

# **COMPETITIVE MARKETS.**

(Partial Equilibrium Analysis)

Consider an economy with:

$I$  consumers,  $i = 1, \dots, I$ .

$J$  firms,  $j = 1, \dots, J$ .

$L$  goods,  $l = 1, \dots, L$ .

Initial endowment of good  $l$  in the economy:  $\omega_l \geq 0$ .

Consumer  $i$ 's:

Consumption set  $X_i \subset \mathbb{R}^L$ .

Utility function:  $u_i : X_i \rightarrow \mathbb{R}$ .

Production technology of firm  $j$  :  $Y_j \subset \mathbb{R}^L$ .

$y_j \in Y_j$  is a production vector  $y_j = (y_{1j}, \dots, y_{Lj}) \in \mathbb{R}^L, j = 1, \dots, J$ .

Total (net) availability of good  $l$  in the economy:  $\omega_l + \sum_{j=1}^J y_{lj}, l = 1, \dots, L$ .

## Pareto Optimality.

**Definition.** An economic allocation  $(x_1, \dots, x_I, y_1, \dots, y_J)$  is a specification of a consumption vector  $x_i \in X_i$  for each consumer  $i = 1, \dots, I$ , and a production vector  $y_j \in Y_j$  for each firm  $j = 1, \dots, J$ .

The allocation is feasible if

$$\sum_{i=1}^I x_{li} \leq \omega_l + \sum_{j=1}^J y_{lj} \text{ for } l = 1, \dots, L.$$

**Definition.** A feasible allocation  $(x_1, \dots, x_I, y_1, \dots, y_J)$  is Pareto optimal (or, Pareto efficient) if there is no other feasible allocation  $(x'_1, \dots, x'_I, y'_1, \dots, y'_J)$  such that

$$u_i(x'_i) \geq u_i(x_i) \text{ for all } i = 1, \dots, I,$$

and

$$u_i(x'_i) > u_i(x_i) \text{ for some } i = 1, \dots, I.$$

## Competitive Equilibria.

Competitive market economy: initial endowments and technological possibilities (firms) are owned by consumers.

Consumer  $i$  initially owns  $\omega_{li} \geq 0$  of good  $l$ ,  $l = 1, \dots, L$ , where

$$\sum_{i=1}^I \omega_{li} = \omega_l.$$

Initial endowment vector of consumer  $i$ :

$$\omega_i = (\omega_{1i}, \dots, \omega_{Li}).$$

In addition each consumer  $i$  owns a share  $\theta_{ij}$  of firm  $j$  giving her a share  $\theta_{ij}$  of the profits of firm  $j$ ,  $j = 1, \dots, J$ .

$$\sum_{i=1}^I \theta_{ij} = 1.$$

A market exists for all goods.

All consumers and producers act as price takers i.e., assume that market prices are unaffected by their actions.

Denote vector of prices:  $p = (p_1, \dots, p_L)$ .

**Definition.** The allocation  $(x_1^*, \dots, x_I^*, y_1^*, \dots, y_J^*)$  and a price vector  $p^* \in \mathbb{R}_+^L$  constitute a competitive (or, Walrasian) equilibrium if the following conditions are satisfied:

(i) Profit maximization: For each firm  $j$ ,  $y_j^*$  solves

$$\max_{y_j \in Y_j} (p^* y_j).$$

(ii) Utility maximization: For each consumer  $i$ ,  $x_i^*$  solves

$$\begin{aligned} & \max u_i(x_i) \\ & \text{s.t.} \\ & p^* x_i \leq p^* \omega_i + \sum_{j=1}^J \theta_{ij} (p^* y_j^*). \end{aligned}$$

(iii) Market clearing: For each good,  $l = 1, \dots, L$ ,

$$\sum_{i=1}^I x_{li}^* = \omega_l + \sum_{j=1}^J y_{lj}^*.$$



Sometimes we permit excess supply in equilibrium with price of the good being zero; assuming free disposal.

If goods are "desirable", for example if marginal utility is always strictly positive, then this possibility is ruled out.

Note: if  $p^* \gg 0$  and  $(x_1^*, \dots, x_I^*, y_1^*, \dots, y_J^*)$  is a competitive equilibrium then so does the allocation  $(x_1^*, \dots, x_I^*, y_1^*, \dots, y_J^*)$  and price vector  $\alpha p^*$  for any  $\alpha > 0$ .

