Preferences and Utility.

X : set of alternatives (choice set or domain).

A preference relation $\succsim$ is a binary relation on X that allows comparison of pairs of alternatives $x, y \in X$.

$x \succsim y$: alternative x is at least as good as alternative y.

From $\succsim$, we can derive two other binary preference relations on X:

(i) Strict preference relation $\succ$ defined by

$$x \succ y \;\Leftrightarrow\; x \succsim y \text{ but not } y \succsim x$$

(ii) Indifference relation $\sim$ defined by

$$x \sim y \;\Leftrightarrow\; x \succsim y \text{ and } y \succsim x$$

A basic assumption on $\succsim$ is that it is *rational.*

Definition: The preference relation $\succsim$ on X is *rational* if it satisfies the following properties:

(1) *Completeness:* for all $x, y \in X$, either $x \succsim y$ or $y \succsim x$ or both.

(2) *Transitivity:*for all $x, y, z \in X$, if $x \succsim y$ and $y \succsim z$, then $x \succsim z$.

Sometimes a rational preference relation is also called an "ordering".

Question:

If the weak preference relation $\succsim$ on X is complete, does it imply that the strict preference relation $\succ$ and the indifference relation $\sim$ are complete?

If the weak preference relation $\succsim$ on X is transitive, does it imply that the strict preference relation $\succ$ and the indifference relation $\sim$ are transitive?

Utility Function.

A utility function assigns a numerical value to each element in X in accordance with the preference.

More formally:

Definition: A function $u : X \to \mathbb{R}$ is a utility function representing preference relation $\succsim$ if, for all $x, y \in X$,

$$x \succsim y \;\Leftrightarrow\; u(x) \geq u(y).$$

A utility function representing a preference relation $\succsim$ is not unique.

Exercise: If u represents $\succsim$ on X, then for any strictly increasing function $f : \mathbb{R} \to \mathbb{R}, v(x) = f(u(x))$ is also a utility function that represents $\succsim$.

Note: v is then said to be a strictly increasing transformation of u.

Properties of utility function that are invariant for all strictly increasing transformations are known as *ordinal* properties.

Exercise: If u represents $\succsim$, then for all $x, y \in X$,

$$x \succ y \;\Leftrightarrow\; u(x) > u(y)$$

and

$$x \sim y \;\Leftrightarrow\; u(x) = u(y).$$

Proposition (1.B.2): *A preference relation $\succsim$ can be represented by a utility function only if it is rational.*

Proof: Suppose there exists a utility function u that represents $\succsim$ on X.

For any pair $x, y \in X$, either

$$u(x) \geq u(y) \text{ or } u(y) \geq u(x) \text{ (or both)}$$

and since u that represents $\succsim$, this must imply:

$$x \succsim y \text{ or } y \succsim x \text{ (or both)}$$

which shows that $\succsim$ is complete.

Next, we show $\succsim$ is transitive. Suppose $x \succsim y$ and $y \succsim z$. We need to show that $x \succsim z$.

As u that represents $\succsim$, this must imply:

$$u(x) \geq u(y) \text{ and } u(y) \geq u(z)$$

so that

$$u(x) \geq u(z)$$

which implies (again because u that represents $\succsim$)

$$x \succsim z.$$

Thus, $\succsim$ is transitive. The proof is complete.

Not every rational preference relation can be represented by a utility function.

Example: Lexicographic preferences (Example 3.C.1 in textbook)

$X = \mathbb{R}^2_+$. For any pair of alternatives $x, y \in X$, where $x = (x_1, x_2), y = (y_1, y_2)$,

$$x \succsim y \text{ if either } x_1 > y_1, \text{or } x_1 = y_1 \text{ and } x_2 \geq y_2.$$

Verify that this particular preference is rational.

Suppose that there is a utility function u that represents this preference.

For every non-negative real number z we can look at the real numbers $u(z, 1)$ and $u(z, 2)$.

Now, from the definition of lexicographic preferences, observe that $(z, 2) \succsim (z, 1)$ but not $(z, 1) \succsim (z, 2)$ which implies $(z, 2) \succ (z, 1)$.

Hence (see exercise above):

$$u(z, 2) > u(z, 1).$$

Between any distinct real numbers there is a rational number. So, for every non-negative real number z there exists a rational number $r(z)$ such that

$$u(z, 2) > r(z) > u(z, 1).$$

Further, from the definition of lexicographic ordering if $z > z'$, then

$$(z, 2) \succ (z, 1) \succ (z', 2) \succ (z', 1)$$

which implies

$$u(z, 2) > r(z) > u(z, 1) > u(z', 2) > r(z') > u(z', 1).$$

Thus, for every non-negative real number z there exists a *distinct* rational number $r(z)$.

But there are uncountable positive real numbers but only countably many rational numbers. A contradiction.

Exercise: If X is finite, then every rational preference relation can be represented by a utility function.

Consumer Behavior and Walrasian Demand.

L commodities, $l = 1, 2, ...L$.

L finite.

A commodity vector (bundle) lists amounts of the different commodities : $x = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_L \end{bmatrix}$

x_l : consumption level of good l

Consumption Set $X \subset \mathbb{R}^L$: Set of all consumption bundles that the consumer can conceivably consume given the physical constraints.

Assume: $X = \mathbb{R}^L_+$

(Any non-negative bundle may be consumed)

Consumer's preference relation $\succsim$ defined on $X = \mathbb{R}^L_+$.

Notation:

For vectors $x = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_L \end{bmatrix}, y = \begin{bmatrix} y_1 \\ y_2 \\ . \\ . \\ y_L \end{bmatrix}$,

$$y \gg x \Longleftrightarrow y_l > x_l, \text{ for all } l = 1, ...L,$$

$$y \geq x \Longleftrightarrow y_l \geq x_l, \text{ for all } l = 1, ...L,$$

Note,

$y \geq x, y \neq x \Longleftrightarrow y_l \geq x_l,$ for all l <u>and</u> $y_l > x_l,$ for some l.

Euclidean norm: For $x \in \mathbb{R}^L$, $\|x\| = \sqrt{\sum_{l=1}^{L}(x_l)^2}$

Euclidean distance: For $x, y \in \mathbb{R}^L$, the (Euclidean) distance between vectors x and y is given by $\|x - y\| = \sqrt{\sum_{l=1}^{L}(x_l - y_l)^2}$

Some common assumptions on $\succsim$ used in classical demand theory:

Desirability assumptions:

Definition: The preference relation $\succsim$ is *monotone if* $x \in X$ and $y \gg x$ implies $y \succ x$.

Definition: The preference relation $\succsim$ is *strongly monotone if* $x \in X$ and $y \geq x, y \neq x$ implies $y \succ x$.

(These imply goods are "good"; no "bads").

Definition: The preference relation $\succsim$ satisfies *local non-satiation* if for every $x \in X$ and every $\epsilon > 0$, there is $y \in X$ such that $\| y - x \| < \epsilon$ and $y \succ x$.

(Implies that arbitrarily close to any consumption bundle, there is a better bundle).

