Four basic elements of a game:

Players

- agents that interact

Rules

- who moves when, what do they know or observe at each point of move, what they choose from....

Outcomes

- for each possible configuration of actions by all players what is the eventual outcome of the interaction - may not be quantifiable

Payoffs

- the players' preferences or utility function defined over possible outcomes.

Games may involve randomness (exogenous uncertainty or randomization in choice of actions)

Players may need to evaluate probability distributions or lotteries over outcomes.

Assume: each agent has preferences over all lotteries over outcomes of the game that are representable by an expected utility function.

The *payoff function* of a player: her Bernoulli utility: $\{$space of outcomes of the game$\} \longrightarrow \mathbb{R}$.

The actual utility levels are called payoffs.

Extensive Form Representation of a Game.

Captures

- who moves when (the sequencing of moves),

- what actions each player may choose from at each point of decision making

- what they know about other players and previous actions chosen by others at each point where they have to move in the game,

- how each configuration of action choices by players through the game generates an outcome....

Finite games: finite number of players, finite number of possible actions, finite number of moves.

Can use game tree to depict the extensive form.

Elements of a game tree:

* Decision nodes (points at which players are required to make decisions):

- Initial Nodes

- Successor Nodes

* Each action at a decision node leads to a distinct branch of the tree.

* Terminal nodes: where game terminates and an outcome of the game is realized.

* Payoff vectors at each terminal node indicating payoffs realized at that outcome.

Exogenous Uncertainty in the play of the game: modeled as move of nature.

Assume: perfect recall.

Player does not forget what she observed at an earlier stage of the game.

Assume: Common knowledge of the structure of the game.

In an extensive form game, this implies all players know the extensive form.

Strategy:

A complete contingent plan or decision rule that specifies how the player will act in each possible *distinguishable* circumstance in which she might be called upon to move i.e., in each information set where she is may be possibly required to make a choice.

A strategy for player i is a function $s_i : \mathcal{H}_i \to \mathcal{A}$ such that $s_i(H) \in C(H)$ for all $H \in \mathcal{H}_i$.

A strategy profile in a game with I players is a vector $s = (s_1, ..., s_I)$ where s_i is the strategy chosen by player i.

Also denoted sometimes as (s_i, s_{-i}) where s_{-i} is a $(I-1)$ vector consisting of a strategy choice for each player other than player i.

Normal Form Representation of an Extensive Form Game:

Every profile of strategies $s = (s_1, ..., s_I)$ induces an outcome of the game:

- a sequence of moves actually taken

$\Rightarrow$a probability distribution over terminal nodes of the game

$\Rightarrow$a probability distribution over payoff realizations of the game

$\Rightarrow$ *expected* payoff (utility) $u_i(s_1, ..., s_I)$ for each player i.

Definition: For a game with I players, the normal form representation Γ_N specifies for each player i a set of strategies S_i (with $s_i \in S_i$) and a payoff function $u_i(s_1, ..., s_I)$ giving the VNM utility levels associated with the (possibly random) outcomes arising from strategies $(s_1, ..., s_I)$.

Formally, $\Gamma_N = [I, \{S_i\}_{i=1}^{I}, \{u_i(.)\}_{i=1}^{I}]$.

Normal form: no information about moves, order of moves, sequencing, how the "strategy" of each player is composed or played or even what it means.

Can be seen as a simultaneous move game where players choose their strategies (rather than actions at various decision nodes), For any extensive form game, unique normal form representation.

Converse not true.

Players may randomize over actions at any decision node.

Choose probability distributions over deterministic or $pure$ strategies.

Such randomized strategies are called $mixed$ strategies.

Suppose that the S_i, the (pure) strategy set of each player i is finite.

A mixed strategy by player i denoted by $\sigma_i : S_i \rightarrow [0, 1]$ assigns to each pure strategy $s_i \in S_i$ a probability $\sigma_i(s_i)$ that it will be played where $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$.

The set of all possible mixed strategies of player i is denoted by $\triangle(S_i)$.

