Choice under Uncertainty.

Decision maker chooses between risky alternatives that are probability distributions over outcomes.

Choice is made before uncertainty is resolved (outcomes are actually realized).

Set of all possible outcomes: C.

Outcomes may be consumption bundle or a monetary payoffs...

They are realized after all uncertainty is resolved.

No uncertainty in the outcomes.

Assume: *C is finite with N elements indexed $n = 1, ...N$.*

A simple lottery L is a probability distribution on the set of outcomes C i.e., a vector $(p_1, ...p_N)$,

$$p_n \geq 0, \sum_{n=1}^{N} p_n = 1,$$

where p_n denotes the probability of outcome n.

A compound lottery is a probability distribution over simple lotteries.

Given K simple lotteries $L_1, ..., L_K$ and probabilities $(\alpha_1, ..., \alpha_K) \geq 0$, $\sum_{k=1}^{K} \alpha_k = 1$, a compound lottery $(L_1, ..., L_K; \alpha_1, ..., \alpha_K)$ is a risky alternative that yields the simple lottery L_k with probability α_k.

Every compound lottery can be reduced to an equivalent simple lottery in the sense that they both generate the same distribution over outcomes in C.

Consider simple lotteries $L_k = (p_1^k, ...p_N^k), k = 1, ...K$.

In the compound lottery $(L_1, ..., L_K; \alpha_1, ..., \alpha_K)$, the probability that any outcome $n \in C$ is realized is given by

$$\begin{aligned} & \sum_{k=1}^{K} \Pr\{L_k \text{ occurs}\} \Pr\{\text{outcome } n \text{ is realized in } L_k\} \\ = & \sum_{k=1}^{K} \alpha_k p_n^k \end{aligned}$$

and so this compound lottery generates the same distribution over C as the simple lottery $L = (p_1, ...p_N)$ where

$$p_n = \alpha_1 p_n^1 + ... + \alpha_K p_n^K, n = 1, ...N.$$

Note that this reduced simple lottery can be obtained by vector addition:

$$
\begin{aligned}
L &= \alpha_1(p_1^1, ...p_N^1) + ... + \alpha_K(p_1^K, ...p_N^K) \\
&= \alpha_1 L_1 + ...\alpha_K L_K.
\end{aligned}
$$

Assume: decision maker cares only about the ultimate probability distribution over outcomes and therefore, for any compound lottery, only the reduced simple lottery is of relevance to the decision maker.

Set of risky alternatives for the decision maker: $\mathcal{L}$ where

$\mathcal{L} =$ *set of all simple lotteries over the set of outcomes* C.

Preferences:

Assume that the decision maker has a binary preference *ordering* $\succsim$ defined on $\mathcal{L}$.

Strict preference and indifference relations: $\succ, \sim$.

Two important axioms.

Continuity: If a sequence of numbers $\{\alpha_i\}_{i=1}^{\infty} \rightarrow \alpha$, (where $\forall i, \alpha_i \in [0, 1]$), then

$$\alpha_i L + (1 - \alpha_i) L' \succsim L", \forall i, \Rightarrow \alpha L + (1 - \alpha) L' \succsim L$$

$$L" \succsim \alpha_i L + (1 - \alpha_i) L', \forall i, \Rightarrow L" \succsim \alpha L + (1 - \alpha) L'.$$

Lexicographic preferences violate continuity.

Example: $C = \{$An Ice-cream; A million dollars; Terrible Accident$\}$.

Let $L" = (1, 0, 0), L = (0, 1, 0), L' = (0, 0, 1)$.

Choose $\{\alpha_i\} \to \alpha = 1, 0 < \alpha_i < 1$.

"Safety first" preference: $L" \succ \alpha_i L + (1 - \alpha_i)L' = (0, \alpha_i, 1 - \alpha_i)$.

$\alpha_i L + (1 - \alpha_i)L' \to L$

But: $L \succ L"$.

Independence: *For all* $L, L', L" \in \mathcal{L}$ *and* $\alpha \in (0, 1)$,

$$L \succsim L' \iff \alpha L + (1 - \alpha)L" \succsim \alpha L' + (1 - \alpha)L".$$

If we mix two lotteries with a third lottery, then the preference ordering of the two mixed lotteries remain unchanged i.e., is independent of the third lottery used.

Consider two states of nature "$rain$" and "$sunny$" that are realized with probabilities α and $1-\alpha$.

Think of the mixed lottery $\alpha L+(1-\alpha)L"$ as the reduced form of the compound lottery $(L, L"; \alpha, 1-\alpha)$ i.e., the lottery that gives the agent the lottery L when it rains and lottery $L"$ when it is sunny,

Likewise the mixed lottery $\alpha L'+(1-\alpha)L"$ is the reduced form of the compound lottery $(L', L"; \alpha, 1-\alpha)$ i.e., the lottery that gives the agent the lottery L' when it rains and lottery $L"$ when it is sunny.

The two compound lotteries yield the same lottery in the state of nature where it is sunny and differ only in the other state.

The independence axiom says the preference between these two compound lotteries (or their reduced forms) should depend only on L and L'; it should be independent of $L"$ - if $L"$ is replaced by some other lottery, the ordering of the two mixed lotteries must remain the same.

Argument: the comparison of the two mixed lotteries ought to depend only on what happens when the state is rainy and in that event, *what might have happened if the state was sunny instead of rainy* should not matter.

Standard theory of consumer behavior in choices between bundles of goods: independence is a severe restriction.