Exercise (Exercise 3.B.1) : Strong monotone $\Rightarrow$ Monotone $\Rightarrow$ Local Non-satiation

A digression:

For any $x \in X$,the set $\{y \in X : y \succsim x\}$ is called the *upper contour set of* x; it is the set of all bundles that are at least as good as x.

For any $x \in X$,the set $\{y \in X : x \succsim y\}$ is called the *lower contour set of* x.

For any $x \in X$,the set $\{y \in X : x \sim y\}$ is called the *indifference set* x.

Local non-satiation rules out "thick" indifference sets.

Convexity:

A set $S \subset \mathbb{R}^L$ is convex if for all $x, y \in S$ and $\alpha \in [0, 1]$,

$$\alpha x + (1 - \alpha)y \in S.$$

(the entire line segment connecting x and y is contained in the set S).

Beware: Convexity of a set is a very different concept from convexity of a function.

Note: $X = \mathbb{R}^L_+$ is a convex set.

Convexity assumptions:

Definition: The preference relation $\succsim$ is *convex* if for every $x \in X$, the upper contour set of x is convex i.e., for all $y, z \in X$ such that $y \succsim x$ and $z \succsim x$, and for all $\alpha \in [0, 1]$,

$$\alpha y + (1 - \alpha) z \succsim x.$$

Definition: The preference relation $\succsim$ is *strictly convex* if for every $x \in X$, and for all $y, z \in X$ such that $y \succsim x$, $z \succsim x, y \neq z$, and for all $\alpha \in (0, 1)$,

$$\alpha y + (1 - \alpha) z \succ x.$$

Convexity captures a taste for diversification. Implies diminishing marginal rates of substitution between any pair of goods.

For $L = 2$, implies that indifference curves are convex to the origin.

Continuity assumption:

Definition: The preference relation $\succsim$ is *continuous* if for any sequence of pairs of commodity bundles $\{(x^n, y^n)\}_{n=1}^{\infty}$, $x^n \in X$, $y^n \in X$ for all n,with $x^n \succsim y^n$ for all n, $x = \lim_{n\to\infty} x^n, y = \lim_{n\to\infty} y^n$, we have $x \succsim y$.

(No reversal of preferences in the limit).

Digression: A set $S \subset \mathbb{R}^L$ is said to be closed if the limit of every sequence of vectors in S is contained in S.

Equivalent definition of continuity: For all $x \in X$, the upper and lower contour sets are closed.

Example (Example 3.C.1): Lexicographic preferences are not continuous.

$x^n = (\frac{1}{n}, 0) \succ y^n = (0, 1)$ for all $n \geq 1$, $x^n \rightarrow (0, 0), y^n \rightarrow (0, 1)$ but $(0, 1) \succ (0, 0)$.

Proposition 3.C.1: Suppose that the rational preference relation is $\succsim$ on X is continuous. Then there is a continuous utility function that represents $\succsim$.

(Does not mean that all utility functions representing the preference relation are continuous)

"Proof": Assume in addition that $\succsim$ is monotone.

Let $e \in \mathbb{R}^L_+$ be the vector $(1, 1, 1, ...1)$.

For every $x \in \mathbb{R}^L_+$, monotone property implies

$$x \succsim 0$$

and for any positive number $\overline{\alpha}$ such that $\overline{\alpha}e \gg x$,

$$\overline{\alpha}e \succsim x.$$

The sets $A^+ = \{\alpha \in \mathbb{R}_+ : \alpha e \succsim x\}$ and $A^- = \{\alpha \in \mathbb{R}_+ : x \succsim \alpha e\}$ are nonempty and closed (why?).

Further, as $\succsim$ is complete, for every $\alpha \in \mathbb{R}_+$, either $\alpha e \succsim x$ or $x \succsim \alpha e$ so that $\alpha \in A^+ \cup A^-$.

Thus, $A^+ \cup A^- = \mathbb{R}_+$.

One mathematical property of $\mathbb{R}_+$ is that it is a connected set which means it cannot be the union of two disjoint, nonempty, closed sets.

So, $A^+ \cap A^- \neq \phi$.

So there exists $\alpha \geq 0$ such that $\alpha e \succsim x$ and $x \succsim \alpha e$ i.e., $\alpha e \sim x$.

By monotonicity, $\alpha_1 e \succ \alpha_2 e$ if $\alpha_1 > \alpha_2$.

Therefore, there can be only one real number $\alpha(x)$ such that $\alpha(x)e \sim x$.

Set $u(x) = \alpha(x)$ for every $x \in X$.

Suppose $u(x) \geq u(y)$.Then, $\alpha(x) \geq \alpha(y)$ so that using monotonicity of preferences $\alpha(x)e \succsim \alpha(y)e$. By construction, $\alpha(x)e \sim x, \alpha(y)e \sim y$ so that using transitivity, $x \succsim y$.

Conversely, suppose $x \succsim y$. Then, by construction, $\alpha(x)e \sim x, \alpha(y)e \sim y$ and so using transitivity $\alpha(x)e \succsim \alpha(y)e$. Using monotonicity of preferences, $\alpha(x) \geq \alpha(y)$ i.e., $u(x) \geq u(y)$.

Thus, $u(x) \geq u(y) \iff x \succsim y$.

So, u represents $\succsim$.

Continuity of u requires a bit more involved argument.

Assumptions on preferences imply certain properties of ALL utility functions that represent them (they are ordinal properties).

Monotone $\succsim \Rightarrow u$ is increasing i.e., $u(x) > u(y)$ if $x \gg y$.

Exercise: Strong monotone $\succsim \Rightarrow$?

Let $S \subset \mathbb{R}^L$ be a convex set.

A function $f : S \to \mathbb{R}$ is said to be quasiconcave if for all $x \in S$,the set $\{y \in S : f(y) \geq f(x)\}$ is a convex set; or alternatively, for all $x, y \in S, \alpha \in [0, 1]$,

$$f(\alpha x + (1 - \alpha)y) \geq \min\{f(x), f(y)\}.$$

A function $f : S \to \mathbb{R}$ is said to be strictly quasiconcave if for all $x, y \in S, x \neq y, \alpha \in (0, 1)$,

$$f(\alpha x + (1 - \alpha)y) > \min\{f(x), f(y)\}.$$

$\succsim$ convex $\Rightarrow u$ is quasi-concave.

$\succsim$ strictly convex $\Rightarrow u$ is strictly quasi-concave.

Exercise: quasi-concavity is an ordinal property (invariant to any strictly increasing transformation of u).

Let $S \subset \mathbb{R}^L$ be a convex set.

A function $f : S \to \mathbb{R}$ is said to be concave if for all $x, y \in S, \alpha \in [0, 1]$,

$$f(\alpha x + (1 - \alpha)y) \geq \alpha f(x) + (1 - \alpha)f(y).$$

Concavity $\Rightarrow$ Quasi-concavity

A function $f : S \to \mathbb{R}$ is said to be strictly concave if for all $x, y \in S, x \neq y, \alpha \in (0, 1)$,

$$f(\alpha x + (1 - \alpha)y) > \alpha f(x) + (1 - \alpha)f(y).$$

Strict concavity $\Rightarrow$ Strict Quasi-concavity

Concavity of the utility function is not an ordinal property. Not invariant to strictly increasing transformations.

