaller than λ_1, as the value of the second lower bound imposed is much larger than the first one. The smaller value of λ_2 is due to the fact the forest cover must be above $(1-\tilde{y}_A^2)$ only once every two periods. This suggests that the forest cover may present large oscillations along the optimal trajectory.

It is also interesting to note that if we take $\lambda_1 = 1.42$, to assure that the lower bound on forest cover (from the social planner's problem) is satisfied in every time period, the value for $\tilde{y}_\lambda^2$ is zero, meaning that *there is* total forest cover at least once every two periods. On the other hand, if we take $\lambda_2 = 1.11$ to assure that the minimal bound once every two periods is satisfied, then the value of $\tilde{y}_\lambda^1$ is one, implying that there could be zero forest cover in some periods (at most once every two time periods).

Finally, if the social planner wants to assure full forest cover in every period, she must offer a subsidy that ensures that $\tilde{y}_\lambda^1 = 1$, i.e., $p + bW'(0) - bf_1[U'(f_2)+\lambda] < 0$ (see (15)), implying that λ must be larger than or equal $1/bf_1(p+bW'(0)) - bU'(f_2) \approx 2.07$.

[20] Salo and Tahvonen (2002) and Wan (1994) contain an analysis of the Mitra–Wan model for the special case of a dual aged forest ($n=2$) which allows for sharper theoretical results.

6. Extensions

Our paper has not addressed the problem of characterization of steady states and their stability in situations where total deforestation is not optimal; this remains an open question in the context of our general model. Our current analysis indicates that in a simplified version of the model, where $n=2$, a characterization of the transition dynamics and comparative dynamics of important parameters, as in Dasgupta and Mitra (2011), may be possible.

More research is needed to examine how our conditions for deforestation are modified if the model is extended to allow for other realistic features such as irreversibility of deforestation, soil erosion and uncertainty.

In particular, irreversibility arises when land under alternative use is rendered unsuitable for future reforestation. Whether or not such an irreversibility is relevant depends on the nature of alternative use as well as the composition of the forest. It is easy to see that introduction of such irreversibility in our model reduces the set of feasible programs; every program that meets the irreversibility constraint is also feasible in our model but the reverse is not true as programs involving diversion of land from alternative use to forests are no longer admissible. This implies that if some program is optimal in our model (where there is no irreversibility) and in addition, satisfies the irreversibility constraint, then it must be optimal in the model with irreversibility. In particular, a program characterized by immediate deforestation in our model involves no reversal of land use from alternative use to forestry and therefore continues to be feasible with irreversibility. Therefore, the conditions under which deforestation is optimal in our model are also sufficient to ensure that deforestation is optimal under irreversibility. However, with irreversibility, deforestation may be optimal under weaker conditions. For instance, if the initial state is one where all land is dedicated to alternative use then the only feasible program under full irreversibility involves deforestation. But (as we have shown) without irreversibility, if our conditions for deforestation do not hold then from the same initial state it is optimal to sustain positive forest cover infinitely often. Thus, introducing irreversibility in diversion of land to alternative use makes deforestation more likely.[21] However, this may be modified if there is uncertainty about future benefits of forest use and irreversibility may create an option value for conservation of forests. Indeed, several previous studies (Arrow and Fisher, 1974; Henry, 1974) have shown that in the presence of irreversibility, uncertainty about the future can make depletion of irreplaceable assets less likely. The study of deforestation under irreversibility and uncertainty remains an important topic for future research.

Our analysis in this paper is carried out under the assumption that the total area of available land for forest and non-forest use is fixed; this allows us to focus on the dynamic tradeoff between forest and non-forest use where the dynamic value of having forest cover is endogenously determined through control of age distribution. However, in many real world situations, diversion of land to alternative use reduces the total amount of land through soil erosion. Intuitively, this should increase the incentive for forest conservation and make deforestation less likely. Mathematically, introducing such a law of motion for the total area of available land makes the optimization problem intrinsically more complicated than the one studied in this paper, but the insights it may provide makes it worth studying in the future.

7. Conclusion

Deforestation is a complex social phenomenon. In this paper, we have tried to understand the broad economic and ecological conditions under which deforestation may occur even though there is no failure of property rights and the forest is managed effectively. Our analysis has been carried out in the context of a very well known economic model of optimal forest management (the Mitra–Wan model) that we have extended to allow for benefits from standing forests (in addition to timber harvests) and alternative use of land. Despite the fact that optimal paths in this class of models may be characterized by cycles and steady states that are not globally stable, we obtain a very sharp characterization of deforestation. We show that if deforestation occurs from any initial state then it must occur in finite time from every initial state. As a result, a "local" condition that simply rules out any incentive to move away from a state of zero forest cover turns out to be necessary and sufficient for global deforestation. The comparative statics of this condition clearly shows that factors that improve profitability of (or more generally the marginal benefit from) harvesting timber such as an increase in price of timber (increase in demand) or reduction in the cost of harvesting tend to promote conservation of forests. Similarly, an increase in the flow benefits from standing forests that are internalized by forest management (for instance, as a consequence of public policy) make deforestation less likely. We also derive a precise bound on forest cover that is relevant to situations where deforestation is not optimal. We demonstrate how our general results can be used to study the wedge between private and social optimality of deforestation, and suggest a simple subsidy on standing forests that can induce a private owner to conserve forests or maintain a minimal level of forest cover.