Complementarity: like to consume one good along with another (or dislike).

However, in decision making under uncertainty, when one looks at a mixed lottery $\alpha L + (1 - \alpha)L"$, the lottery $L"$ is not consumed with the lottery L because in the state in which $L"$ is realized, L is not realized and vice-versa (after the realization of initial uncertainty about which lottery is consumed, you consume L instead of $L"$ or $L"$ instead of L).

So, the usual argument for dependence in standard consumer theory is not relevant.

Independence is the most controversial and strong restriction in the theory of decision making under uncertainty.

It is also the most important axiom needed for the expected utility theorem.

Representation: A utility function $U : \mathcal{L} \to \mathbb{R}$ represents the preference ordering $\succsim$ if for every $L, L' \in \mathcal{L}$

$$L \succsim L' \Longleftrightarrow U(L) \geq U(L').$$

Can be shown that if $U : \mathcal{L} \to \mathbb{R}$ represents the preference ordering $\succsim$, then

$$\begin{aligned} L &\succ L' \Longleftrightarrow U(L) > U(L'). \\ L &\sim L' \Longleftrightarrow U(L) = U(L'). \end{aligned}$$

Mathematically, $\mathcal{L}$ is just like a finite dimensional commodity space with $N - tuples$ of numbers.

Just as in standard consumer theory, the continuity axiom is sufficient to ensure the existence of a utility function that represents the preference ordering $\succsim$.

In fact, there is a continuous utility function.

Every positive monotonic transformation of the utility function also represents $\succsim$.

von-Neumann and Morgenstern: Expected utility form.

Economists and decision theorists have long been interested in a specific kind of utility representation which is linear.

The utility function $U : \mathcal{L} \rightarrow \mathbb{R}$ *has an* expected utility form *if there are fixed numbers* $(u_1, ..., u_N)$ *so that for every lottery* $L = (p_1, p_2, .., p_N) \in \mathcal{L}$,

$$U(L) = u_1 p_1 + ... + u_N p_N.$$

i.e., $U(L)$ *is a linear function of the probabilities with fixed coefficients* $(u_1, ..., u_N)$.

Note that if $L = (0, 0, .., 1, ...0)$ where the probability one is assigned to the $i - th$ outcome (i.e., L is the degenerate lottery that realizes outcome i with certainty), then such an expected utility form would imply, $U(L) = u_i$.

So, one can interpret the coefficients u_i as the "utility" of the deterministic outcome i.

Thus, $u_1 p_1 + ... + u_N p_N$ is the mathematical expectation (or, weighted average) of the utility that can be attained through the lottery L.

A utility function $U : \mathcal{L} \rightarrow \mathbb{R}$ with the expected utility form is called a *von Neumann-Morgenstern (vNM) expected utility function.*

A utility function $U : \mathcal{L} \rightarrow \mathbb{R}$ *is said to be linear if for any* K *lotteries* $(L_1, ...L_K)$ *and for any* $K-$*vector of probabilities*$(\alpha_1, ...\alpha_K), \alpha_k \geq 0, \sum_{k=1}^{K} \alpha_k = 1$, *if* $L = \alpha_1 L_1 + ... + \alpha_K L_K$, *then*

$$U(L) = \alpha_1 U(L_1) + ... + \alpha_K U(L_K).$$

Proposition. *A utility function* $U : \mathcal{L} \rightarrow \mathbb{R}$ *has an expected utility form* $\Longleftrightarrow$ *it is linear.*

Proof. Suppose U has an expected utility form as described earlier.

Let $L_k = (p_1^k, ..., p_N^k), k = 1, ...K.$

Let $L = \alpha_1 L_1 + ... + \alpha_K L_K$ where , $\alpha_k \geq 0$, $\sum_{k=1}^{K} \alpha_k = 1$

Then, the lottery $L = (p_1, ...p_N)$ where

$$p_n = \alpha_1 p_n^1 + ... + \alpha_K p_n^K, n = 1, ...N.$$

Therefore,

$$\begin{aligned} U(L) &= \sum_{n=1}^{N} p_n u_n \\ &= \sum_{n=1}^{N} (\alpha_1 p_n^1 + ... + \alpha_K p_n^K) u_n \\ &= \sum_{n=1}^{N} \alpha_1 p_n^1 u_n + ... + \sum_{n=1}^{N} \alpha_K p_n^K u_n \\ &= \alpha_1 U(L_1) + ... + \alpha_K U(L_K). \end{aligned}$$

Suppose that U is linear in the above sense.

For $n = 1, ...N$, let $\widehat{L}_n = (0, ...1, ...0)$ be the specific degenerate lottery where probability 1 is on outcome n.

Fix $u_n = U(\widehat{L}_n), .n = 1, ..N$.

Then, for any lottery $L = (p_1, ...p_N) \in \mathcal{L}$, one can write

$$L = p_1\widehat{L}_1 + ... + p_N\widehat{L}_N$$

and using linearity of U,

$$\begin{aligned} U(L) &= p_1U(\widehat{L}_1) + ... + p_NU(\widehat{L}_N) \\ &= u_1p_1 + ... + u_Np_N. \end{aligned}$$

The proof is complete.

The expected utility property of the utility function is not necessarily preserved under any positive monotonic transformation.

Suppose $U : \mathcal{L} \to \mathbb{R}$ is a utility function that represents $\succeq$ and has the expected utility form.

