General Competitive Equilibrium: A Brief Introduction

Economy:

I consumers, $i = 1, ...I$.

J firms, $j = 1, ...J$.

L goods, $l = 1, ...L$

Initial Endowment of good l in the economy: $\overline{\omega}_l, l = 1, ...L$.

Total endowment vector: $\overline{\omega} = (\overline{\omega}_1, ..., \overline{\omega}_L) \in \mathbb{R}^L$.

Consumer i : preferences over consumption set $X_i \subset \mathbb{R}^L$
.

Typical consumption bundle of consumer i :

$$x_i = (x_{i1}, ..., x_{iL}) \in X_i$$

$\succsim_i$: binary preference relation of consumer i defined on X_i

Preferences are assumed to be rational (complete and transitive).

Firm j: technology summarized by a production possibility set $Y_j \subset \mathbb{R}^L$.

Typical production vector of firm j : $y_j = (y_{1j}, ..., y_{Lj}) \in \mathbb{R}^L$.

Y_j assumed to be nonempty and closed.

Total (net) amount of good l available for consumption in the economy, $l = 1, ...L$:

$$\overline{\omega}_l + \sum_{j=1}^{J} y_{lj}.$$

Definition: An allocation $(x, y) = (x_1, ...x_I, y_1, ..., y_J)$ is a specification of a consumption bundle $x_i \in X_i$ for each consumer $i = 1, ...I$, and a production plan $y_j \in Y_j$ for each firm $j = 1, ...J$.

Definition: An allocation $(x_1, ...x_I, y_1, ..., y_J)$ is feasible if

$$\sum_{i=1}^{I} x_{il} = \overline{\omega}_l + \sum_{j=1}^{J} y_{lj}, l = 1, ...L.$$

or in vector notation

$$\sum_{i=1}^{I} x_i = \overline{\omega} + \sum_{j=1}^{J} y_j. \qquad (1)$$

Utility possibility set: Suppose preferences of each consumer i can be represented by utility function u_i.

Set of all utility profiles of consumers $i = 1, ...I$ that can be feasibly generated in this economy.

$$\begin{aligned} U &= \{(u_1, .., u_I) : \exists \text{ a feasible allocation} \\ &\quad (x_1, ..x_I, y_1, ., y_J) \\ \text{such that } u_i(x_i) &\geq u_i \; \forall \; i = 1, ..I.\} \end{aligned}$$

Efficiency

Definition: An allocation $(x', y') = (x'_1, ...x'_I, y'_1, ..., y'_J)$ is said to Pareto-dominate an allocation $(x, y) = (x_1, ...x_I,$ $y_1, ..., y_J)$ if

$$x'_i \succsim x_i \text{ for all } i = 1, ..., I, \text{AND } x'_i \succ x_i \text{ for some } i.$$

Definition: A feasible allocation (x, y) is Pareto optimal or Pareto efficient if it is not Pareto-dominated by any other feasible allocation.

In other words, an allocation is Pareto efficient if it is not possible to make any one better off without making someone worse off.

Private Ownership (or Market) Economy.

Initial endowments and technological possibilities (firms) are owned by consumers.

Consumer i initially owns a (endowment) vector $\omega_i = (\omega_{1i}, ..., \omega_{Li})$ where

$$\sum_{i=1}^{I} \omega_i = \overline{\omega}.$$

Share of firm j owned by consumer i : $\theta_{ij} \geq 0, j = 1, ...J$, where

$$\sum_{i=1}^{I} \theta_{ij} = 1, j = 1, ...J.$$

The share θ_{ij} entitles consumer i to a fraction θ_{ij} of firm $j's$ profit.

(Perfectly) Competitive Economy

A market exists for each of the L goods.

All consumers and firms act as price takers i.e., assume that their individual actions do not affect market prices.

Price vector: $p = (p_1, ..., p_L) \in \mathbb{R}^L$.

Note that given price vector p, the budget set of each consumer i is given by:

$$\{x_i \in X_i : px_i \leq p\omega_i + \sum_{j=1}^{J} \theta_{ij}(p \cdot y_j)\}$$

Consumer's wealth depends on prices as prices determine the value of initial endowment as well as firms' profits (value of consumer's shareholding).