It is not based on any property of the underlying preference structure.

Law of diminishing marginal utility (implied by concavity, but not necessarily by quasi-concavity) is not an ordinal property.

The Utility Maximization Problem

Assume: $\succsim$ is a rational, continuous and locally non-satiated preference relation represented by a continuous utility function $u(x)$ on $X = \mathbb{R}^L_+$.

We also assume:

(i) the L commodities are all traded in the market at dollar prices that are publicly quoted (complete markets assumption). In particular, price vector

$$p = \begin{bmatrix} p_1 \\ p_2 \\ . \\ . \\ p_L \end{bmatrix} \gg 0$$

(ii) consumers are price taking.

Affordability of a commodity bundle depends on price vector $p \in \mathbb{R}^L_{++}$ and wealth $w \in \mathbb{R}_{++}$.

Definition: The Walrasian, or competitive budget set $B_{p,w} = \{x \in X : p \bullet x \leq w\}$ is the set of all feasible consumption bundles for the consumer who faces market prices p and has wealth w.

The upper boundary of the budget set $\{x \in X : p \bullet x = w\}$ is called the budget hyperplane (or budget line for the case $L = 2$).

Digression:

A set $S \subset \mathbb{R}^L$ is said to be *bounded* if there exists $N > 0$ such $\|x\| < N$ for all $x \in S$.

A set $S \subset \mathbb{R}^L$ is *compact* if it is both closed and bounded.

Exercise: $B_{p,w}$ is a compact and convex set.

Utility Maximization Problem (UMP):

Given $p \gg 0$ and $w > 0$,

$$\begin{aligned} &\max_{x \in X} u(x) \\ \text{subject to } \; p \bullet x \; &\leq \; w. \end{aligned}$$

or equivalently,

$$\max_{x \in B_{p,w}} u(x).$$

Weirstrass' Theorem: Let $f : K \to \mathbb{R}$ be a continuous function and K is a compact set. Then, there exists $x', x'' \in K$ such that

$$f(x') \leq f(x) \leq f(x'') \text{ for all } x \in K.$$

In other words, f attains a maximum and a minimum in the set K.

Proposition 3.D.1: If $p \gg 0$ and $w > 0$, then UMP has a solution.

The proof follows from continuity of u and compactness of $B_{p,w}$.

Note: Solution need not be unique.

Let $x(p,w) = \{x' \in B_{p,w} : u(x') \geq u(x)$ for all $x \in B_{p,w}\}$

In other words, $x(p,w)$ is the set of all solutions to the UMP.

We can view $x(p,w)$ a set valued mapping or a correspondence that associates a set of optimal consumption bundles with each (p,w). We call this the Walrasian demand correspondence.

If $x(p,w)$ is single valued for each (p,w) i.e., there is a unique solution to UMP for every (p,w), then $x(p,w)$ is a function often called the Walrasian demand function.

Proposition 3.D.2: Suppose that $u(.)$ is a continuous utility function representing a locally non-satiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. Then the Walrasian demand correspondence $x(p, w)$ has the following properties:

(i) Homogeneity of degree zero in (p, w) : $x(\alpha p, \alpha w) = x(p, w)$ for any $p \gg 0, w > 0$ and scalar $\alpha > 0$.

(ii) Walras' law: $p \bullet x = w$ for all $x \in x(p, w)$.

(iii) Convexity/uniqueness: If $\succsim$ is convex so that u is quasiconcave, then $x(p, w)$ is a convex set for every $(p, w) \gg 0$. If $\succsim$ is strictly convex so that u is strictly quasiconcave, then $x(p, w)$ consists of single point.

Proof.

(i) Homogeneity of degree zero in (p, w) : $x(\alpha p, \alpha w) = x(p, w)$ for any $p \gg 0, w > 0$ and scalar $\alpha > 0$.

This follows from

$$\{x \in X : p \bullet x \leq w\} = \{x \in X : \alpha p \bullet x \leq \alpha w\}$$

and therefore, the UMP

$$\begin{aligned} & \max_{x \in X} u(x) \\ \text{subject to } \; p \bullet x \;\; \leq \;\; & w. \end{aligned}$$

is equivalent to

$$\begin{aligned} & \max_{x \in X} u(x) \\ \text{subject to } \; \alpha p \bullet x \;\; \leq \;\; & \alpha w. \end{aligned}$$

so that the set of solutions to the two max problems are identical.

(ii) Walras' law: $p \bullet x = w$ for all $x \in x(p, w)$

For any bundle x such that $p \bullet x < w$, there exists $\epsilon > 0$ small enough such that $p \bullet x' < w$ for all $x' \in X$ such that $\|x' - x\| < \epsilon$.

Using local non-satiation, there exists x' such that $\|x' - x\| < \epsilon$ and $x' \succ x$ and as that x' is in the budget set, it contradicts the optimality of x.

(iii)i) Convexity/uniqueness: If $\succsim$ is convex so that u is quasiconcave, then $x(p,w)$ is a convex set for every $(p,w) \gg 0$.

If $x(p,w)$ has only one element, then it is trivially a convex set. So suppose that there are at least two distinct elements $x \neq x'$ in $x(p,w)$ i.e., both x, x' solve UMP.

Let

$$u(x) = u(x') = u^*.$$

For any $\alpha \in [0,1]$,consider the bundle $x'' = \alpha x + (1-\alpha)x'$.

As $B_{p,w}$ is a convex set, $x'' \in B_{p,w}$.

As u is quasiconcave,

$$u(x'') = u(\alpha x + (1-\alpha)x') \geq \min\{u(x), u(x')\} = u^*$$

so that x'' must also be optimal i.e., $x'' \in x(p,w)$. Thus, $x(p,w)$ is a convex set.

Next we show that if $\succsim$ is strictly convex so that u is strictly quasiconcave, then $x(p, w)$ consists of single point.

Suppose to the contrary that there are at least two distinct solutions to UMP $x, x' \in x(p, w), x \neq x'$.

Let

$$u(x) = u(x') = u^*.$$

Consider the bundle $x'' = \frac{1}{2}x + \frac{1}{2}x'$.

As $B_{p,w}$ is a convex set, $x'' \in B_{p,w}$.

As u is strictly quasiconcave,

$$u(x'') = u(\frac{1}{2}x + \frac{1}{2}x') > \min\{u(x), u(x')\} = u^*$$

which contradicts the fact that x solves UMP and u^* is the maximum utility attainable on $B_{p,w}$.

Applications:

Suppose $x(p, w)$ is a differentiable function.