To the best of our knowledge, this is the first paper to attempt to derive analytical conditions for deforestation and in doing so, it contributes to the general economic theory of extinction and conservation of natural resources.

[21] Instead of full irreversibility, we can think of a situation where reforestation is possible at some cost. Then, a term representing the cost of conversion of land from alternative use to forests must be included in the objective function. With partial irreversibility the perturbation analysis performed in the proof of Lemma 1 can be done, with new terms appearing in Eqs. (5) and (6) whenever land is converted to forestry. This asymmetry breaks the equivalence obtained in Proposition 1.

Acknowledgments

Adriana Piazza is grateful of the hospitality of the Southern Methodist University where this research was initiated and acknowledges the financial support from Fondecyt 1140720, Programa Basal 03, CMM, U. de Chile and Project Anillo ACT-1106.

Appendix A. Proofs

Proof of Lemma 1. We define the unitary vector $\boldsymbol{e}_j \in \mathrm{IR}^n$ such that $e_j = 1$ and $e_k=0$ for all $k \neq j$. Consider an alternative program $\{\widehat{\boldsymbol{x}}_s\}_{s=0}^{\infty}$ such that

$$\widehat{\boldsymbol{x}}_{t+j} = \boldsymbol{x}_{t+j} + \epsilon \boldsymbol{e}_j \quad \forall j = 1, \dots, a.$$
$$\widehat{\boldsymbol{x}}_s = \boldsymbol{x}_s \quad \text{else}$$

In words, whenever $\epsilon > 0$ ($\epsilon > 0$), the alternative program is one where the area under alternative use is reduced (increased) by ϵ at time period $t+1$ and replanted to yield more (less) young forest next period. This modification of the young forest area is let to propagate until age a at which point the consumption of forest of age a is modified. Of course, the area under alternative use, can only be reduced if it is strictly positive along the periods involved, i.e., $\min_{j=1,\dots,a}\{y_{t+j}\} > 0$. On the other hand, to increase the land under alternative use along the time periods $t+1, \dots, t+a$, we decrease the harvest of age class a at time $t+a$, hence we need $c_{a,t+a} > 0$. In conclusion, the alternative program is feasible for $-c_{a,t+a} < \epsilon < \min\{y_{t+1}, y_{t+2}, \dots, y_{t+a}\}$.

As $\{\boldsymbol{x}_t\}_{t=0}^{\infty}$ is optimal, the modification must give a smaller benefit, hence:

$$\sum_{t=1}^{\infty} b^{t-1}[U(c_t) + W(y_t) + S(\boldsymbol{x}_t) - px_{1,t+1}] \geq \sum_{t=1}^{\infty} b^{t-1}[U(\widehat{c}_t) + W(\widehat{y}_t) + S(\widehat{\boldsymbol{x}}_t) - p\widehat{x}_{1,t+1}]$$

which is equivalent to

$$b^a U(c_{t+a}) + \sum_{j=1}^{a} b^j [W(y_{t+j}) + S(\boldsymbol{x}_t)] - px_{1,t+1}$$
$$\geq b^a U(c_{t+a} + f_a \epsilon) + \sum_{j=1}^{a} b^j [W(y_{t+j} - \epsilon) + S(\boldsymbol{x}_t + \epsilon \boldsymbol{e}_a)] - p(x_{1,t+1} + \epsilon),$$

and reordering we get

$$p\epsilon + \sum_{j=1}^{a} b^j [W(y_{t+j}) - W(y_{t+j} - \epsilon)]$$
$$\geq b^a [U(c_{t+a} + f_a \epsilon) - U(c_{t+a})] + \sum_{j=1}^{a} b^j [S(\boldsymbol{x}_t + \epsilon \boldsymbol{e}_a) - S(\boldsymbol{x}_t)] \tag{19}$$

If $\min_{j=1\dots a}\{y_{t+j}\} > 0$ we can consider $\epsilon > 0$. Dividing through by ϵ and taking the limit as $\epsilon \to 0^+$, we obtain inequality (5).

If $c_{a,t+a} > 0$ we can consider $\epsilon < 0$. Dividing through by $\epsilon < 0$ and taking the limit as $\epsilon \to 0^-$, we obtain inequality (6). □

Proof of Proposition 1. We see first that 1, 2, 3 and 4 are equivalent by showing that $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 1$.