If $f : \mathbb{R} \to \mathbb{R}$ is any strictly increasing function then $W = f(U) : \mathcal{L} \to \mathbb{R}$ also represents $\succeq$, but W need not have the expected utility form (could be non-linear).

The expected utility property is not an ordinal property of utility functions on the space of lotteries - it is a cardinal property.

It is preserved only under linear transformations.

Proposition. *Suppose that* $U : \mathcal{L} \to \mathbb{R}$ *is a vNM expected utility function representing the preference ordering* $\succsim$ *on* $\mathcal{L}$*. Then* $W : \mathcal{L} \to \mathbb{R}$ *is another vNM expected utility function representing the preference ordering* $\succsim$ *on* $\mathcal{L}$ <u>if and only if</u> *there exists scalars* $\beta > 0$ *and* γ *such that*

$$W(L) = \gamma + \beta U(L), \forall L \in \mathcal{L}.$$

Proof. We will show the "if" part.

Suppose there exist scalars $\beta > 0$ and γ such that $W(L) = \gamma + \beta U(L), \forall L \in \mathcal{L}$.

We will show that W has the expected utility property.

Since $U : \mathcal{L} \to \mathbb{R}$ is a vNM expected utility function, there exists $(u_1, ..., u_N)$ such that for any lottery $L = (p_1, p_2, .., p_N) \in \mathcal{L}$,

$$U(L) = u_1 p_1 + ... + u_N p_N$$

Define a fixed set of constants $(w_1, ...w_N)$ where $w_n = (\gamma + \beta u_n), n = 1, ...N$.

Then, for any lottery $L = (p_1, p_2, .., p_N) \in \mathcal{L}$

$$\begin{aligned}
W(L) &= \gamma + \beta U(L) \\
&= \gamma + \beta(u_1 p_1 + ... + u_N p_N) \\
&= \gamma(p_1 + p_2 + .. + p_N) + \beta(u_1 p_1 + ... + u_N p_N) \\
&= (\gamma + \beta u_1)p_1 + (\gamma + \beta u_2)p_2 + ... + (\gamma + \beta u_N)p_N \\
&= w_1 p_1 + ... + w_n p_n.
\end{aligned}$$

Thus, W is a vNM expected utility function. //

Proposition. *If the preference relation* $\succeq$ *on* $\mathcal{L}$ *is represented by a utility function* U *that has the expected utility form, then* $\succeq$ *satisfies the continuity and independence axioms.*

Proof. First we show continuity. Consider any sequence $\{\alpha_i\}_{i=1}^{\infty} \to \alpha$, (where $\forall i, \alpha_i \in [0,1]$) and $\alpha_i L + (1-\alpha_i)L' \succsim L", \forall i$.

Then,

$$U(\alpha_i L + (1-\alpha_i)L') \geq U(L"), \forall i,$$

and using the linearity of U

$$\alpha_i U(L) + (1-\alpha_i)U(L') \geq U(L"), \forall i,$$

which implies (taking limit as $i \to \infty$)

$$\alpha U(L) + (1-\alpha)U(L') \geq U(L")$$

and linearity implies

$$U(\alpha L + (1-\alpha)L') \geq U(L")$$

so that $\alpha L + (1-\alpha)L' \succsim L$.

Next, we show independence.

Consider $L, L', L" \in \mathcal{L}$ *and* $\alpha \in (0,1)$.

Need to show: $L \succsim L' \Longleftrightarrow \alpha L + (1-\alpha)L" \succsim \alpha L' + (1-\alpha)L"$.

Suppose $L \succsim L'$.

Then, $U(L) \geq U(L')$ so that

$$\alpha U(L) + (1-\alpha)U(L") \geq \alpha U(L') + (1-\alpha)U(L")$$

and linearity implies

$$U(\alpha L + (1-\alpha)L") \geq U(\alpha L' + (1-\alpha)L")$$

which implies

$$\alpha L + (1-\alpha)L" \succsim \alpha L' + (1-\alpha)L".$$

Suppose that $\alpha L + (1-\alpha)L" \succsim \alpha L' + (1-\alpha)L"$.Then,

$$U(\alpha L + (1-\alpha)L") \geq U(\alpha L' + (1-\alpha)L")$$

and using linearity of U,

$$\alpha U(L) + (1-\alpha)U(L") \geq \alpha U(L') + (1-\alpha)U(L")$$

which implies that $U(L) \geq U(L')$.

Expected Utility Theorem.

Proposition. Suppose that the preference ordering $\succsim$ *on* $\mathcal{L}$ *satisfies the continuity and independence axioms. There* $\succsim$ *admits a utility representation of the expected utility form.*

Allais paradox.

$N = 3$.

$C = \{\$2.5 \text{ million}, \$0.5 \text{ million}, \$0\}$

Consider $L_1 = (0, 1, 0)$,

$L_1' = (0.10, 0.89, 0.01)$.

$L_2 = (0, 0.11, 0.89)$,

$L_2' = (0.10, 0, 0.90)$.

$$L_1 = (0, 1, 0) \succ L_1' = (0.10, 0.89, 0.01)$$

implies

$$U(0.5) > 0.1U(2.5) + 0.89U(0.5) + 0.01U(0)$$

which implies

$$0.11U(0.5) > 0.1U(2.5) + 0.01U(0)$$

so that

$$0.11U(0.5) + 0.89U(0) > 0.1U(2.5) + 0.90U(0)$$

i.e.,

$$(0, 0.11, 0.89) \succ (0.1, 0, 0.9).$$

Machina's paradox.