So, we can always normalize prices without loss of generality.

**Lemma.** *If the allocation  $(x_1, \dots, x_I, y_1, \dots, y_J)$  and price vector  $p \gg 0$  satisfy the market clearing condition (iii) for all goods  $l \neq k$ , and if every consumer's budget constraint is satisfied with equality so that*

$$px_i = p\omega_i + \sum_{j=1}^J \theta_{ij}(py_j), \text{ for all } i = 1 \dots I, \quad (1)$$

*then the market for good  $k$  also clears (i.e., (iii) holds for  $l = k$ ).*

**Proof.** Adding (1) over  $i = 1 \dots L$  we get

$$p \left[ \sum_{i=1}^I (x_i - \omega_i - \left( \sum_{j=1}^J \theta_{ij} y_j \right)) \right] = 0$$

so that

$$\sum_{l=1}^L p_l \left( \sum_{i=1}^I x_{li} - \omega_l - \sum_{i=1}^I \sum_{j=1}^J \theta_{ij} y_{lj} \right) = 0$$

i.e.,

$$\sum_{l=1}^L p_l \left( \sum_{i=1}^I x_{li} - \omega_l - \sum_{j=1}^J y_{lj} \right) = 0$$

or,

$$\sum_{l \neq k} p_l \left( \sum_{i=1}^I x_{li} - \omega_l - \sum_{j=1}^J y_{lj} \right) = -p_k \left( \sum_{i=1}^I x_{ki} - \omega_k - \sum_{j=1}^J y_{kj} \right)$$

and as the left hand side is zero, the RHS is zero and since  $p_k > 0$ , we have

$$\left( \sum_{i=1}^I x_{ki} - \omega_k - \sum_{j=1}^J y_{kj} \right) = 0.$$

## Partial Equilibrium Analysis.

Analysis of market for one (or several) goods that form a small part of the economy.

Marshall (1920): consider one good that accounts for small fraction of consumer's total expenditure.

The wealth (or income) effect on the demand for the good can be negligible.

Substitution effect of change in the price of the good is dispersed among all goods and so prices of other goods are approximately unaffected.

So, for the analysis of this market, we can take prices of all other goods as fixed.

Expenditure on all other goods taken to be a composite commodity - the *numeraire*.

The Basic Quasi-linear Model:

Consumers  $i = 1, \dots, I$ .

Two commodities: good  $l$  and the *numeraire*.

$x_i$  : consumer  $i$ 's consumption of good  $l$ .

$m_i$  : consumer  $i$ 's consumption of the *numeraire* (i.e., expenditure on all other goods).

Consumption set of consumer  $i$  :  $\mathbb{R} \times \mathbb{R}_+$ .

(Allow negative consumption of the *numeraire* good - "borrowing" - assumption avoids dealing with corner solution).

Utility function:

$$u_i(m_i, x_i) = m_i + \phi_i(x_i), i = 1, \dots, I$$

Assume:

$\phi_i(\cdot)$  is bounded above, twice continuously differentiable,

$$\phi_i(0) = 0,$$

$$\phi_i'(x_i) > 0, \phi_i''(x_i) < 0, \forall x_i \geq 0.$$

Quasi-linear formulation: no wealth effect.



Normalize price of the *numeraire* good to equal 1.

Let  $p$  be the price (relative price) of good  $l$ ,

Then, one can think of  $\phi_i(x_i)$  as measuring utility in terms of the numeraire good

Firm  $j = 1, \dots, J$ , produces  $q_j$  units of good  $l$  using (at least) amount  $c_j(q_j)$  of the *numeraire* good.

$c_j(q_j)$  : "cost function" of firm  $j$ .

Technology of firm  $j$ :

$$Y_j = \{(-z_j, q_j) : q_j \geq 0, z_j \geq c_j(q_j)\}.$$

Assume:

$c_j : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is twice differentiable.

$c'_j(q_j) > 0$  and  $c''_j(q_j) \geq 0$  at all  $q_j \geq 0$ .

[Think of  $c_j(q_j)$  as derived from a cost minimization problem with fixed input prices.]

Non-decreasing marginal cost curve (allows for constant and decreasing returns to scale).