Homogeneity of degree zero of the function $x_l(p, w)$ implies:

$$\sum_{k=1}^{L} \frac{\partial x_l}{\partial p_k} p_k + \frac{\partial x_l}{\partial w} w = 0 \text{ for all } l = 1, ...L.$$

If $x_l(p, w) > 0$,dividing through by x_l :

$$\sum_{k=1}^{L} \frac{\partial x_l}{\partial p_k} \frac{p_k}{x_l} + \frac{\partial x_l}{\partial w} \frac{w}{x_l} = 0$$

so that:

$$\sum_{k=1}^{L} \varepsilon_{lk}(p, w) + \varepsilon_{lw}(p, w) = 0$$

where $\varepsilon_{lk}(p, w)$ is the (cross) price elasticity of demand for good l with respect to price of good $k, \varepsilon_{lw}(p, w)$ is the wealth (or income) elasticity of demand for good l. An equi-proportionate change in all prices and wealth has no net effect on demand.

Walras Law:

$$p \cdot x(p,w) = w \text{ for all } p \gg 0, w > 0$$

gives us an identity in p, w.Differentiating through with respect to p_k we have

$$\sum_{l=1}^{L} \frac{\partial x_l(p,w)}{\partial p_k} p_l + x_k(p,w) = 0 \text{ for all } k = 1,...L.$$

If $x(p,w) \gg 0$, multiplying through by p_k and dividing through by w

$$\sum_{l=1}^{L} \frac{\partial x_l}{\partial p_k} \frac{p_k}{x_l} \frac{p_l x_l}{w} + \frac{p_k x_k}{w} = 0 \text{ for all } k = 1,...L.$$

so that

$$\sum_{l=1}^{L} \varepsilon_{lk}(p,w) b_l(p,w) + b_k(p,w) = 0 \text{ for all } k = 1,...L.$$

where $b_l(p,w) = \frac{p_l x_l}{w}$ is the share of expenditure (or budget share) on good l.

$$p \cdot x(p, w) = w \text{ for all } p \gg 0, w > 0$$

Differentiating through with respect to w we have

$$\sum_{l=1}^{L} \frac{\partial x_l(p, w)}{\partial w} p_l = 1.$$

If $x(p, w) \gg 0$,

$$\sum_{l=1}^{L} \frac{\partial x_l}{\partial w} \frac{w}{x_l} \frac{p_l x_l}{w} = 1$$

so that

$$\sum_{l=1}^{L} \varepsilon_{lw}(p, w) b_l(p, w) = 1.$$

Continuity of Demand:

In general, there may not be any continuous demand function that one can select from the correspondence $x(p, w)$. Even though utility is continuous.

Example:$L = 2$.

$$u(x_1, x_2) = x_1 + x_2$$

$$\begin{aligned} x(p, w) &= (\frac{w}{p_1}, 0), \text{ if } p_1 < p_2 \\ &= \{x : px = w\} \text{ if } p_1 = p_2 \\ &= (0, \frac{w}{p_2}), \text{ if } p_1 > p_2. \end{aligned}$$

Maximum Theorem $\Rightarrow$ If there is a unique solution to UMP for every (p, w), then $x(p, w)$ is continuous at every $(p, w) \gg 0$.

Thus, strict quasi-concavity of u ensures continuity of demand function.

Assume: u is continuously differentiable on X

Kuhn-Tucker necessary condition: If $x^* \in x(p, w)$, then there exists a (scalar) multiplier $\lambda \geq 0$ such that for all $l = 1, ...L$,

$$\frac{\partial u(x^*)}{\partial x_l} \leq \lambda p_l$$

and

$$\frac{\partial u(x^*)}{\partial x_l} = \lambda p_l \text{ if } x_l^* > 0.$$

These are the first order conditions.

If

$$\nabla u(x) = [\frac{\partial u(x)}{\partial x_1}, ..., \frac{\partial u(x)}{\partial x_L}]$$

then the above conditions can be written as

$$\nabla u(x^*) \leq \lambda p$$

and

$$x^* \cdot [\nabla u(x^*) - \lambda p] = 0.$$

If $\nabla u(x^*) \geq 0$, $\nabla u(x^*) \neq 0$ and $x^* \gg 0$, then $\lambda > 0$ (why?) and for any two goods l, k

$$\frac{\frac{\partial u(x^*)}{\partial x_l}}{\frac{\partial u(x^*)}{\partial x_k}} = \frac{p_l}{p_k}.$$

The left hand side the marginal rate of substitution (MRS) between goods l and k.

Note MRS need not equal price ratio if we have corner solution.

Interpretation of multiplier λ :

Suppose $x(p,w)$ is a differentiable function and $x(p,w) \gg 0$.

Then, the maximum utility is $u(x(p,w))$.

$$\begin{aligned}\frac{\partial u(x(p,w))}{\partial w} &= \sum_{l=1}^{L} \frac{\partial u(x(p,w))}{\partial x_l} \frac{\partial x_l}{\partial w} \\ &= \lambda \sum_{l=1}^{L} p_l \frac{\partial x_l}{\partial w} \\ &= \lambda,\end{aligned}$$

using an implication of Walras' Law. Thus, λ is the marginal value of wealth. This result holds much more generally (do not need $x(p,w)$ to be a differentiable or even continuous function); all one needs is that the maximum utility $u(x(p,w))$ should be differentiable in wealth.

Sufficiency of first order conditions.

Suppose $x^* \geq 0$ satisfies the first order conditions

$$\frac{\partial u(x^*)}{\partial x_l} \leq \lambda p_l$$

and

$$\frac{\partial u(x^*)}{\partial x_l} = \lambda p_l \text{ if } x_l^* > 0.$$

for some $\lambda \geq 0$. Further,

$$px^* = w.$$

Under what conditions is x^* optimal (i.e., $x^* \in x(p, w)$)?

Answer: if u is quasi-concave, monotone and $\nabla u(x) \neq 0$ for all $x \in \mathbb{R}^L_+$.

Concept of Open Set

A set $S \subset \mathbb{R}^L$ is open if its complement is closed.

A set $S \subset \mathbb{R}^L$ is open if for every $x \in S$ there exists $\epsilon > 0$ such that $\{y : \|x - y\| < \epsilon\} \subset S$.

$\mathbb{R}^L$ and $\mathbb{R}^L_{++}$ are open sets (in $\mathbb{R}^L$).

A set may be neither open nor closed.

A set may be both open and closed - for instance, ϕ and $\mathbb{R}^L$.

Verifying quasi-concavity: a useful result.

An interesting characterization of twice continuously differentiable quasi-concave functions can be given in terms of the "bordered" Hessian matrix associated with the functions.

Let A be an open subset of $\mathbb{R}^n$, and $f : A \rightarrow \mathbb{R}$ be a twice continuously differentiable function on A.

The *bordered Hessian matrix of* f at $x \in A$ is denoted by $G_f(x)$ and is defined as the following $(n+1) \times (n+1)$ matrix

$$G_f(x) = \begin{bmatrix} 0 & \nabla f(x) \\ \nabla f(x) & H_f(x) \end{bmatrix}$$

where $H_f(x)$ is the Hessian matrix of second order partial derivatives whose (i, j)-th element is $\frac{\partial^2 f}{\partial x_i \partial x_j}$.