$C = \{$Trip to Venice, Watching an excellent movie about Venice, Staying home$\}$

Suppose decision maker prefers first to second to third outcome.

$L = \{99.9\%, 0.1\%, 0\}$

$L' = \{99.9\%, 0, 0.1\%\}$

$U(L) = 99.9\% U$(trip to Venice for sure) $+0.1\% U$(watching a movie about Venice for sure)

$>$ $99.9\% U$(trip to Venice for sure) $+0.1\% U$(staying home for sure)

$= U(L')$.

But you may be severely disappointed if you don't get to go to Venice and in that event, you may be better off if you stayed home - anticipating this you may choose L' over L.

Here, not being able to get to Venice may change your preference between the other two alternatives.

Regret.

Influence of what might have been on the utility obtained.

Violates independence.

Money Lotteries and Risk Aversion.

Need lotteries with continuum of outcomes.

$C = \mathbb{R}$.

A monetary lottery is probability distribution on $\mathbb{R}$ summarized by a distribution function $F : \mathbb{R} \to [0, 1]$,

$$F(x) = \text{Pr}\{\text{Realized monetary payoff } \leq x\}.$$

If the distribution function is absolutely continuous, then it has a density function f associated with it and

$$F(x) = \int_{-\infty}^{x} f(t)dt.$$

Consider K lotteries $L_1, ..., L_K$ with respective distribution functions $F_1, ...F_K$ and the compound lottery $L = (L_1, ..., L_K; \alpha_1, ...\alpha_K)$. Then, the final distribution of money induced by this compound lottery is given by the distribution function F where:

$$\begin{aligned} & F(x) \\ = \; & \text{Pr}\{\text{lottery } L \text{ yields payoff} \leq x\} \\ = \; & \sum_{k=1}^{K} \text{Pr}\{L_k \text{ is realized}\} \, \text{Pr}\{L_k \text{ yields payoff} \leq x\} \\ = \; & \sum_{k=1}^{K} \alpha_k F_k(x). \end{aligned}$$

So, as before, we can just work with (simple) lotteries that are just distribution functions on the real line.

$\mathcal{L} = \{$set of all distribution functions over an interval $[a, \infty)\}$

where a will be usually taken to be equal to zero.

Assume that there exists a utility function U with the expected utility form.

This implies that for each monetary outcome $x \in [a, \infty)$, there is a fixed real number $u(x)$ such that for any (lottery) distribution function F on $[a, \infty)$, we have

$$U(F) = \int_a^\infty u(x) dF(x)$$

which is the *expectation of the random variable* $u(x)$ when x follows the probability distribution F.

If in particular F is a discrete probability distribution assuming values $\{x_1, x_2, ...\}$ with probabilities $\{p_1, p_2, ...\}$, then

$$U(F) = \sum_i u(x_i) p_i$$

If F is an absolutely continuous probability distribution with density function f, then

$$U(F) = \int_a^{\infty} u(x) f(x) dx$$

The fixed real number $u(x)$ is interpreted as the utility of the deterministic (sure) monetary outcome x and $u : [a, \infty) \rightarrow \mathbb{R}$ is called the Bernoulli utility function.

Assume: Bernoulli utility function $u : [a, \infty) \to \mathbb{R}$ is increasing and continuous.

Risk Aversion.

A decision maker exhibits risk aversion if for any lottery F, the degenerate lottery that yields the expected amount $\overline{x} = \int xF(x)dx$ with certainty is at least as good as the lottery F itself i.e.,

$$\begin{aligned} u(\overline{x}) &= u(\int x dF(x)) \\ &\geq \int u(x) dF(x), \forall F. \end{aligned}$$

"*Utility of any lottery does not exceed the utility of getting the average monetary payoff of the lottery with certainty.*"

If the decision maker is always indifferent between the two, we say that she is risk neutral.

The agent is strictly risk averse if

(i) he is risk averse

(ii) he is indifferent between any lottery F and the certainty amount $\overline{x} = \int x dF(x)$ only if F is the degenerate distribution that assumes value $\overline{x}$ with probability one.

The inequality:

$$u(\int x dF(x)) \geq \int u(x) dF(x), \forall F.$$

defining risk aversion is called Jensen's inequality and holds *if and only if* u is a concave function.

Risk aversion is equivalent to the concavity of the Bernoulli utility function u.

(diminishing marginal utility for differentiable u).

Strict risk aversion:

$$u(\int x dF(x)) > \int u(x) dF(x), \forall F \text{ non-degenerate}$$

if and only if u is strictly concave.

(strictly diminishing marginal utility for differentiable u).

Risk neutrality

$$u(\int xdF(x)) = \int u(x)dF(x), \forall F.$$

if and only if u is linear.

Can also define risk loving behavior:

$$u(\int x dF(x)) \leq \int u(x) dF(x), \forall F.$$

which is equivalent to convexity of u.

For a given Bernoulli utility function u :

Define:

CERTAINTY EQUIVALENT of a lottery with distribution F, denoted by $c(F, u)$ by

$$u(c(F, u)) = \int u(x) dF(x)$$

i.e., the consumer is indifferent between the (possibly uncertain) lottery F and receiving the amount of money $c(F, u)$ with certainty.

First note that if decision maker is risk averse (u concave), then

$$c(F, u) \leq \int x dF(x)$$

i.e., the certainty equivalent does not exceed the expected payoff from the lottery.

