Production Theory.

Description of Production Technology

L commodities, $l = 1, ...L$.

A production vector (production plan) $y = (y_1, ..., y_L) \in \mathbb{R}^L$ describes the (net) outputs of the L commodities from a production process where negative numbers denote net inputs (zero implies no input or output of that commodity).

The set of all feasible production vectors (i.e., vectors that are producible by the firm) is called the production set, denoted by $Y \subset \mathbb{R}^L$.It incorporates technological, regulatory, contractual and other constraints on the firm.

In general, the same good that appears as an input in one production vector may appear with a negative sign as an input in another production vector.

An alternative way to describe the production set Y is by looking at the boundary of the set.

Transformation function: $F(y) : \mathbb{R}^L \to \mathbb{R}$

$$Y = \{y \in \mathbb{R}^L : F(y) \leq 0\}$$

$F(y) = 0$ if and only if F is on the boundary of Y.

Transformation frontier:

$$\{y \in \mathbb{R}^L : F(y) = 0\}$$

If F is differentiable, then for any production plan $\overline{y}$ on the transformation frontier, the marginal rate of transformation between goods l, k

$$MRT_{l,k}(\overline{y}) = \frac{\partial F(\overline{y})/\partial y_l}{\partial F(\overline{y})/\partial y_k}$$

measures how much the net output of good k can increase if the firm decreases net output of good l by a unit (all else constant). It is the slope of the boundary of the production set.

Sometimes it is realistic to think of production sets where each commodity is either always an input or always an output (each coordinate is either nonnegative in all production plans in Y or non-positive in all production plans in Y.)

Suppose that $M < L$ commodities are outputs and $L - M$ are inputs. $(q_1, ..., q_M) \geq 0$ is the vector of quantity of outputs and $(z_1, ...z_{L-M}) \geq 0$ is the vector of quantity of inputs.

Most frequently used special structure: single output, multiple inputs; $M = 1$.

In this case, we can describe the technology by a *production function* $f(z_1, ..., z_{L-1})$ that gives us the maximum amount q of output (only one good that can be an output) that can be produced using input amounts $z = (z_1, ..., z_{L-1}) \geq 0$.

If the $L - th$ good is the output, then

$$Y = \{(-z, q) \in \mathbb{R}^{L-1} \times \mathbb{R} : 0 \leq q \leq f(z), z \geq 0\}.$$

If f is differentiable, then for any two pairs of inputs l, k, the marginal rate of technical substitution (MRTS) of input l for input k at fixed input level $\overline{z} \geq 0$ given by

$$MRTS_{l,k}(\overline{z}) = \frac{\frac{\partial f}{\partial z_l}}{\frac{\partial f}{\partial z_{lk}}}$$

measures the amount by which input k must be increased for each marginal unit decrease in input l in order to keep output unchanged at level $f(\overline{z})$. (Absolute) slope of isoquant when $L = 3$.

Some common assumptions on the production set Y (only some of them may be imposed in any instance):

(i) Y is nonempty.

(ii) Y is closed.

[Single output, multiple inputs case: production function f is continuous)

(iii) No free lunch (cannot produce outputs without any inputs): $y \geq 0, y \neq 0 \Rightarrow y \notin Y$.

[Single output, multiple inputs case: $f(0) = 0$.]

(iv)Possibility of inaction: $\mathbf{0} \in Y$.

Not satisfied if some decisions have been made or irreversible contracts for delivery of some inputs have been signed (fixed costs are sunk).

(v) Free disposal: $y \in Y, y' \leq y \Rightarrow y' \in Y$.

(can always produce "less of outputs" with "more of inputs").

[Single output, multiple inputs case: $f(z)$ is non-decreasing in each input].

(vi) Nonincreasing returns to scale: $y \in Y \Rightarrow \alpha y \in Y$ for all $\alpha \in [0, 1]$.

Always possible to scale down the production process.

[Increasing return to scale: if levels of all inputs are reduced by a certain common fraction, then every output must decline by a larger fraction so that it is not possible to scale down the production process.]

It implies the possibility of inaction.

[Single output, multiple inputs case: $f(\alpha z) \geq \alpha f(z)$ for all $\alpha \in [0, 1]$,or equivalently, $f(\lambda z) \leq \lambda f(z)$ for all $\lambda \geq 1$]

(vii) Nondecreasing returns to scale: $y \in Y \Rightarrow \alpha y \in Y$ for all $\alpha > 1$.

Always possible to scale up the production process.