Every profile of mixed strategies (one for each player) generates a probability distribution over outcomes and payoffs of the game.

As players have VNM utility on the space of lotteries over outcomes, the payoff to each player from a mixed strategy profile is the expected utility (or payoff) generated.

Let $S = S_1 \times S_2 \times ... \times S_I$.

Let $\sigma = (\sigma_1, ..., \sigma_I)$ be a profile of mixed strategies where players randomize independently (not correlated strategies).

Player i's VNM utility or payoff from this mixed strategy profile, denoted by $u_i(\sigma)$, is given by

$$u_i(\sigma) = \sum_{(s_1,...,s_I) \in S} [\sigma_1(s_1)....\sigma_I(s_I)] u_i(s_1, ..., s_I)$$

If strategy set is not finite, each mixed strategy is captured by a probability distribution function and the payoffs can be similarly defined.

Normal form game allowing for mixed strategies: denoted by $\Gamma_N = [I, \{\triangle(S_i)\}, \{u_i\}]$

Consider normal form game $\Gamma_N = [I, \{S_i\}_{i=1}^I, \{u_i(.)\}_{i=1}^I]$ where we confine players to use only pure strategies.

Prisoner's Dilemma

$$\begin{bmatrix} 1\downarrow, 2\rightarrow & Not\ Confess & Confess \\ Not\ Confess & -2,-2 & -10,-1 \\ Confess & -1,-10 & -5,-5 \end{bmatrix}$$

(Strictly) Dominant Strategy for each player: Confess.

Let $S_{-i} = S_1 \times S_{i-1} \times S_{i+1} ... \times S_I$ denote the product of strategy sets of all players other than player i.

Definition: A strategy $s_i \in S_i$ is a strictly dominant strategy for player i in a game $\Gamma_N = [I, \{S_i\}_{i=1}^I, \{u_i(.)\}_{i=1}^I]$ if for all $s'_i \neq s_i, s'_i \in S_i$, we have

$$u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i}) \text{ for all } s_{-i} \in S_{-i}.$$

If a player has a strictly dominant strategy, it is individually optimal for the player to play it irrespective of her belief about what other players play.

In fact, it is the unique individually optimal strategy.

If every player has a strictly dominant strategy, it is obvious that all players should play this.

However, the outcome obtained as a result may be "collectively or jointly suboptimal" or "Pareto inefficient" in the sense that all players could have been better off if they had played according to a different strategy profile.

An example of how self interested individual behavior may not be collectively good.

Reason: each player determines his or her "optimal" strategy by looking at his or her own payoff ignoring the payoffs of other players.

"Externality".

It is rare for strictly dominant strategies to exist.

What strategy is optimal for a player often depends on what other players play.

However, a rational player will never play a strategy that is dominated by some other strategy (i.e., leads to strictly lower payoff no matter what other players play).

Definition: A strategy $s_i \in S_i$ is a *strictly dominated* strategy for player i in a game $\Gamma_N = [I, \{S_i\}_{i=1}^{I}, \{u_i(.)\}_{i=1}^{I}]$ if there exists another strategy $s_i' \in S_i$ such that

$$u_i(s_i', s_{-i}) > u_i(s_i, s_{-i}) \text{ for all } s_{-i} \in S_{-i}.$$

In this case, we say s_i' strictly dominates s_i.

A strictly dominated strategy should not be played by a rational player no matter what he believes about the strategy choice of other players.

$$\begin{bmatrix} 1\downarrow, 2 \rightarrow & L & R \\ U & 1,-1 & -1,1 \\ M & -1,1 & 1,-1 \\ D & -2,5 & -3,2 \end{bmatrix}$$

Both U and M strictly dominate D.

Note that if there is a strictly dominant strategy for a player , it strictly dominates every other strategy of the player (and vice-versa).