This follows from the fact that risk aversion implies:

$$\begin{aligned} u(\int x dF(x)) &\geq \int u(x) dF(x) \\ &= u(c(F, u)) \end{aligned}$$

so that

$$u(\int x dF(x)) \geq u(c(F, u))$$

and since u is increasing, we have the result.

The converse is also true.

Suppose that for all lotteries F

$$c(F,u) \leq \int x dF(x)$$

then

$$u(c(F,u)) \leq u(\int x dF(x))$$

$$\int u(x) dF(x) \leq u(\int x dF(x))$$

so that u must be concave.

Thus, RISK AVERSION

$\Longleftrightarrow$ concave u

$\Longleftrightarrow c(F,u) \leq \int x dF(x), \forall F.$

Can show, Strict risk aversion $\Longleftrightarrow$strictly concave u

$\Longleftrightarrow c(F,u) < \int x dF(x), \forall F$ non-degenerate.

Risk neutrality: $\Longleftrightarrow$ linear u

$\Longleftrightarrow c(F,u) = \int x dF(x), \forall F$.

Risk loving

$\Longleftrightarrow$ convex u

$\Longleftrightarrow c(F,u) \geq \int x dF(x), \forall F$.

Given Bernoulli utility function u, a fixed amount of money x and number $\epsilon > 0$, consider the lottery that assigns:

probability mass $\frac{1}{2}$ to realization $x + \epsilon$

probability mass $\frac{1}{2}$ to realization $x - \epsilon$.

This is a lottery whose expected payoff is x.

If the agent is risk averse, then

$$u(x) \geq \frac{1}{2}u(x + \epsilon) + \frac{1}{2}u(x - \epsilon).$$

Define the probability premium $\pi(x, \epsilon, u)$ by

$$u(x) = (\frac{1}{2}+\pi(x, \epsilon, u))u(x+\epsilon)+(\frac{1}{2}-\pi(x, \epsilon, u))u(x-\epsilon).$$

RISK AVERSION $\Longleftrightarrow \pi(x,\epsilon,u) \geq 0, \forall x,\epsilon.$

Some simple applications.

Demand for Insurance.

Strictly risk averse agent with initial wealth $\$w$

Runs risk of losing $\$D$ with probability $\pi \in (0, 1)$.

Can buy insurance.

Each unit of insurance pays \$1 if the loss occurs.

It costs $\$q$ to buy one unit of insurance.

Suppose agent buys α units of insurance.

Then his expected utility is:

$$f(\alpha) = (1 - \pi)u(w - \alpha q) + \pi u(w - D - \alpha q + \alpha)$$

Agent maximizes $f(\alpha)$ with respect to $\alpha \geq 0$.

Assume u is differentiable.

Then, u' is (strictly) decreasing.

FOC: Optimal α^* satisfies

$$\begin{aligned} & -q(1-\pi)u'(w-\alpha^* q) \\ & +\pi(1-q)u'(w-D+\alpha^*(1-q)) \\ \leq \; & 0 \\ = \; & 0, \text{ if } \alpha^* > 0. \end{aligned}$$

Suppose that insurance is sold in a competitive market.

Expected Profit of a firm per unit of insurance sold at market price q:

$$(1-\pi)q + \pi(q-1)$$

which is independent of quantity sold so that in equilibrium, the market price q must be such that

$$(1-\pi)q + \pi(q-1) = 0$$

so that

$$q = \pi.$$

This is called *an actuarially fair insurance premium* (equal to expected cost to a firm of providing a unit of insurance).

If we use this in consumer's FOC:

$$\begin{aligned} & u'(w - D + \alpha^*(1-\pi)) \\ \leq \ & u'(w - \alpha^*\pi) \\ = \ & u'(w - \alpha^*\pi), \text{ if } \alpha^* > 0. \end{aligned}$$

Since $w > w - D, u'(w) < u'(w-D)$,hence α^* must be > 0.

This implies

$$u'(w - D + \alpha^*(1-\pi)) = u'(w - \alpha^*\pi)$$

and since u' is (strictly) decreasing

$$w - D + \alpha^*(1 - \pi) = w - \alpha^*\pi$$

which yields,

$$\alpha^* = D.$$

Thus, with an actuarially fair insurance premium, the agent insures completely.

Intuition: With actuarially fair premium $q = \pi$, the agent's *expected* final wealth for any choice of α is

$$\begin{aligned} & (1 - \pi)(w - \alpha\pi) + \pi(w - D - \alpha\pi + \alpha) \\ = \; & w - \pi D \end{aligned}$$

which is independent of choice of α.

However, choosing $\alpha = D$, enables him to reach $w - \pi D$ with certainty and clearly is the best option for a risk averse agent. If $q > \pi$, the agent insures incompletely.

Risk Portfolio Choice.

A risk-averse agent with differentiable Bernoulli utility u divides his wealth $w > 0$ between two assets: -a safe asset that guarantees a return of \$1 per dollar invested

- a risky asset whose return per dollar invested is given by a random variable z with distribution function $F(z)$ where

$$E(z) = \int z dF(z) > 1.$$

Let α, β denote the amounts invested in the risky and safe assets respectively.