Consider the private ownership perfectly competitive economy specified by

$$\left(\{(X_i, \succsim_i)\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J}, \{(\omega_i, \theta_{i1}, ...\theta_{iJ})\}_{i=1}^{I}\right).$$

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Definition: An allocation $(x^*, y^*) = (x_1^*, .., x_I^*, y_1^*, ., y_J^*)$ and a price vector $p = (p_1, ..., p_L) \in \mathbb{R}^L$ constitute a competitive or Walrasian equilibrium for the economy described above if the following hold:

(a) For every j, y_j^* maximizes profits in Y_j i.e.,

$$p \cdot y_j^* \geq p \cdot y_j \text{ for all } y_j \in Y_j$$

(b) For every i, $x_i^* \succsim x_i$ for all x_i in consumer i's budget set:

$$\{x_i \in X_i : p \cdot x_i \leq p \cdot \omega_i + \sum_{j=1}^{J} \theta_{ij}(p \cdot y_j^*)\}.$$

(c) Market Clearing:

$$\sum_{i=1}^{I} x_i^* = \overline{\omega} + \sum_{j=1}^{J} y_j^*.$$

Note that if the allocation $(x_1^*, .., x_I^*, y_1^*, ., y_J^*)$ and price vector $p^* \gg 0$ constitute a competitive equilibrium, then so do the allocation $(x_1^*, .., x_I^*, y_1^*, ., y_J^*)$ and price αp^* for any scalar $\alpha > 0$.

In this case, we can normalize prices without any loss of generality - for instance, by setting one of the prices equal to 1 (the corresponding good is called the *numeraire* good).

Lemma 10.B.1: If the allocation $(x_1, ...x_I, y_1, ..., y_J)$ and a price vector $p \gg 0$ satisfies the market clearing condition for all goods other than some good k,

$$\sum_{i=1}^{I} x_{li} = \omega_l + \sum_{j=1}^{J} y_{lj}, l = 1, ...L, l \neq k,$$

and if every consumer's budget constraint is satisfied with equality:

$$p \cdot x_i = p \cdot \omega_i + \sum_{j=1}^{J} \theta_{ij}(p \cdot y_j), i = 1, ...I.$$

then the market for good k also clears.

Proof: The budget equality for each consumer i can be written as

$$\sum_{l=1}^{L} p_l x_{li} = \sum_{l=1}^{L} p_l \omega_{li} + \sum_{j=1}^{J} \theta_{ij} \left(\sum_{l=1}^{L} p_l y_{lj} \right)$$

so that

$$\begin{aligned}
& p_k x_{ki} - p_k \omega_{ki} - \sum_{j=1}^{J} \theta_{ij} p_k y_{kj} \\
= & -\left[\sum_{l \neq k} p_l x_{li} - \sum_{l \neq k} p_l \omega_{li} - \sum_{j=1}^{J} \theta_{ij} \left(\sum_{l \neq k} p_l y_{lj} \right) \right] \\
= & -\left[\sum_{l \neq k} p_l x_{li} - \sum_{l \neq k} p_l \omega_{li} - \sum_{l \neq k} p_l \left(\sum_{j=1}^{J} \theta_{ij} y_{lj} \right) \right] \\
= & -\sum_{l \neq k} p_l \left[x_{li} - \omega_{li} - \left(\sum_{j=1}^{J} \theta_{ij} y_{lj} \right) \right]
\end{aligned}$$

i.e.,

$$
\begin{aligned}
& p_k(x_{ki} - \omega_{ki} - \sum_{j=1}^{J} \theta_{ij} y_{kj}) \\
= \;& -\sum_{l \neq k} p_l \left[x_{li} - \omega_{li} - \left(\sum_{j=1}^{J} \theta_{ij} y_{lj} \right) \right]
\end{aligned}
$$