Also continuity of  $c_j$  at 0 rules out any fixed cost that is not sunk

(cost can be avoided by producing zero).

Initial endowment: No initial endowment of good  $l$ .

Consumer  $i$ 's initial endowment of the numeraire good :  
 $\omega_{mi} > 0$ .

Let

$$\omega_m = \sum_{i=1}^I \omega_{mi}$$

be the total endowment of the numeraire good in the economy.

Competitive Equilibrium:

Profit max.

Given equilibrium price  $p^*$  for good  $l$ , firm  $j$ 's equilibrium output  $q_j^*$  solves

$$\max_{q_j \geq 0} [p^* q_j - c_j(q_j)]$$

Necessary and sufficient first order condition:

$$p^* \leq c'_j(q_j^*), \text{ if } q_j^* = 0 \quad (2)$$

$$= c'_j(q_j^*), \text{ if } q_j^* > 0. \quad (3)$$

Utility max.

Given  $p^*$  and the solution to the firms' profit maximization problems, consumer  $i$ 's equilibrium consumption  $(m_i^*, x_i^*)$  solves:

$$\begin{aligned} & \max_{m_i \in \mathbb{R}, x_i \in \mathbb{R}_+} [m_i + \phi_i(x_i)] \\ & \text{s.t.} \\ & m_i + p^* x_i \leq \omega_{mi} + \sum_{j=1}^J \theta_{ij} (p^* q_j^* - c_j(q_j^*)) \end{aligned}$$

Budget constraint holds with equality in any solution to the above problem.

Rewrite the problem without loss of generality as one of choosing only the consumption of good  $l$ :

$$\max_{x_i \in \mathbb{R}_+} [\omega_{mi} + \sum_{j=1}^J \theta_{ij}(p^* q_j^* - c_j(q_j^*)) - p^* x_i + \phi_i(x_i)]$$

or equivalently,  $x_i^*$  must solve

$$\max_{x_i \geq 0} [\phi_i(x_i) - p^* x_i]$$

and  $m_i^*$  is determined by

$$m_i^* = \omega_{mi} + \sum_{j=1}^J \theta_{ij}(p^* q_j^* - c_j(q_j^*)) - p^* x_i^*.$$

A necessary and sufficient first order condition:

$$p^* \geq \phi'_i(x_i^*), \text{ if } x_i^* = 0, \quad (4)$$

$$= \phi'_i(x_i^*), \text{ if } x_i^* > 0. \quad (5)$$

Thus, an equilibrium allocation is characterized fully by a price  $p^*$  of good  $l$  and the vector  $(x_1^*, \dots, x_I^*, q_1^*, \dots, q_J^*)$  of consumption and production of good  $l$ .



Finally, market clearing for good  $l$  requires:

$$\sum_{i=1}^I x_i^* = \sum_{j=1}^J q_j^*. \quad (6)$$

**Proposition:** *The allocation  $(x_1^*, \dots, x_I^*, q_1^*, \dots, q_J^*)$  and price  $p^*$  constitutes a competitive equilibrium if and only if.*

$$\begin{aligned} p^* &\leq c'_j(q_j^*), \text{ if } q_j^* = 0 \\ &= c'_j(q_j^*), \text{ if } q_j^* > 0, j = 1, \dots, J \end{aligned}$$

$$\begin{aligned} p^* &\geq \phi'_i(x_i^*), \text{ if } x_i^* = 0, \\ &= \phi'_i(x_i^*), \text{ if } x_i^* > 0, .i = 1, \dots, I \end{aligned}$$

$$\sum_{i=1}^I x_i^* = \sum_{j=1}^J q_j^*.$$

The above  $(I+J+1)$  conditions determine the  $(I+J+1)$  equilibrium values  $(x_1^*, \dots, x_I^*, q_1^*, \dots, q_J^*, p^*)$ .

*The equilibrium allocation and price of good  $l$  are entirely independent of the distribution of initial endowments and ownership shares of firms.*

Observe: since  $\phi'_i(x_i) > 0, \forall x_i \geq 0$ , it follows that the equilibrium price  $p^* > 0$ .