Definition: A strategy $s_i \in S_i$ is a *weakly dominated* strategy for player i in a game $\Gamma_N = [I, \{S_i\}_{i=1}^I, \{u_i(.)\}_{i=1}^I]$ if there exists another strategy $s_i' \in S_i$ such that

$$u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i}) \text{ for all } s_{-i} \in S_{-i}.$$

and further, there exists $\widehat{s}_{-i} \in S_i$ such that

$$u_i(s_i', \widehat{s}_{-i}) > u_i(s_i, \widehat{s}_{-i}).$$

In this case, we say s_i' weakly dominates s_i.

$$\begin{bmatrix} 1 \downarrow, 2 \rightarrow & L & R \\ U & 5,1 & 4,0 \\ M & 6,0 & 3,1 \\ D & 6,4 & 4,4 \end{bmatrix}$$

D weakly dominates U and M.

If a strategy for a player weakly dominates every other strategy in the strategy set of the player, we say it is a *weakly dominant* strategy.

Unlike a strictly dominated strategy, a rational player may play a weakly dominated strategy (if he/she has certain kind of belief about what the other players play).

Cannot be ruled out ex ante.

Rationality $\Rightarrow$ Rules out strictly dominated strategies.

Common knowledge of rationality

$\Rightarrow$ *Iterated Elimination of Strictly Dominated Strategies*.

Prisoner's Dilemma Modified (bias in favor of prisoner 1).

$$\begin{bmatrix} 1\downarrow, 2\rightarrow & Not\ Confess & Confess \\ Not\ Confess & 0, -2 & -10, -1 \\ Confess & -1, -10 & -5, -5 \end{bmatrix}$$

1 ↓, 2 →	L	M	R
T	−1, 7	4, 5	4, 10
C	0, 11	1, 4	3, 2
B	−1, 19	2, 10	1, −1

Order of deletion does not affect the set of strategies that survive iterated elimination of strictly dominated strategies.

Can generalize strictly dominated and dominant strategy concepts to normal form games that allow for mixed strategies in a straightforward way.

$$\begin{bmatrix} 1 \downarrow, 2 \rightarrow & L & R \\ U & 10, 1 & 0, 4 \\ M & 4, 2 & 4, 3 \\ D & 0, 5 & 10, 2 \end{bmatrix}$$

Playing U and D with probability $\frac{1}{2}$ each strictly dominates M.

Nash Equilibrium.

Consider normal form game $\Gamma_N = [I, \{S_i\}, \{u_i\}]$ where players restrict themselves to pure strategies.

Definition 1 *A strategy profile* $s^* = (s_1^*, s_2^*, ..., s_I^*) \in S$ *constitutes a Nash Equilibrium (NE) if for every* $i = 1, ...I,$

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$$

for all $s_i \in S_i$.

Each player's strategy is *a* best response to the strategies actually played by rivals.

$$s_i^* \in b_i(s_{-i}^*), i = 1, ...I$$

where $b_i(s_{-i}^*)$ is the best-response (or best reply or "reaction") correspondence defined by

$$b_i(s_{-i}^*) = \{s_i \in S_i : s_i \text{ solves } \max_{s_i' \in S_i} u_i(s_i', s_{-i}^*)\}.$$

* No player has a (strict) incentive to *unilaterally* deviate from playing according to strategy profile s^* (does not rule out gainful deviation by a coalition of multiple players).

In a NE, players play rationally holding correct conjectures (or forecasts) of rivals' play.

Example:

$$\begin{bmatrix} & b_1 & b_2 & b_3 \\ a_1 & 0,7 & 2,5 & 7,0 \\ a_2 & 5,2 & 3,3 & 5,2 \\ a_3 & 7,0 & 2,5 & 0,7 \end{bmatrix}$$

Unique NE: (a_2, b_2).

* Let N denote the set of NE strategy profiles,

IED the set of strategy profiles that survive iterated elimination of strictly dominated strategies and

U the set of strategy profiles consisting of strategies that are strictly undominated.

Then,

$$N \subset IED \subset U.$$

The concept of NE is based on the concept of mutually correct expectations.

Quite often, there can be multiple NE.

Coordination problems.