We denote the $(k+1)th$ leading principal minor of $G_f(x)$ by $\left| G_f(x; k) \right|$, where $k = 1, ..., n$. The $(k+1)th$ leading principal minor is the determinant of the matrix obtained after deleting all but the first $k+1$ rows and $k+1$ columns of the matrix.

Result: Let A be an open convex set in $\mathbb{R}^n$, and $f : A \longrightarrow \mathbb{R}$ be a twice continuously differentiable function on A. If

$$(-1)^k \left| G_f(x;k) \right| > 0$$

for $x \in A$, and $k = 1, ..., n$, then f is strictly quasi-concave on A

Result: If $h : \mathbb{R}^n_+ \longrightarrow \mathbb{R}$ is continuous on $\mathbb{R}^n_+$ and quasi-concave on $\mathbb{R}^n_{++}$, then it is quasi-concave on $\mathbb{R}^n_+$.

Indirect Utility Function: value of the utility maximization problem i.e., the maximum utility as a function of prices and wealth.

$$\begin{aligned} v(p, w) &= \max_{x \in B_{p,w}} u(x) \\ &= u(x^*) \text{ where } x^* \in x(p, w). \end{aligned}$$

Proposition 3.D.3: Suppose that $u(.)$ is a continuous utility function representing a locally non-satiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. Then the indirect utility function $v(p, w)$ has the following properties:

(i) Homogeneity of degree zero in (p, w).

(ii) Strictly increasing in w and nonincreasing in p_l for any l.

(iii) Quasiconvex: the set $\{(p, w) : v(p, w) \leq \overline{v}\}$ is convex for any $\overline{v}$.

(iv) Continuous in p, w.

Note: indirect utility function depends on the specific utility function chosen to represent the preferences.

The Expenditure Minimization Problem (EMP)

For $p \gg 0$ and $u > u(0)$,

$$\begin{aligned} & \min_{x \in \mathbb{R}^L_+} p \cdot x \\ s.t.\ u(x) & \geq u. \end{aligned}$$

Here we continue to assume $u(.)$ is a continuous utility function representing a locally nonsatiated preference relation on $\mathbb{R}^L_+$.

Existence of solution to EMP: all we need is to ensure that there exists x such that $u(x) \geq u$.

Why?

The Expenditure Function: Given prices $p \gg 0$ and required utility level $u > u(0)$, the expenditure function is given by

$$\begin{aligned} e(p,u) = & \min_{x \in \mathbb{R}^L_+} p \cdot x \\ s.t.\ u(x) \geq & \ u. \end{aligned}$$

Proposition 3.E.2: Suppose that $u(.)$ is a continuous utility function representing a locally nonsatiated preference relation on $X = \mathbb{R}^L_+$. The expenditure function $e(p, u)$ is:

(i) Homogenous of degree one in p.

(ii) Strictly increasing in u and nondecreasing in p_l for any l

(iii) Concave in p

(iv) Continuous in p and u.

Set of optimal solutions to EMP: $h(p, u)$

Called the Hicksian or compensated demand correspondence (or function if $h(p, u)$ is single valued).

Proposition 3.E.3: Suppose that $u(.)$ is a continuous utility function representing a locally nonsatiated preference relation on $X = \mathbb{R}^L_+$. Then for any $p \gg 0, u > u(0)$, the Hicksian demand correspondence $h(p, u)$ satisfies:

(i) Homogenous of degree zero in p : $h(\alpha p, u) = h(p, u)$ for any $\alpha > 0$

(ii) For any $x \in h(p, u), u(x) = u$.

(iii) Convexity/uniqueness: If $\succsim$ is convex so that u is quasiconcave, then $h(p, u)$ is a convex set. If $\succsim$ is strictly convex so that u is strictly quasiconcave, then $h(p, u)$ consists of single point.

Suppose that $u(.)$ is continuously differentiable.

Kuhn-Tucker first order necessary conditions: : If $x^* \in h(p, u)$, then there exists a (scalar) multiplier $\lambda \geq 0$ such that for all $l = 1, ...L$,

$$p_l \geq \lambda \frac{\partial u(x^*)}{\partial x_l}$$

and

$$p_l = \lambda \frac{\partial u(x^*)}{\partial x_l} \text{ if } x_l^* > 0.$$

Also,

$$u(x^*) = u.$$

The first set of conditions can be written as

$$p \geq \lambda \nabla u(x^*)$$

and

$$x^* \cdot [p - \lambda \nabla u(x^*)] = 0.$$

If $h(p, u)$ is single valued everywhere, then it is a continuous function.

First order conditions are sufficient for optimality in the EMP if u is quasi-concave.

Proposition 3.E.1: Suppose that $u(.)$ is a continuous utility function representing a locally nonsatiated preference relation on $X = \mathbb{R}^L_+$ and that the price vector $p \gg 0$. Then

(i) If x^* solves UMP when wealth is $w > 0$, then x^* solves EMP when the required utility level is $u(x^*)$. Moreover, the minimum expenditure in the latter EMP is exactly w.

(ii) If x^* solves EMP when the required utility level is $u > u(0)$, then x^* solves UMP when wealth is $p \cdot x^*$. Moreover, the maximum utility in the latter UMP is exactly u.

Proof. (i) To show: If x^* solves UMP when wealth is $w > 0$, then x^* solves EMP when the required utility level is $u(x^*)$.

Suppose to the contrary that x^* does not solve EMP when the required utility level is $u(x^*)$.

Then there exists x' such that $p \cdot x' < p \cdot x^*$ and $u(x') \geq u(x^*)$.

As x^* solves UMP, $p \cdot x^* = w$.

Thus, $p \cdot x' < w$.

By local nonsatiation, there exists $x'' \in B_{p,w}$ such that $u(x'') > u(x^*)$ which contradicts the optimality of x^* in the UMP.

Thus, x^* solves EMP when the required utility level is $u(x^*)$ and the minimum expenditure is $p \cdot x^* = w$.

(ii) To show: If x^* solves EMP when the required utility level is $u > u(0)$, then x^* solves UMP when wealth is $p \cdot x^*$. Moreover, the maximum utility in the latter UMP is exactly u.

Since x^* solves EMP given $u > u(0)$, $x^* \neq 0$.

Hence, $p \cdot x^* > 0$.

Suppose x^* does not solve UMP when wealth is $p \cdot x^*$.

Then, there exists x' such $p \cdot x' \leq p \cdot x^*$ and $u(x') > u(x^*)$.

By continuity, there exists $x'' = \alpha x'$ where $\alpha \in (0, 1)$ such that $p \cdot x'' < p \cdot x^*$ and $u(x'') > u(x^*)$.

This contradicts optimality of x^* in EMP.

Thus, x^* solves UMP when wealth is $p \cdot x^*$ and the maximized utility level is therefore $u(x^*) = u$.

Implication of Proposition 3.E.1: Identities.