Adding this over $i = 1, ...I$:

$$\begin{aligned}
& p_k \sum_{i=1}^{I} \left[x_{ki} - \omega_{ki} - \sum_{j=1}^{J} \theta_{ij} y_{kj} \right] \\
= & -\sum_{i=1}^{I} \left[\sum_{l \neq k} p_l \left\{ x_{li} - \omega_{li} - \left(\sum_{j=1}^{J} \theta_{ij} y_{lj} \right) \right\} \right] \\
= & -\sum_{l \neq k} p_l \left[\sum_{i=1}^{I} \left\{ x_{li} - \omega_{li} - \left(\sum_{j=1}^{J} \theta_{ij} y_{lj} \right) \right\} \right] \\
= & -\sum_{l \neq k} p_l \left[\sum_{i=1}^{I} x_{li} - \sum_{i=1}^{I} \omega_{li} - \sum_{i=1}^{I} \sum_{j=1}^{J} \theta_{ij} y_{lj} \right] \\
= & -\sum_{l \neq k} p_l \left[\sum_{i=1}^{I} x_{li} - \omega_l - \sum_{j=1}^{J} \left(\sum_{i=1}^{I} \theta_{ij} \right) y_{lj} \right] \\
= & -\sum_{l \neq k} p_l \left[\sum_{i=1}^{I} x_{li} - \omega_l - \sum_{j=1}^{J} y_{lj} \right] = 0,
\end{aligned}$$

using the fact that all markets other than k clear. As $p_k > 0$, it follows

$$\sum_{i=1}^{I} \left[x_{ki} - \omega_{ki} - \sum_{j=1}^{J} \theta_{ij} y_{kj} \right] = 0$$

i.e.,

$$\sum_{i=1}^{I} x_{ki} - \omega_k - \sum_{i=1}^{I} \sum_{j=1}^{J} \theta_{ij} y_{kj} = 0$$

which implies

$$\sum_{i=1}^{I} x_{ki} - \omega_k - \sum_{j=1}^{J} y_{kj} = 0$$

i..e., the market for good k also clears.

First Fundamental Theorem of Welfare Economics: **If preferences are locally non-satiated, any Walrasian equilibrium allocation is Pareto-optimal.**

Proof. Suppose to the contrary that there is a Walrasian equilibrium allocation $(x^*, y^*) = (x_1^*, .., x_I^*, y_1^*, ., y_J^*)$ that is not Pareto-optimal.

Let $p = (p_1, ..., p_L)$ be the associated Walrasian equilibrium price vector.

As (x^*, y^*) is not Pareto-optimal, there exists another feasible allocation $(\widehat{x}, \widehat{y}) = (\widehat{x}_1, .., \widehat{x}_I, \widehat{y}_1, ., \widehat{y}_J)$ that Pareto-dominates (x^*, y^*) i.e.,

$$\widehat{x}_i \succsim x_i^* \text{ for all } i = 1, ..., I, \quad (2)$$

$$\text{AND } \widehat{x}_{i'} \succ x_{i'}^* \text{ for some } i'. \quad (3)$$

As $(\widehat{x}, \widehat{y}) = (\widehat{x}_1, .., \widehat{x}_I, \widehat{y}_1, ., \widehat{y}_J)$ is feasible allocation:

$$\sum_{i=1}^{I} \widehat{x}_i = \overline{\omega} + \sum_{j=1}^{J} \widehat{y}_j \quad (4)$$

From the definition of Walrasian equilibrium, for all $i = 1, ...I$, $x_i^* \succsim x_i$ for all x_i in consumer i's budget set:

$$\{x_i \in X_i : p \cdot x_i \leq p \cdot \omega_i + \sum_{j=1}^{J} \theta_{ij}(p \cdot y_j^*)\}.$$

If $\widehat{x}_i$ is "below the budget line at price vector p" for any i:

$$p \cdot \widehat{x}_i < p \cdot \omega_i + \sum_{j=1}^{J} \theta_{ij}(p \cdot y_j^*)$$

then by local non-satiation, there is a consumption bundle $\widetilde{x}_i$ close enough to $\widehat{x}_i$ that is in the budget set and strictly preferred to $\widehat{x}_i$:

$$\widetilde{x}_i \in \{x_i \in X_i : p \cdot x_i \leq p \cdot \omega_i + \sum_{j=1}^{J} \theta_{ij}(p \cdot y_j^*)\}$$

AND $\widetilde{x}_i \succ \widehat{x}_i$.

Using (2) we have $\widetilde{x}_i \succ x_i^*$, a contradiction.