[Note:

Decreasing returns to scale: if levels of all inputs are expanded by a certain common fraction, then outputs can expand by only a smaller fraction so it is not possible to scale up the production process.

Reflects certain unlisted factors of production that may be scarce or fixed .]

[Single output, multiple inputs case: $f(\lambda z) \geq \lambda f(z)$ for all $\lambda \geq 1$ or equivalently, $f(\alpha z) \leq \alpha f(z)$ for all $\alpha \in [0, 1]$]

(viii) Constant returns to scale: $\alpha y \in Y$ for all $\alpha \geq 0$.

Satisfies both nonincreasing and nondecreasing returns to scale.

(Geometrically, Y is a cone.]

[Single output, multiple inputs case: $f(\lambda z) = \lambda f(z)$ for all $\lambda \geq 0$ i.e., f is homogenous of degree one.]

(ix) Y is convex: $y, y' \in Y \Rightarrow \alpha y + (1 - \alpha)y' \in Y$ for all $\alpha \in [0, 1]$.

Convexity + Possibility of inaction $\Rightarrow$ nonincreasing returns to scale.

Implies, more "balanced" input combinations are at least as productive as "unbalanced" combinations.

[Single output, multiple inputs case: equivalent to $f(z)$ is concave.]

(ix)' Y is strictly convex.

Digression: A set $S \subset \mathbb{R}^L$ is said to be strictly convex if for any two distinct elements $y^1, y^2 \in S, y^1 \neq y^2$ and $\alpha \in (0,1), z = \alpha y^1 + (1-\alpha)y^2$ is an interior point of S i.e., there exists $\epsilon > 0$ such that $\{y : \|y - z\| < \epsilon\} \subset S$.

[Single output, multiple inputs case: equivalent to $f(z)$ is strictly concave.]

Profit Maximization by an individual "price taking" firm.

Given vector of market prices $p = (p_1, ..., p_L) \gg 0$ where p_l is the price of the $l-th$ good.

Prices are independent of the production plan of the firm.

Assume: $Y \subset \mathbb{R}^L$ is nonempty, closed and satisfies free disposal.

For any production plan $y = (y_1, ..., y_L) \in Y$,

$$p \cdot y = \sum_{l=1}^{L} p_l y_l$$

is the *profit* of the firm where:

the sum of the product of positive y_l (outputs) with their prices yields the *revenue* earned by the firm

and the sum of the product of negative y_l (inputs) with their prices yields the production cost incurred by the firm.

Profit maximization problem (PMP): Given $p \gg 0$,

$$\max_y p \cdot y$$
$$\text{subject to } y \in Y.$$

Equivalently: Given $p \gg 0$,

$$\max_y p \cdot y$$
$$\text{subject to } F(y) \leq 0.$$

Note there is no separate non-negativity constraint in this maximization problem.

The profit function $\pi(p)$ is the value of this maximization problem i.e., the maximum profit that can be earned given price vector p, and defined by:

$$\pi(p) = \max\{p \cdot y : y \in Y\}.$$

If there is no solution to the PMP, one can still define $\pi(p)$ as $\sup\{p \cdot y : y \in Y\}$.

Let $y(p)$ be the set of solutions to the PMP:

$$y(p) = \{y \in Y : p \cdot y = \pi(p)\}.$$

$y(p)$ is the firm's (net) supply correspondence. If there is a unique solution to the PMP for every p, then $y(p)$ is a function (the supply function).

Note: negative entries in any vector in $y(p)$ reflect (net) demand for inputs.

Y is often not a bounded set and therefore not compact. So a solution to the PMP may not exist and so $y(p)$ may be an empty set.

Example: $L = 2$. One input, one output. $Y = \{(y_1, y_2) : y_1 \leq 0, y_2 \leq -2y_1\}$.

Consider $(p_1, p_2) \gg 0$, where $2p_2 > p_{1.}$ Then for any $z > 0$,a production plan of the type $(-z, 2z) \in Y$ and yields profit:

$$\begin{aligned} & 2zp_2 - zp_1 \\ = \;& z(2p_2 - p_1) \\ \rightarrow \;& +\infty \text{ as } z \rightarrow +\infty. \end{aligned}$$

There is no solution to the PMP.

[The technology in this example satisfies constant returns to scale; it is equivalent to having a production function of the form $f(z) = 2z$ where z is the quantity of input of good 1].

Exercise 5.C.1: If Y exhibits nondecreasing returns to scale then either $\pi(p) \leq 0$ or $\pi(p) = +\infty$;there is no solution to PMP in the latter case.