The agent solves:

$$\begin{aligned} & \max_{\alpha,\beta \geq 0} \int u(\alpha z + \beta) dF(z) \\ s.t., \alpha + \beta &= w. \end{aligned}$$

equivalent to

$$\begin{aligned} & \max \int u(w+\alpha(z-1))dF(z) \\ s.t., 0 \leq \ & \alpha \leq w. \end{aligned}$$

FOC: Optimal α^* satisfies:

$$\begin{aligned} & \int u'(w+\alpha^*(z-1))(z-1)dF(z) \\ \leq \ & 0, \text{ if } \alpha^* < w \\ = \ & 0, \text{if } 0 < \alpha^* < w \\ \geq \ & 0, \text{ if } 0 < \alpha^*. \end{aligned}$$

Note that $\alpha^* = 0$ cannot satisfy FOC since

$$\begin{aligned} & \int u'(w)(z-1)dF(z) \\ = \ & u'(w)[\{\int zdF(z)\} - 1] > 0. \end{aligned}$$

So, optimal portfolio has $\alpha^* > 0$ with investment in the risky asset.

If risk is actuarially favorable, a risk averse agent will accept at least a small part of it.

Measurement of Risk Aversion.

Let u be a twice differentiable concave Bernoulli utility function of an agent, $u' > 0, u'' \leq 0$.

The *Arrow-Pratt* coefficient of *absolute* risk aversion at x is defined as

$$r_A(x) = -\frac{u''(x)}{u'(x)}.$$

It is a *local* measure of the curvature of u.

Cannot use just u'' as that would not be invariant to positive linear transformations of u.

For any $\epsilon > 0$ consider the probability premium $\pi(x, \epsilon, u)$, denoted hereafter as $\pi(\epsilon)$, defined by:

$$u(x) = (\frac{1}{2} + \pi(\epsilon))u(x + \epsilon) + (\frac{1}{2} - \pi(\epsilon))u(x - \epsilon)$$

Differentiating this identity twice w.r.t. ϵ and evaluating it at zero yields:

$$4\pi'(0) = -\frac{u''(x)}{u'(x)} = r_A(x)$$

so that $r_A(x)$ measures the rate at which the probability premium increases at certainty when existing wealth is x and with a *small additive risk* of the order of ϵ.

Ex. $u(x) = -\alpha \exp(-ax) + \beta, a > 0, \alpha > 0.$

Then,

$$r_A(x) = a \text{ for all } x.$$

Constant absolute risk aversion (CARA).

Ex

$$\begin{aligned} u(x) &= \frac{x^{1-\sigma}}{1-\sigma}, \sigma \neq 1, \sigma > 0 \\ u(x) &= \ln x \ (\sigma \to 1). \end{aligned}$$

$$r_A(x) = \frac{\sigma}{x}$$

Decreasing absolute risk aversion (DARA).

Given two Bernoulli utility functions u_1 and u_2 when can we say that u_2 exhibits (strictly) higher risk aversion globally compared to u_1?

(i) $r_A(x, u_2) \geq (>) r_A(x, u_1), \forall x.$

(ii) There exists an increasing, (strictly) concave function Ψ such that

$$u_2(x) = \Psi(u_1(x))$$

i.e., u_2 is a (strictly) concave transformation of u_1 (u_2 is more concave than u_1).

Ex. $u(x) = x^{0.25}$ is more concave than $u(x) = x^{0.5}$.

(iii) $c(F, u_2)(<) \leq c(F, u_1)$, $\forall F$ (non-degenerate).

(iv) $\pi(x, \epsilon, u_2) \geq (>) \pi(x, \epsilon, u_1)$ for any x and $\epsilon > 0$.

(v)

$$\int u_2(x)dF(x) \geq u_2(\overline{x})$$

for any (non-degenerate) F,$\overline{x}$ implies

$$\int u_1(x)dF(x) \geq (>) u_1(\overline{x}).$$

Any risk that would be accepted by u_2 starting from a position of certainty would also be accepted (strictly preferred) by u_1.

Proposition: *Definitions (i) - (v) of (strictly) more-risk-averse-than relation are equivalent.*

Note that two Bernoulli utility functions need not be comparable in terms of risk aversion - it could easily be the case that $r_A(x, u_2) \geq r_A(x, u_1)$ at some x but $r_A(x', u_2) < r_A(x', u_1)$ at some other x'.

Application. Portfolio choice between two assets - a safe asset with return 1 for each dollar invested and a risky asset with return given by z per dollar where $\int z dF(z) > 1$.

Suppose we have two individuals with strictly concave differentiable Bernoulli utility functions u_1 and u_2.

Let α_1^* and α_2^* denote their optimal investment in the risky asset..

Suppose u_2 is more risk averse than u_1.

We will show:

$$\alpha_2^* \leq \alpha_1^*.$$

Suppose to the contrary that

$$\alpha_2^* > \alpha_1^*$$

Since $\alpha_2^* \leq w$,

$$\alpha_1^* < w.$$

We know that $\int z dF(z) > 1$ implies

$$\alpha_1^* > 0, \alpha_2^* > 0.$$

FOCs:

$$\int (z-1) u_1'(w + \alpha_1^*(z-1)) dF(z) = 0$$

$$\int (z-1) u_2'(w + \alpha_2^*(z-1)) dF(z) \geq 0$$

Let $\phi(\alpha)$ be the function

$$\phi(\alpha) = \int (z-1) u_2'(w + \alpha(z-1)) dF(z)$$

As u_2 is concave, $\phi(\alpha)$ is decreasing.