From the definitions of total deforestation and eventual deforestation it is evident that $1 \Rightarrow 2$.

Let us show that $2 \Rightarrow 3$. Let $\{\boldsymbol{x}_t\}_{t=0}^{\infty}$ be an optimal program characterized by eventual deforestation. As $\{y_t\} \to 1$, there exists T such that $y_t > 0$ for all $t \geq T$. Using Lemma 1, we have that (5) holds for all a and $t \geq T$. Observe that as $\{y_t\} \to 1$, $\{x_{a,t}\} \to 0$ for all a so that taking the limit as $t \to \infty$ on both sides of the inequalities (5) and rearranging the terms we obtain (7).

We turn now to the proof that $3 \Rightarrow 4$. We want to show that under (7), it is optimal to never re-plant, i.e., for any $\boldsymbol{x}_0 \in \mathcal{D}$, the optimal policy is such that $x_{1,t} = 0$ for all $t > 0$. To show the optimality of such a policy, suppose to the contrary that there exists $\boldsymbol{x}_0 \in D$ such that there is an optimal program from this initial state where

$$x_{1,1} > 0.$$

The trees that are one year old at $t=1$ must be harvested at some time period. Hence there is at least one value of $a = 1, \dots, n$ such that $c_{a,a} > 0$. From Lemma 1, we have that (6) holds for $t=0$ i.e.,

$$p + \sum_{j=1}^{a} b^j W'(y_j) \leq b^a f_a U'(c_a) + \sum_{j=1}^{a} b^j S_j(\boldsymbol{x}_j)$$

and using the *strict* concavity of U on a neighborhood of zero, A_1 and A_2 we have

$$p+\sum_{j=1}^{a} b^j W'(1) < b^a f_a U'(0) + \sum_{j=1}^{a} b^j S_j(\mathbf{0})$$

which violates (7).

Finally, to see that $4 \Rightarrow 1$, simply observe that if $x_{1,t}=0$ for all $t>0$ all forest is transferred to alternative use, after at most n periods, and stays that way forever.

This completes the first part of the proof. The equivalence of the last two assertions follows almost directly. Indeed, it is evident that $5 \Rightarrow 6$. It is also easy to see that $1 \Rightarrow 5$. Finally, if 6 holds then the program $\boldsymbol{x}_t=0$ and $y_t=1\ \forall t$ is optimal and characterized by eventual deforestation, yielding that $6 \Rightarrow 2$. □

Remark 1. While it is very easy to see that the terms $p/\sum_j b^j f_j$ and $-b^a f_a U'(0)$ in (12) are decreasing with b, proving that $((1-b^a)/(1-b)\sum_j b^j f_j) bW'(1)$ is decreasing entails a more much complicated calculation and depends on $f_1 \le f_2 \le \cdots \le f_n$. Indeed, it can be proved that the derivative of this last term is

$$W'(1)\frac{1}{(\sum_{j=1}^{n-1} b^{j-1} f_j)^2}\sum_{a=1}^{n-1}\left[\underbrace{(f_a-f_{a+1})}_{\le 0}\underbrace{\sum_{i=a}^{n-1}\left(ib^{i-1}+\sum_{j=1}^{a-1}(i-j)b^{i+j-1}\right)}_{\ge 0}\right] \le 0. \quad \square$$

Proof of Lemma 2. Let us denote by $y=\min_{j=1,\ldots,a}\{y_{t+j}\}$. If $y=0$ the inequality holds trivially. If $y>0$, we know thanks to (5) that

$$\begin{aligned}
0 &\le p+\sum_{j=1}^{a} b^j W'(y_{t+j}) - b^a f_a U'(c_{t+a}) - \sum_{j=1}^{a} b^a S_j(\boldsymbol{x}_{t+j}) \\
&\le p+\sum_{j=1}^{a} b^j W'(y_{t+j}) - b^a f_a U'\left(f_m \sum_{a=1}^{n} x_{a,t+j}\right) - \sum_{j=1}^{a} b^a S_j\left(\boldsymbol{e}_m \sum_{a=1}^{n} x_{a,t+j}\right) \\
&= p+\sum_{j=1}^{a} b^j W'(y_{t+j}) - b^a f_a U'(f_m(1-y_{t+j})) - \sum_{j=1}^{a} b^a S_j(\boldsymbol{e}_m(1-y_{t+j}) \\
&\le p+\sum_{j=1}^{a} b^j W'(y) - b^a f_a U'(f_m(1-y)) - \sum_{j=1}^{a} b^a S_j(\boldsymbol{e}_m(1-y)) = g^a(y).
\end{aligned}$$

This implies that $g^a(\min_{j=1,\ldots,a}\{y_{t+j}\}) \ge 0$, and hence, $\min_{j=1,\ldots,a}\{y_{t+j}\} \le \tilde{y}^a$. □

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