For any $p > 0, w > 0, u > u(0)$

$$\begin{aligned} e(p, v(p, w)) &= w \\ v(p, e(p, u)) &= u \end{aligned}$$

For a fixed price vector $p = \overline{p}$,

$$\begin{aligned} e(\overline{p}, u) &= v^{-1}(\overline{p}, u) \\ v(\overline{p}, w) &= e^{-1}(\overline{p}, w) \end{aligned}$$

Two more identities: for any $p > 0, w > 0, u > u(0)$

$$\begin{aligned} h(p, u) &= x(p, e(p, u)) \\ x(p, w) &= h(p, v(p, w)) \end{aligned}$$

Note $x(p, e(p, u))$ is the Walrasian demand if for every price vector, wealth is adjusted to a level that allows the agent to just attain utility u.

This is the Hicksian compensation use to decompose effect of price change into substitution and income effect.

The first identity therefore explains why $h(p, u)$ is called the compensated demand.

Compensated Law of Demand

Modified Proposition 3.E.4: Suppose that $u(.)$ is a continuous utility function representing a locally nonsatiated preference relation on $X = \mathbb{R}^L_+$. Then, for all $p', p'' \gg 0, u > u(0), x' \in h(p', u), x'' \in h(p'', u)$, the following holds

$$(p' - p'')(x' - x'') \leq 0.$$

Proof: As $x' \in h(p', u), x'' \in h(p'', u)$

$$\begin{aligned} p'x' &\leq p'x'' \\ p''x'' &\leq p''x' \end{aligned}$$

and subtracting the second from the first inequality yields the result.

Implication of the proposition: If $p'_k = p''_k$ for all $k \neq l$,

$$\begin{aligned} & (p' - p'')(x' - x'') \\ = \; & (p'_l - p''_l)(x'_l - x''_l) \end{aligned}$$

so that we have

$$\begin{aligned} p'_l &> p''_l \Rightarrow x'_l \leq x''_l \\ p'_l &< p''_l \Rightarrow x'_l \geq x''_l \end{aligned}$$

i.e., other things being equal, compensated demand for any commodity is non-increasing in own price.

Restated as: "substitution effect of price increase is negative"

Note: Law of demand does not apply to Walrasian demand. Income effect can overtake substitution effect.

Some Useful Relationships between Demand, Indirect Utility and Expenditure Functions

Continue to assume: $u(.)$ is a continuous utility function representing a locally nonsatiated preference relation on $X = \mathbb{R}^L_+$.

Also, assume u is strictly quasi-concave.

Thus, $x(p, w)$ and $h(p, u)$ are single valued at each $p \gg 0, w > 0, u > u(0)$.

$x(p, w)$ and $h(p, u)$ are functions.

The maximum theorem can be used to show that the functions $v(p, w), x(p, w), e(p, u)$ and $h(p, u)$ are continuos at every $p \gg 0, w > 0, u > u(0)$.

Further, it can be shown that $e(p, u)$ is differentiable at every $p \gg 0, u > u(0)$.

Proposition 3.G.1: At every $p \gg 0, u > u(0)$,

$$h_l(p, u) = \frac{\partial e(p, u)}{\partial p_l}, l = 1, .., L.$$

"Proof": Easy to show under additional restriction that the utility function u is differentiable, $h(p, u)$ is differentiable in p and $h(p, u) \gg 0$.

First order necessary condition for EMP: for some $\lambda \geq 0$ (check λ is in fact > 0)

$$p_k = \lambda \frac{\partial u(h(p, u))}{\partial x_k}, k = 1, ...L.$$

Now,

$$u(h(p, u)) = u$$

As the latter is identity that holds for all $p \gg 0, u > u(0)$, we have by differentiating with respect to p_l :

$$\sum_{k=1}^{L} \frac{\partial u(h(p, u))}{\partial x_k} \frac{\partial h_k(p, u)}{\partial p_l} = 0$$

and using the first order condition:

$$\begin{aligned} & \sum_{k=1}^{L} \frac{\partial u(h(p,u))}{\partial x_k} \frac{\partial h_k(p,u)}{\partial p_l} \\ = & \sum_{k=1}^{L} \frac{p_k}{\lambda} \frac{\partial h_k(p,u)}{\partial p_l} \end{aligned}$$

so that

$$\sum_{k=1}^{L} p_k \frac{\partial h_k(p,u)}{\partial p_l} = 0.$$

Further, identity:

$$e(p,u) = p \cdot h(p,u), \text{ for all } p \gg 0, u > u(0)$$

Differentiating with respect to p_l :

$$\begin{aligned} \frac{\partial e(p,u)}{\partial p_l} &= h_l(p,u) + \sum_{k=1}^{L} p_k \frac{\partial h_k(p,u)}{\partial p_l} \\ &= h_l(p,u). \end{aligned}$$

Can also be shown to be a direct consequence of the "envelope theorem".

Implication.

Suppose that $h(p,u)$ is continuously differentiable at (p,u).

Then, $e(p,u)$ is twice continuously differentiable in p and

$$\frac{\partial^2 e(p,u)}{\partial p_k \partial p_l} = \frac{\partial h_l(p,u)}{\partial p_k}.$$

Young's theorem:

$$\frac{\partial^2 e(p,u)}{\partial p_k \partial p_l} = \frac{\partial^2 e(p,u)}{\partial p_l \partial p_k}$$

(for a twice continuously differentiable function, the second order partial derivatives are independent of the order of differentiation).

For a twice differentiable concave function, the matrix of second order cross-partial derivatives (i.e., the Hessian matrix) is negative semi-definite matrix. Thus,

$$\begin{bmatrix} \frac{\partial^2 e}{\partial p_1^2} & \frac{\partial^2 e}{\partial p_1 \partial p_2} & & \frac{\partial^2 e}{\partial p_1 \partial p_L} \\ \frac{\partial^2 e}{\partial p_2 \partial p_1} & \frac{\partial^2 e}{\partial p_2^2} & & \frac{\partial^2 e}{\partial p_2 \partial p_L} \\ & & & \\ \frac{\partial^2 e}{\partial p_L \partial p_1} & & & \frac{\partial^2 e}{\partial p_L^2} \end{bmatrix}$$

is a symmetric negative semi-definite matrix. This implies, the (Jacobian) matrix of first order derivatives of h with respect to prices (i.e., the matrix of substitution effects)

$$\begin{bmatrix} \frac{\partial h_1}{\partial p_1} & \frac{\partial h_2}{\partial p_1} & & \frac{\partial h_L}{\partial p_1} \\ \frac{\partial h_1}{\partial p_2} & \frac{\partial h_2}{\partial p_2} & & \frac{\partial h_L}{\partial p_2} \\ & & & \\ \frac{\partial h_1}{\partial p_L} & & & \frac{\partial h_L}{\partial p_L} \end{bmatrix}$$

is a symmetric negative semi-definite matrix. This matrix is sometimes called the substitution matrix.

As the diagonal terms of any negative semi-definite matrix is non-positive,

$$\frac{\partial h_l}{\partial p_l} \leq 0, l = 1, ...L.$$

which the own price effect in the law of compensated demand.