Assume:

$$\max_i \phi'_i(0) > \min_j c'_j(0).$$

Then, in equilibrium, total consumption and production

of good  $l$  : 
$$\sum_{i=1}^I x_i^* = \sum_{j=1}^J q_j^* > 0.$$

[If all consumers consume 0 and all firms produce 0, then

$$c'_j(0) \geq p^* \geq \phi'_i(0), i = 1, \dots, I, j = 1, \dots, J,$$

so that

$$\max_i \phi'_i(0) \leq \min_j c'_j(0),$$

a contradiction.]

One can derive the equilibrium through traditional Marshallian demand-supply analysis.

Demand :

For any  $p$ , each consumer's first order condition:

$$\begin{aligned} p &\geq \phi'_i(x_i), \text{ if } x_i = 0, \\ &= \phi'_i(x_i), \text{ if } x_i > 0. \end{aligned}$$

Note  $\phi'_i$  is a continuous and strictly decreasing function on  $\mathbb{R}_+$  with range  $[0, \phi'_i(0)]$ .

Individual Walrasian demand function of consumer  $i$  :

$$\begin{aligned} x_i(p) &= 0, p \geq \phi'_i(0), \\ &= \phi_i'^{-1}(p), p \in (0, \phi'_i(0)). \end{aligned}$$

Note individual demand  $x_i(p)$  is independent of wealth, continuous & non-increasing in  $p$  and strictly decreasing in  $p$  on  $(0, \phi'_i(0))$ .

Aggregate (market) demand for good  $l$ :

$$x(p) = \sum_{i=1}^I x_i(p)$$

- independent of endowment & the distribution of endowments,
- continuous & non-increasing in  $p$  and
- strictly decreasing in  $p$  on  $(0, \max_i \phi'_i(0))$ .

Note that individual and aggregate demand is infinite at zero price.

Also, aggregate demand is zero for  $p \geq \max_i \phi'_i(0)$ .

Supply:

For any  $p$ , firm  $j$ 's profit max yields the following first order condition:

$$\begin{aligned} p &\leq c'_j(q_j), \text{ if } q_j = 0 \\ &= c'_j(q_j), \text{ if } q_j > 0. \end{aligned}$$

If  $p < c'_j(0)$ , firm's supply is zero.

Suppose  $c_j$  is strictly convex (upward sloping marginal cost) and  $c'_j(q) \rightarrow \infty$  as  $q \rightarrow \infty$ .

Then, for each  $p > c'_j(0)$ , there is a unique  $q_j$  such that  $p = c'_j(q_j)$ .

The firm's supply curve in that case:

$$\begin{aligned}q_j(p) &= 0, p \leq c'_j(0), \\ &= c_j'^{-1}(p), p > c'_j(0).\end{aligned}$$

$q_j(p)$  is

- continuous and non-decreasing in  $p$  &
- strictly increasing for  $p > c'_j(0)$ .



The aggregate (market) supply curve is given by:

$$q(p) = \sum_{j=1}^J q_j(p).$$

Note that  $q(p) = 0$  for  $p \leq \min_j c'_j(0)$ .

For  $p > \min_j c'_j(0)$ ,  $q(p)$  is strictly positive, strictly increasing and continuous.

The market equilibrium price  $p^*$  is given by the point where aggregate demand and supply intersect i.e.,

$$x(p^*) - q(p^*) = 0 \quad (7)$$

Let  $z(p) = x(p) - q(p)$ . Then,

$$\begin{aligned} z(p) &= x(p) > 0, p \leq \min_j c'_j(0) \\ &= -q(p) < 0, p \geq \max_i \phi'_i(0). \end{aligned}$$

$z(p)$  is continuous and strictly decreasing in  $p$  on  $(\min_j c'_j(0), \max_i \phi'_i(0))$ .

Unique  $p^* \in (\min_j c'_j(0), \max_i \phi'_i(0))$ , such that  $z(p^*) = 0$  i.e., (7) holds.

The equilibrium allocation is given by setting  $x_i^* = x_i(p^*)$ ,  $i = 1, \dots, I$ ,  $q_j^* = q_j(p^*)$ ,  $j = 1, \dots, J$ .

If  $c_j$  is convex but not strictly convex (for example, linear), there may not be unique solution to the profit max problem and so  $q_j(p)$  is a correspondence (upper hemi-continuous, using Maximum Theorem).

Similar analysis goes through with more technical arguments.