To see this observe that the function:

$$\phi'(\alpha) = \int (z-1)^2 u_2''(w + \alpha(z-1))dF(z) \leq 0.$$

As $\phi(\alpha_2^*) \geq 0$, if we show that $\phi(\alpha_1^*) \leq 0$, then it follows that $\alpha_2^* \leq \alpha_1^*$.

Using definition (ii) of more risk averse,

$u_2(x) = \Psi(u_1(x))$ for some increasing, concave function Ψ.

Assume for simplicity that Ψ is differentiable.

$$\phi(\alpha_1^*) = \int (z-1)\Psi'(u_1(w+\alpha_1^*(z-1)))u_1'(w+\alpha_1^*(z-1))dF(z)$$

Note that $\Psi'(u_1(w+\alpha_1^*(z-1)))$ is a decreasing, positive function of z.

For $z \geq 1$, $\Psi'(u_1(w+\alpha_1^*(z-1))) \leq \Psi'(u_1(w))$ so that

$$(z-1)\Psi'(u_1(w+\alpha_1^*(z-1))) \leq (z-1)\Psi'(u_1(w))$$

For $z \leq 1$, $\Psi'(u_1(w+\alpha_1^*(z-1))) \geq \Psi'(u_1(w))$ so that

$$(z-1)\Psi'(u_1(w+\alpha_1^*(z-1))) \leq (z-1)\Psi'(u_1(w))$$

which implies that $\forall z$,

$$(z-1)\Psi'(u_1(w+\alpha_1^*(z-1))) \leq (z-1)\Psi'(u_1(w))$$

and therefore,

$$\begin{aligned} & \phi(\alpha_1^*) \\ = & \int (z-1)\Psi'(u_1(w+\alpha_1^*(z-1)))u_1'(w+\alpha_1^*(z-1))dF(z \\ \leq & \int (z-1)\Psi'(u_1(w))u_1'(w+\alpha_1^*(z-1))dF(z) \\ = & \Psi'(u_1(w))\int (z-1)u_1'(w+\alpha_1^*(z-1))dF(z) \\ = & 0. \end{aligned}$$

Thus, $\alpha_2^* \leq \alpha_1^*$ which contradicts the supposition that $\alpha_2^* > \alpha_1^*$.

If we assume that u_2 is strictly more risk averse than u_1, then $\mathbf{\Psi}$ is strictly concave in which the same arguments as above show that $\alpha_2^* < \alpha_1^*$.

Comparison across wealth levels.

"wealthier people more willing to bear risk"?

u exhibits decreasing absolute risk aversion (DARA):

$$r_A(x, u) \text{ is a decreasing function of } x.$$

Consider two levels of wealth $x_1 > x_2$.

Consider a random risk z faced by the agent (increment or decrement to wealth).

The individual evaluates the risk at wealth level x_1 according to the Bernoulli utility $u_1(z) = u(x_1 + z)$.

The individual evaluates the risk at wealth level x_1 according to the Bernoulli utility $u_2(z) = u(x_2 + z)$.

Comparing attitude towards risk at two different wealth levels x_1 and $x_2$is equivalent to comparing the utility functions u_1 and u_2.

DARA implies

$$r_A(z, u_2) = r_A(x_2+z, u) \geq r_A(x_1+z, u) = r_A(z, u_1), \forall z$$

Proposition. The following are equivalent:

(i) The Bernoulli utility function exhibits DARA

(ii) For any $x_2 < x_1, u(x_2 + z)$ is a concave transformation of $u(x_1 + z)$

(iii) For any risk $F(z)$, the certainty equivalent c_x of the lottery formed by adding risk z to wealth level x

$$u(c_x) = \int u(x + z) dF(z)$$

satisfies $(x - c_x)$ is decreasing in x.The higher wealth is, the less the agent is willing to pay to get rid of the risk.

(iv) The probability premium $\pi(x, \epsilon, u)$ is decreasing in x

(v) For any $F(z)$, if

$$\int u(x_2 + z) dF(z) \geq u(x_2)$$

and $x_2 < x_1$, then

$$\int u(x_1 + z) dF(z) \geq u(x_1).$$

Relative risk aversion.

Think of risky projects whose outcomes are proportionate gains or losses of current wealth.

Let $t > 0$ denote the proportional change of wealth.

An individual with Bernoulli utility u and initial wealth x can evaluation a random percentage risk t by the utility function

$$\widetilde{u}(t) = u(tx).$$

The initial wealth corresponds to $t = 1$.

For small risk around $t = 1$, the Arrow-Pratt measure of absolute risk aversion for $\widetilde{u}$

$$r_A(1, \widetilde{u}) = -\frac{\widetilde{u}''(1)}{\widetilde{u}'(1)}$$

captures the degree of risk aversion and the latter is equal to

$$-x\frac{u''(x)}{u'(x)}.$$

Arrow-Pratt measure of relative risk aversion at x :

$$r_R(x, u) = -x\frac{u''(x)}{u'(x)}.$$

Nonincreasing relative risk aversion: as wealth increases, individual is less risk averse regarding gambles that are proportional to wealth.

Stronger than DARA.

As $r_R(x, u) = x r_A(x, u)$,nonincreasing relative risk aversion implies decreasing absolute risk aversion but the converse may not hold.

Ex.

$$
\begin{aligned}
u(x) &= \frac{x^{1-\sigma}}{1-\sigma}, \sigma \neq 1, \sigma > 0 \\
u(x) &= \ln x \ (\sigma \to 1).
\end{aligned}
$$

$$
r_R(x) = \sigma.
$$

Constant relative risk aversion.