$\frac{\partial h_l}{\partial p_k} \geq 0$: goods l, k are substitutes

$\frac{\partial h_l}{\partial p_k} \leq 0$:goods l, k are complements

As $h(p, u)$ is homogenous of degree zero in p, Euler's theorem:

$$\sum_{k=1}^{L} \frac{\partial h_l}{\partial p_k} p_k = 0, l = 1, ..., L.$$

As $\frac{\partial h_l}{\partial p_l} \leq 0, \frac{\partial h_l}{\partial p_k} \geq 0$ for some k (every good has at least one substitute).

Proposition 3.G.3: (Slutsky equation) For all $p, w \gg 0$ and $u = v(p, w)$

$$\frac{\partial h_l(p,u)}{\partial p_k} = \frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w} x_k(p,w) \text{ for all } l, k$$

Proof. Choose any $\overline{p}, \overline{w} \gg 0$.Fix Let $\overline{u} = v(\overline{p}, \overline{w})$. Then, $e(\overline{p}, \overline{u}) = \overline{w}$. Identity in p:

$$h_l(p, \overline{u}) = x_l(p, e(p, \overline{u}))$$

Differentiating through with respect to p_k and evaluating at $\overline{p}, \overline{u}$:

$$\begin{aligned}
\frac{\partial h_l(\overline{p},\overline{u})}{\partial p_k} &= \frac{\partial x_l(\overline{p}, e(\overline{p},\overline{u}))}{\partial p_k} + \frac{\partial x_l(\overline{p}, e(\overline{p},\overline{u}))}{\partial w} \frac{\partial e(\overline{p},\overline{u})}{\partial p_k} \\
&= \frac{\partial x_l(\overline{p},\overline{w})}{\partial p_k} + \frac{\partial x_l(\overline{p},\overline{w})}{\partial w} h_k(\overline{p},\overline{u}) \\
&= \frac{\partial x_l(\overline{p},\overline{w})}{\partial p_k} + \frac{\partial x_l(\overline{p},\overline{w})}{\partial w} x_k(\overline{p}, e(\overline{p},\overline{u})) \\
&= \frac{\partial x_l(\overline{p},\overline{w})}{\partial p_k} + \frac{\partial x_l(\overline{p},\overline{w})}{\partial w} x_k(\overline{p},\overline{w}).
\end{aligned}$$

Implication: Effect of change of own price on Walrasian demand

$$\begin{aligned} & \frac{\partial x_l(p,w)}{\partial p_l} \\ = & \frac{\partial h_l(p,u)}{\partial p_l} - \frac{\partial x_l(p,w)}{\partial w} x_l(p,w) \\ = & \text{substitution effect} + \text{income effect} \end{aligned}$$

For a normal good, $\frac{\partial x_l(p,w)}{\partial w} \geq 0$, so that income and substitution effects work in same direction (negative).

You can recover Walrasian demand function from the indirect utility function.

Proposition 3.G.4. Suppose that the indirect utility function $v(p, w)$ is differentiable at $p, w \gg 0$.

$$x_l(p, w) = -\frac{\partial v(p, w)/\partial p_l}{\partial v(p, w)/\partial w}.$$

Proof. Choose any $\overline{p}, \overline{w} \gg 0$.Fix Let $\overline{u} = v(\overline{p}, \overline{w})$. Then, $e(\overline{p}, \overline{u}) = \overline{w}$.Identity in p:

$$v(p, e(p, \overline{u})) = \overline{u}$$

Differentiating through with respect to p_l and evaluating at $p = \overline{p}$:

$$\frac{\partial v(\overline{p}, e(\overline{p}, \overline{u}))}{\partial p_l} + \frac{\partial v(\overline{p}, e(\overline{p}, \overline{u}))}{\partial w} \frac{\partial e(\overline{p}, \overline{u})}{\partial p_l} = 0$$

so that

$$\frac{\partial v(\overline{p}, \overline{w})}{\partial p_l} + \frac{\partial v(\overline{p}, \overline{w})}{\partial w} h_l(\overline{p}, \overline{u}) = 0$$

and as $h_l(p,u) = h_l(p, v(p,w)) = x_l(p,w)$, we have

$$\frac{\partial v(\overline{p}, \overline{w})}{\partial p_l} + \frac{\partial v(\overline{p}, \overline{w})}{\partial w} x_l(\overline{p}, \overline{w}) = 0$$

and this yields the result.

Welfare Evaluation of Economic Changes

Fix wealth $w > 0$.

Suppose that price vector changes from p^0 to p^1.

Consumer is better off as a result of this change if

$$\Delta v = v(p^1, w) - v(p^0, w) > 0$$

and worse off if the opposite is true. The specific amount of this change in welfare depends on the choice of utility function (indirect utility function depends on u).

How then do we measure this welfare change?

Use money metric (indirect) utility.

Choose any utility function u, derive the indirect utility function v. Now, choose any arbitrary price vector $\overline{p} \gg 0$.

Consider the function

$$e(\overline{p}, v(p, w)).$$

It is the (minimum) amount of money you need to spend at price vector $\overline{p}$ to attain same utility level as the maximum utility you can attain where price vector is p and wealth is w. As $e(\overline{p}, v(p, w))$ is strictly increasing in the second argument, it is nothing but a strictly increasing transformation of the indirect utility function. So, there is a utility function representing the same preference ordering for which $e(\overline{p}, v(p, w))$ is in fact the indirect utility (why?).

Also note that $e(\overline{p}, v(p, w))$ does not depend on which indirect utility v is chosen.

A dollar measure of the welfare change:

$$e(\overline{p}, v(p^1, w)) - e(\overline{p}, v(p^0, w))$$

This measure is independent of which indirect utility v is chosen.

If we choose $\overline{p} = p^0$, we obtain a measure of welfare change called Equivalent Variation (EV):

$$\begin{aligned} EV(p^0, p^1, w) &= e(p^0, v(p^1, w)) - e(p^0, v(p^0, w)) \\ &= e(p^0, v(p^1, w)) - w \end{aligned}$$

It is the dollar amount that the consume would be indifferent about accepting in lieu of the price change. Let

$$u^0 = v(p^0, w), u^0 = v(p^1, w),$$

Then,

$$EV(p^0, p^1, w) = e(p^0, u^1) - w$$

If we choose $\overline{p} = p^1$, we obtain a measure of welfare change called Compensating Variation (CV):

$$\begin{aligned} CV(p^0, p^1, w) &= e(p^1, v(p^1, w)) - e(p^1, v(p^0, w)) \\ &= w - e(p^1, u^0) \end{aligned}$$

It is the amount by which the agent must be compensated after a price change to make him as well off as before the price change.

Both CV and EV can be interpreted as area to the left of the Hicksian demand curves.