Therefore,

$$p \cdot \widehat{x}_i \geq p \cdot \omega_i + \sum_{j=1}^{J} \theta_{ij}(p \cdot y_j^*), \text{ for all } i = 1, ...I. \quad (5)$$

Further, $\widehat{x}_{i'} \succ x_{i'}^*$ implies $\widehat{x}_{i'}$ is outside the budget set at price vector p :

$$p \cdot \widehat{x}_{i'} > p \cdot \omega_i + \sum_{j=1}^{J} \theta_{i'j}(p \cdot y_j^*) \quad (6)$$

Adding up (5) and (6):

$$\sum_{i=1}^{I} p \cdot \widehat{x}_i > \sum_{i=1}^{I} p \cdot \omega_i + \sum_{i=1}^{I} \sum_{j=1}^{J} \theta_{ij}(p \cdot y_j^*)$$

and as y_j^* is profit maximizing for firm j,

$$(p \cdot y_j^*) \geq (p \cdot \widehat{y}_j),$$

so that

$$
\begin{aligned}
\sum_{i=1}^{I} p \cdot \widehat{x}_i &> \sum_{i=1}^{I} p \cdot \omega_i + \sum_{i=1}^{I} \sum_{j=1}^{J} \theta_{ij}(p \cdot \widehat{y}_j) \\
&= p \cdot \overline{\omega} + \sum_{j=1}^{J} \sum_{i=1}^{I} \theta_{ij}(p \cdot \widehat{y}_j) \\
&= p \cdot \overline{\omega} + \sum_{j=1}^{J} p \cdot \widehat{y}_j. \qquad (7)
\end{aligned}
$$

However, using (4):

$$
\begin{aligned}
\sum_{i=1}^{I} p \cdot \widehat{x}_i &= p \cdot \sum_{i=1}^{I} \widehat{x}_i \\
&= p \cdot (\overline{\omega} + \sum_{j=1}^{J} \widehat{y}_j) \\
&= p \cdot \overline{\omega} + \sum_{j=1}^{J} p \cdot \widehat{y}_j,
\end{aligned}
$$

thus contradicting (7).

Second Fundamental Theorem of Welfare Economics

Goal: Can every Pareto-optimal allocation can be attained by a decentralized competitive market economy as a Walrasian or competitive equilibrium?

Answer: May not be possible without redistribution of wealth (or endowment).

Definition: Given an economy specified by

$$\left(\{(X_i, \succsim_i)\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J}, \overline{\omega}\right),$$

an allocation (x^*, y^*) and price vector $p = (p_1, ..., p_L)$ constitute a *price equilibrium with transfers* if there is an assignment of wealth levels $(w_1, ..., w_I)$ with

$$\sum_{i=1}^{I} w_i = p \cdot \overline{\omega} + \sum_{j=1}^{J} p \cdot y_j^*$$

such that:

(i) For every j, y_j^* maximizes profits in Y_j i.e.,

$$p \cdot y_j \geq p \cdot y_j \text{ for all } y_j \in Y_j$$

(ii) For every i, $x_i^* \succsim x_i$ for all x_i in consumer i's budget set $\{x_i \in X_i : p \cdot x_i \leq w_i\}$

(iii) Market Clearing:

$$\sum_{i=1}^{I} x_i^* = \overline{\omega} + \sum_{j=1}^{J} y_j^*.$$

Note that: in a private ownership economy, $(w_1, ..., w_I)$ can be thought of as a reallocation of the vector $(p \cdot \omega_i + \sum_{j=1}^{J} \theta_{ij}(p \cdot y_j^*), i = 1, ..I)$ of wealth of consumers before they spend it on consumption (for instance through some tax -transfer scheme).

Second Fundamental Theorem of Welfare Economics:

Consider an economy specified by

$$\left(\{(X_i, \succsim_i)\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J}, \overline{\omega}\right)$$

Assume that:

(a) Y_j *is convex for all* $j = 1, ...J,$

(b) $\succsim_i$*is continuous, convex and strongly monotone for all* $i = 1, ...I.$

(c) there exists $(y_{1,}...y_J), y_j \in Y_j$ *such that* $\sum_{j=1}^{J} y_j + \overline{\omega} >> 0$ *i.e., a strictly positive vector of goods can be supplied potentially.*

Then for every Pareto-optimal allocation (x^*, y^*) *there exists a price vector*

$$p = (p_1, ...p_L) \neq 0$$

such that (x^*, y^*, p) *is a price equilibrium with transfers.*