If F is differentiable, $y^* \in y(p)$ implies the following first order necessary condition: for some $\lambda \geq 0$

$$p_l = \lambda \frac{\partial F(y^*)}{\partial y_l}, l = 1, ...L.$$

Note that if the technology is such that certain goods can only be produced as output (have non-negative sign in all feasible production plans) and/or certain goods certain goods can only be used as inputs (have non-positive sign in all feasible production plans), then F is not typically differentiable on $\mathbb{R}^L$ (kink near zero). In these cases, one can effectively re-write the profit maximization problem with non-negativity and non-positivity constraints; the first order conditions would then take the form of inequalities when these constraints are binding (use or output of a good is zero).

Law of supply: For all $p, p' \gg 0, y \in y(p), y' \in y(p')$

$$(p - p')(y - y') \geq 0$$

Proof: Follows from

$$\begin{aligned} py &\geq py' \\ p'y' &\geq p'y. \end{aligned}$$

If only one price changes i.e., $p_l \neq p'_l, p_k = p'_k$ for all $k \neq l$, then we have

$$(p_l - p'_l)(y_l - y'_l) \geq 0$$

Thus, $p_l > p'_l \Rightarrow y_l \geq y'_l$.

Other things being equal, an increase in the price of an output increases its supply and an increase in the price of an input reduces the demand for the input.

Efficiency & Profit maximization.

Definition 5.F.1: A production plan $y \in Y$ is said to be efficient if there is no $y' \in Y$ such that $y' \geq y$ and $y' \neq y$.

Proposition 5.F.1: If $y^* \in y(p)$ for some $p \gg 0$,then y^* is efficient.

Special case: Single Output Technology with Production function f.

Let $z = (z_1, ...z_{L-1})$ be the vector of inputs and $w = (w_1, ..., w_{L-1}) \gg 0$ be the vector of prices of these inputs. Let the <u>scalar</u> $p > 0$ be the price of output (good 1)

PMP reduces to

$$\max_{z \geq 0}[pf(z) - w \cdot z]$$

Then,

$$y(p) = \{(-z^*, f(z^*)) : z^* \text{ solves above PMP}\}$$

Also, if z^* solves the above PMP,

$$\pi(p) = pf(z^*) - w \cdot z^*.$$

If f is differentiable, then one has the following first order necessary condition for $z^*$to be optimal: For $l = 1, ...L$

$$
\begin{aligned}
p\frac{\partial f(z^*)}{\partial z_l} &\leq w_l \\
&= w_l, \text{ if } z_l^* > 0.
\end{aligned}
$$

The left hand side is the value of marginal product of input l and it is compared to the price of input l i.e., the marginal cost of employing this input.

If f is concave, then these first order conditions are also sufficient.

If both inputs l and k are used i.e., $z_l^* > 0, z_k^* > 0$, then:

$$MRTS_{l,k}(z^*) = \frac{\frac{\partial f(z^*)}{\partial z_l}}{\frac{\partial f(z^*)}{\partial z_k}} = \frac{w_l}{w_k}$$

The marginal rate at which one can technologically substitute one input for another is equal to their price ratio i.e., rate at which the market allows the firm to substitute one input for another.

Some important properties of profit function $\pi(p)$ and supply correspondence $y(p)$:

Proposition 5.C.1 Assume Y is closed and satisfies free disposal. Then:

(i) $\pi(p)$ is homogenous of degree one; $y(p)$ is homogenous of degree zero

(ii) $\pi(p)$ is a convex function

(iii) If Y is convex, then $y(p)$ is a convex set for all $p \gg 0$. If Y is strictly convex, then $y(p)$ is single valued if it is nonempty.

(iv) Hotelling's lemma: If $y(\overline{p})$ consists of a single point, then π is differentiable at $\overline{p}$ and

$$\frac{\partial \pi(\overline{p})}{\partial p_l} = y_l(\overline{p}), l = 1, ...L.$$

(v) If $y(p)$ is a differentiable function at $p = \overline{p}$, then the supply substitution matrix

$$Dy(\overline{p}) = \begin{bmatrix} \frac{\partial y_1}{\partial p_1} & \frac{\partial y_2}{\partial p_1} & & \frac{\partial y_L}{\partial p_1} \\ \frac{\partial y_1}{\partial p_2} & \frac{\partial y_2}{\partial p_2} & & \frac{\partial y_L}{\partial p_2} \\ & & & \\ & & & \\ \frac{\partial y_1}{\partial p_L} & & & \frac{\partial y_L}{\partial p_L} \end{bmatrix}$$

(where all partials are evaluated at $p = \overline{p}$) is a symmetric and positive semi-definite matrix. Further,

$$Dy(\overline{p})\overline{p} = 0.$$

As diagonal terms of a positive semidefinite matrix are non-negative, this implies the law of supply:

$$\frac{\partial y_l}{\partial p_l} \geq 0.$$

Cost Minimization

A necessary consequence of profit maximization.