Comparing Payoff Distributions: Return and Risk.

For simplicity, confine attention to distributions $F()$ such that $F(0) = 0$ and $F(x) = 1$ for some x.

When can we say that a distribution (or lottery) F yields unambiguously *higher returns* than the distribution G?

Two answers:

1. If every expected utility maximizer who likes to have more money rather than less (i.e., Bernoulli utility u is nondecreasing) prefers F over G.

2. For every amount of money x,the probability of getting at least x is higher under distribution F than under G.

As it turns out: 1 and 2 are equivalent.

Definition: The distribution F first order stochastically dominates G, if for every nondecreasing function $u : \mathbb{R}_+ \to \mathbb{R}$,

$$\int u(x)dF(x) \geq \int u(x)dG(x).$$

Proposition 6.D.1: The distribution of monetary payoffs $F(.)$ first order stochastically dominates the distribution $G(.)$ if, and only if,

$$F(x) \leq G(x), \text{ for all } x.$$

Proof. Suppose F first order stochastically dominates G. We want to show $F(x) \leq G(x)$, for all x. Suppose to the contrary there exists $\overline{x}$ such that $F(\overline{x}) > G(\overline{x})$. Then, obviously $\overline{x} > 0$. Define the specific nondecreasing function $u(.)$ by:

$$\begin{aligned} u(x) &= 0, \text{ for } x \leq \overline{x} \\ &= 1, \text{ for } x > \overline{x}. \end{aligned}$$

Then,

$$\begin{aligned} \int u(x)dF(x) &= \int_{\{x>\overline{x}\}} dF(x) = 1 - F(\overline{x}) \\ \int u(x)dG(x) &= \int_{\{x>\overline{x}\}} dG(x) = 1 - G(\overline{x}) \end{aligned}$$

so that $F(\overline{x}) > G(\overline{x})$ implies:

$$\int u(x)dF(x) < \int u(x)dG(x),$$

which contradicts the fact that F first order stochastically dominates G.

Next, suppose that $F(x) \leq G(x)$, for all x.We want to show that for all nonincreasing function $u(.)$,

$$\int u(x)dF(x) \geq \int u(x)dG(x).$$

Easy version of proof: assume that in addition u is differentiable. Integrating by parts:

$$\begin{aligned} \int u(x)dF(x) &= [u(x)F(x)]_0^\infty - \int u'(x)F(x)dx \\ \int u(x)dG(x) &= [u(x)G(x)]_0^\infty - \int u'(x)G(x)dx \end{aligned}$$

so that

$$\begin{aligned} & \int u(x)dF(x) - \int u(x)dG(x) \\ = & \{[u(x)F(x)]_0^\infty - [u(x)G(x)]_0^\infty\} \\ & + \int u'(x)G(x)dx - \int u'(x)F(x)dx \\ = & \{[u(x)(F(x) - G(x))]_0^\infty\} + \int u'(x))(G(x) - F(x))dx \\ = & 0 + \int u'(x))(G(x) - F(x))dx \\ \geq & 0 \end{aligned}$$

as $u'(x) \geq 0, G(x) - F(x) \geq 0$ for all x.

Note that if F first order stochastically dominates G, then the mean or expected monetary payoff under distribution F is at least as large as that under G :

$$\int x dF(x) \geq \int x dG(x)$$

(Just choose $u(x) = x$).

However, the converse is not necessarily true; having higher expected payoff does not imply first order stochastic dominance.

Comparing probability distributions based on riskiness or dispersion $\Rightarrow$Second Order Stochastic Dominance

What is a "less risky" distribution? It is one that would be preferred by every risk-averse agent.

To abstract from comparison based on "returns": assume that we comparing distributions with the same mean.

Definition: For any two probability distributions $F(.), G(.)$ with the same mean, the distribution F second order stochastically dominates (SOSD) G i.e., F is less risky than G, if for every nondecreasing *concave* function $u : \mathbb{R}_+ \rightarrow \mathbb{R}$,

$$\int u(x)dF(x) \geq \int u(x)dG(x).$$

An alternative way of looking at SOSD: Mean-preserving spreads (MPS)

Consider a lottery whose monetary payoff x is distributed according to F.

Now add another layer of risk in the following manner: consider a compound lottery where for each realization x of the lottery F, the agent obtains total payoff $x + z$ where z is distributed according to a distribution $H_x(z)$ with mean zero i.e.,

$$\int z dH_x(z) = 0.$$

Thus, the (conditional) mean of $x + z$ is x.

The reduced simple lottery of this compound lottery generates a distribution $G(y)$.

We say that $G(.)$ is a mean preserving spread of F.

Observe that if $u(.)$ is concave:

$$\begin{aligned} & \int u(y)dG(y) \\ = & \int \left(\int u(x+z)dH_x(z) \right) dF(x) \\ \leq & \int u \left(\int (x+z)dH_x(z) \right) dF(x), \text{ using Jensen's inequality} \\ = & \int u(x)dF(x). \end{aligned}$$

so that $F(.)$ SOSD $G(.)$.

The converse is also true.

So that : F SOSD G if, and only if, G is a MPS of F.

It can also be shown that for any two probability distributions $F(.), G(.)$ with the same mean, the distribution F second order stochastically dominates (SOSD) G if, and only if,

$$\int_0^x G(t)dt \geq \int_0^x F(t)dt \text{ for all } x \geq 0$$

i.e., the total area under distribution function G (weakly) exceeds that under distribution F.