Suppose $p_1^0 \neq p_1^1$ and $p_l^0 = p_l^1 = \overline{p}_l$ for all $l \neq 1$.Let $\overline{p}_{-1} = (\overline{p}_2, ...\overline{p}_L)$.Then (assuming appropriate differentiability), as

$$h_1(p, u) = \frac{\partial e}{\partial p_1},$$

$$\begin{aligned} EV(p^0, p^1, w) &= e(p^0, u^1) - w \\ &= e(p^0, u^1) - e(p^1, u^1) \\ &= \int_{p_1^1}^{p_1^0} \frac{\partial e(p_1, \overline{p}_{-1}, u^1)}{\partial p_1} dp_1 \\ &= \int_{p_1^1}^{p_1^0} h_1(p_1, \overline{p}_{-1}, u^1) dp_1. \end{aligned}$$

Similarly,

$$CV(p^0, p^1, w) = \int_{p_1^1}^{p_1^0} h_1(p_1, \overline{p}_{-1}, u^0) dp_1$$

Suppose good 1 is a normal good (income effect is strictly positive). If $p_1^1 < p_1^0$. In that case, for all $p_1 \in (p_1^1, p_1^0\ )$,

$$h_1(p_1, \overline{p}_{-1}, u^1) > x_1(p_1, \overline{p}_{-1}, w) \ > h_1(p_1, \overline{p}_{-1}, u^0)$$

and

$$\begin{aligned} h_1(p_1^1, \overline{p}_{-1}, u^1) &= x_1(p_1^1, \overline{p}_{-1}, w) > h_1(p_1^1, \overline{p}_{-1}, u^0) \\ h_1(p_1^0, \overline{p}_{-1}, u^1) &> x_1(p_1^0, \overline{p}_{-1}, w) = h_1(p_1^0, \overline{p}_{-1}, u^0). \end{aligned}$$

To see these inequalities, suppose that the agent is initially facing prices vector p^1 and enjoying utility u^1.

If price of good 1 now increases to $p_1 > p_1^1$, then the quantity bought will decline to $h_1(p_1, \overline{p}_{-1}, u^1)$ due to substitution effect and then decline further to $x_1(p_1, \overline{p}_{-1}, w)$

due to income effect (as the good is a normal good) so that $h_1(p_1, \overline{p}_{-1}, u^1) > x_1(p_1, \overline{p}_{-1}, w)$.

Next, suppose that the agent is initially facing prices vector p^0 and enjoying utility u^0.

If price of good 1 now decreases to $p_1 < p_1^0$, then the quantity bought will increase to $h_1(p_1, \overline{p}_{-1}, u^0)$ due to substitution effect, and then increase further to $x_1(p_1, \overline{p}_{-1}, w)$ due to income effect (as the good is a normal good), so that $h_1(p_1, \overline{p}_{-1}, u^0) < x_1(p_1, \overline{p}_{-1}, w)$.

It follows therefore, that

$$\begin{aligned} & EV(p^0, p^1, w) \\ = & \int_{p_1^1}^{p_1^0} h_1(p_1, \overline{p}_{-1}, u^1) dp_1 \\ > & CV(p^0, p^1, w) \\ = & \int_{p_1^1}^{p_1^0} h_1(p_1, \overline{p}_{-1}, u^0) dp_1. \end{aligned}$$

It is easy to check that the same inequality holds if $p_1^1 > p_1^0$ (keep in mind that the integrals equal to EV and CV are now negative numbers).

If good 1 is an inferior good, $EV < CV$.

If there is no income effect, $EV = CV$; in particular, the Hicksian and Walrasian demands coincide (as a function of prices) and

$$EV(p^0, p^1, w) = \int_{p_1^1}^{p_1^0} x_1(p_1, \overline{p}_{-1}, w) dp_1 = CV(p^0, p^1, w)$$

i.e., as area to the left of the Walrasian demand curve for good 1 (Marshallian consumer surplus).

This is what happens when the utility function is quasi-linear.

If the utility function is one where income effect is not zero then using the area to the left of the Walrasian demand curve can still be an approximation of CV or EV provided price changes are very small.

Revealed Preference Approach to Law of Demand

(Samuelson)

Preferences are not observable, only choices are.

If choices made by individuals always satisfy some basic consistency axioms, then we can obtain certain patterns of economic behavior including the law of demand.

Instead of imposing structures on unobservable preferences, we need to think about what consistency requirements on choices can generate.

Assume that for each (p, w), the consumer chooses a unique bundle $x(p, w)$.[We say nothing about why the consumer makes that choice.]

We call $x(p, w)$ a demand function.

Further, assume $x(p, w)$ is homogenous of degree zero and satisfies Walras' law.

Weak Axiom of Revealed Preference (WARP): For any two price-wealth situations (p, w) and (p', w'), the following holds: if $p \cdot x(p', w') \leq w$ and $x(p', w') \neq x(p, w)$, then $p' \cdot x(p, w) > w'$.

Reasoning behind the axiom:

$p \cdot x(p', w') \leq w$ and $x(p', w') \neq x(p, w)$

$\Rightarrow$ the bundle $x(p', w')$ was affordable in the situation where consumer chose a different bundle $x(p, w)$

$\Rightarrow$ consumer revealed a preference for $x(p, w)$ over $x(p', w')$

$\Rightarrow$ consumer should choose $x(p, w)$ over $x(p', w')$ whenever both are affordable

$\Rightarrow$ since consumer chooses $x(p', w')$ in situation (p', w'), the bundle $x(p, w)$ must not be affordable in this situation

$\Rightarrow p' \cdot x(p, w) > w'$.

Now, fix w and consider a price change from p to p'.

To remove income effect, adjust wealth to w' so that at price p', the consumer can just afford the bundle $x(p, w)$ chosen prior to price change..

(Slutsky compensation criterion)

$$w' = p' \cdot x(p, w)$$

Then, $x(p', w')$ is the (Slutsky) compensated demand at price p'.

(Proposition 2.F.1) Compensated Law of Demand.

If $x(.,.)$ satisfies homogeneity of degree zero, Walras' Law and WARP, then the following property holds:

for any compensated price change from a initial price-wealth situation (p, w) to a new situation $(p', w') = (p', p' \cdot x(p, w))$, we have

$$(p' - p)[x(p', w') - x(p, w)] \leq 0$$

with strict inequality whenever $x(p, w) \neq x(p', w')$ i.e.,

Proof: The inequality is immediate if $x(p, w) = x(p', w')$. So, suppose $x(p, w) \neq x(p', w')$.

Then,

$$\begin{aligned} & (p' - p)[x(p', w') - x(p, w)] \\ = \;& p' \cdot x(p', w') - p \cdot x(p', w') - p' \cdot x(p, w) + p \cdot x(p, w) \\ = \;& w' - p \cdot x(p', w') - w' + w, \end{aligned}$$

using Walras Law and Slutsky compensation criterion, and the latter expression is

$$= w - p \cdot x(p', w').$$

If $w - p \cdot x(p', w') \geq 0$, then WARP implies $p' \cdot x(p, w) > w'$ which violates the fact that $w' = p' \cdot x(p, w)$. Thus,

$$\begin{aligned} & (p' - p)[x(p', w') - x(p, w)] \\ = \;& w - p \cdot x(p', w') < 0. \end{aligned}$$

This concludes the proof.