Focus on single output technology with production function f.

Continue to assume Y is closed and satisfies free disposal.

Suppose firm is price taking in the input market (can have market power in the output market).

Cost Minimization Problem (CMP): Given the input price vector $w = (w_1, ..., w_{L-1}) \gg 0$,output $q > f(0)$

$$\begin{aligned} & \min_{z \geq 0} w \cdot z \\ \text{subject to } f(z) & \geq q. \end{aligned}$$

Mathematically, identical to the expenditure minimization problem in consumer theory.

The value of the CMP i.e., the minimized level of cost is the cost function $c(w, q)$.

The set of optimal input vectors, denoted by $z(w, q)$ is known as the *conditional factor demand correspondence* (or *function*, if it is always single valued).

Solution always exists.

Even if technology is characterized by increasing returns to scale.

$z^* \in z(w, q), w \gg 0, q > 0$ implies

$$f(z^*) = q.$$

If the production function f is differentiable, then $z^* \in z(w,q)$ implies that for some $\lambda \geq 0$ and for $l = 1, ...L$:

$$\begin{aligned} w_l &\geq \lambda \frac{\partial f(z^*)}{\partial z_l} \\ &= \lambda \frac{\partial f(z^*)}{\partial z_l} \text{ if } z_l^* > 0. \end{aligned}$$

These conditions (along with $f(z^*) = q$) are sufficient for cost minimization if the production function f is *quasi-concave*.

If both inputs l and k are used i.e., $z_l^* > 0, z_k^* > 0$, then:

$$MRTS_{l,k}(z^*) = \frac{\frac{\partial f(z^*)}{\partial z_l}}{\frac{\partial f(z^*)}{\partial z_k}} = \frac{w_l}{w_k}$$

The marginal rate at which one can technologically substitute one input for another is equal to their price ratio.

Interpretation of λ :

Marginal cost

$$\frac{\partial c}{\partial q} = \lambda$$

Law of (Conditional Input) Demand:

For $q > f(0), w \gg 0, z \in z(w,q), z' \in z(w',q)$

$$(w - w')(z - z') \leq 0$$

Properties (Proposition 5.C.2)

(i) $c(w, q)$ is homogenous of degree one in w and $z(w, q)$ is homogenous of degree zero in w

(ii) $c(w, q)$ is non-decreasing in q.

(ii) $c(w, q)$ is concave in w.

(iii) If f is quasi-concave, then $z(w, q)$ is a convex set for every $w \gg 0, q > f(0)$. If f is strictly quasi-concave, then $z(w, q)$ is single valued.

(iv) (Shephard's lemma) If $z(\overline{w}, q)$ is single valued, then $c(w, q)$ is differentiable at $w = \overline{w}$ and

$$\frac{\partial c(\overline{w}, q)}{\partial w_l} = z_l(\overline{w}, q).$$

(v) If $f(.)$ is homogenous of degree $k > 0$, then $c(w, q)$ and $z(w, q)$ are homogenous of degree $\frac{1}{k}$ in q; in particular, $c(w, q) = q^{\frac{1}{k}}\phi(w)$. Further, if $z(w, q)$ is single valued, then $z(w, q) = q^{\frac{1}{k}}\theta(w)$ where ϕ, θ depend only on w.

(Constant returns to scale $\Rightarrow c(w, q) = q\phi(w)$ so that average cost and marginal cost are both independent of q.)

(vi) If f is concave, then $c(w, q)$ is a convex function of q.(If, in addition, c is differentiable in q, then marginal cost is nondecreasing in q).

Restating the profit maximization problem (PMP2) for a price taking firm: Given output price $p > 0$, input prices $w \gg 0$,

$$\max_{q \geq 0} pq - c(w, q)$$

If $c(w, q)$ is differentiable in q, the first order condition for output q^* to solve PMP2:

$$\begin{aligned} p &\leq \frac{\partial c(w, q^*)}{\partial q} \\ &= \frac{\partial c(w, q^*)}{\partial q}, \text{ if } q^* > 0. \end{aligned}$$

Familiar comparison of price and marginal cost.

If $c(w, q)$ is convex in q (which holds when f is concave), first order conditions are sufficient for a maximum.