Example: Coordination game.

$$\begin{bmatrix} & L & R \\ U & 100, 100 & 0, 0 \\ D & 0, 0 & 1000, 1000 \end{bmatrix}$$

The two NE are Pareto-ranked (both players better off in (D, R) compared to (U, L).

Example: (Pure coordination game)

$$\begin{bmatrix} & L & R \\ U & 100, 100 & 0, 0 \\ D & 0, 0 & 100, 100 \end{bmatrix}$$

Example: Battle of Sexes

$$\begin{bmatrix} & Opera & Game \\ Opera & 100, 1000 & 0, 50 \\ Game & 50, 0 & 1000, 100 \end{bmatrix}$$

Example: Cake eating.

A cake is to be divided among two players.

Players 1 and 2 simultaneously choose the shares (s_1, s_2), $0 \leq s_i \leq 1$, of the cake they demand.

The payoff of each player i is the share of the cake obtained be her and is given by:

$$\begin{aligned} x_i &= s_i, \text{ if } s_i + s_j \leq 1, \\ &= 0, \text{ if } s_i + s_j > 1. \end{aligned}$$

Set of NE $= \{(s_1, s_2) : s_1 + s_2 = 1, 0 \leq s_i \leq 1, i = 1, 2\}$

Continuum of NE. Conflict of objectives across NE.

Why should we expect conjectures to be correct?

Certainly not a necessary consequence of rationality or common knowledge of rationality and payoffs.

* If there is a unique predicted outcome for a game (a unique obvious way to play the game), then it must be a Nash equilibrium.

* If certain outcomes are *focal* (Schelling) for cultural or other reasons (having to do with information not contained within the description of the game), then such an outcome can be a prediction only if it is Nash equilibrium.

* If players make a *non-binding* agreement prior to play about how they are going to play the game, then such an agreement is credible only if it is a Nash equilibrium (the pre-game communication makes the agreement focal).

*Stable social convention (norm): If the game is played repeatedly, then some stable social convention about how to play the game may emerge (a limit of some dynamic adjustment process); such a stable social convention or norm must be a NE.

Mixed Strategy Nash Equilibrium.

Consider the normal form game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i\}]$

Definition 2 *A strategy profile* $\sigma^* = (\sigma_1^*, \sigma_2^*, ..., \sigma_I^*) \in \prod_{i=1}^{I} \Delta(S_i)$ *constitutes a Nash Equilibrium (NE) if for every* $i = 1, ...I$,

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*)$$

for all $\sigma_i \in \Delta(S_i)$.

Example (Matching Pennies)

$$\begin{bmatrix} & H & T \\ H & +1, -1 & -1, +1 \\ T & -1, +1 & +1, -1 \end{bmatrix}$$

There is no NE in pure strategies.

Each player playing H and T with probability $\frac{1}{2}$ each constitutes a mixed strategy NE.

Given this strategy of rival, each player indifferent between playing H or T.

In any mixed strategy NE, each player is indifferent between pure strategies that she plays with strictly positive probability

i.e., given the mixed strategies played by other players, all such pure strategies must yield her exactly her the same expected utility or payoff (which would also be her NE payoff).

Further, no pure strategy that is played with probability zero by a player can yield strictly higher payoff than the payoff from the pure strategies that are played with strictly positive probability.

[In case the strategy set is not finite, the above must be true for *almost every* strategy in *the support* of the mixed strategy of each player].

Example:

$$\begin{bmatrix} & L & R \\ U & 100, 100 & 0, 0 \\ D & 0, 0 & 1000, 1000 \end{bmatrix}$$

Suppose player 1 plays U and D with probability p and $1 - p$, respectively.

For player 2, playing L yields expected payoff $100p$ and playing R yields $1000(1-p)$.These two expected payoffs are equal only if $p = \frac{1}{11}$. By a symmetric argument, player 1 is indifferent between U and D if and only if player 2 plays L and R with probabilities $\frac{1}{11}$ and $\frac{10}{11}$, respectively.

Thus, these mixed strategies constitute a NE.