Important case: Constant returns to scale.  $c_j(q_j) = c_j q_j$  where  $c_j > 0$  is the constant average as well as marginal cost ("unit cost").

Firm  $j$ 's supply function

$$\begin{aligned} q_j(p) &= 0, p < c_j \\ &\in [0, \infty), p = c_j \\ &= \infty, p > c_j. \end{aligned}$$

The aggregate supply is infinite (not well defined as a real number) for  $p > c_j$ .

If all firms have constant returns to scale technology with cost functions  $c_j(q_j) = c_j q_j, j = 1, \dots, J$ ,

then the unique equilibrium price  $p^* = \min_j c_j$ .

Only firms with the minimum unit cost can produce in equilibrium (such firms are indifferent between all levels of output at that price).

The total quantity of good  $l$  produced and consumed is given by the aggregate demand function and equals  $x(p^*)$ .

If there are multiple firms with the minimum unit cost, the way the total quantity demanded  $x(p^*)$  is produced across these firms is not uniquely determined.

Recall, industry supply:

$$q(p) = \sum_{j=1}^J q_j(p)$$

where

$$q_j(p) = c_j'^{-1}(p), \forall j \text{ such that } q_j(p) > 0.$$

For any  $y > 0$ , the inverse of the aggregate supply function given by  $q^{-1}(y)$  indicates the equalized marginal cost of all firms that produce this output:  $q^{-1}(y)$  is the industry's marginal cost curve.

Define the industry's aggregate cost of producing any level of total output  $y$  by:

$$C(y) = \min_{q_j, j=1, \dots, J} \sum_{j=1}^J c_j(q_j)$$
$$s.t. \sum_{j=1}^J q_j = y, q_j \geq 0, j = 1, \dots, J.$$

Lagrangian:

$$L(q_1, \dots, q_J, \lambda) = \sum_{j=1}^J c_j(q_j) + \lambda(y - \sum_{j=1}^J q_j)$$

First order necessary and sufficient conditions:

$$\begin{aligned}c'_j(\hat{q}_j) &= \lambda, \hat{q}_j > 0 \\ &\geq \lambda, \hat{q}_j = 0\end{aligned}$$

- all firms that produce strictly positive output, marginal cost is equalized to  $\lambda$

-all firms that produce zero output, marginal cost at zero is no larger than  $\lambda$

For any  $p$ , letting  $\lambda = p$ , we can see that  $\hat{q}_j = q_j(p)$ ,  $j = 1, \dots, J$ , must minimize industry's aggregate cost of producing  $y = \sum_{j=1}^J q_j(p)$ .



Further, using envelope theorem:

$$\begin{aligned} C'(y) &= \lambda \\ &= c'_j(\hat{q}_j), \forall j \text{ such that } \hat{q}_j > 0. \end{aligned}$$

Thus, industry's marginal cost of producing  $y = \sum_{j=1}^J q_j(p)$  is given by  $c'_j(q_j(p))$ , for all  $j$  such that  $q_j(p) > 0$  i.e.,  $q^{-1}(y)$ , the inverse aggregate supply curve.

Therefore,

$$\begin{aligned} C(q) &= C(0) + \int_0^q C'(y)dy \\ &= C(0) + \int_0^q q^{-1}(y)dy \end{aligned}$$

so that

$$\int_0^q q^{-1}(y)dy = C(q) - C(0)$$

The area under the aggregate supply curve is equal to the (minimized) total variable cost of the industry (i.e., the total cost excluding the sunk cost).

The area under individual supply curve is the total variable cost of the firm.

Recall,

$$x(p) = \sum_{i=1}^I x_i(p)$$

where

$$x_i(p) = \phi_i'^{-1}(p), \forall i \text{ such that } x_i(p) > 0.$$

Let  $P(y) = x^{-1}(y)$  be the inverse aggregate demand function.

Then, for any  $y$ ,  $P(y) = \phi_i'(x_i(P(y)))$ ,  $\forall i$  such that  $x_i(p) > 0$ .

In other words,  $P(y)$  represents the marginal benefit from consumption of good  $l$  to a consumer that consumes strictly positive quantity when the price is  $P(y)$  and total quantity  $y$  is consumed.

So,  $P(y)$  represents the marginal benefit to society from total consumption of amount  $y$ .

For any  $y > 0$ , define the (maximum) social benefit from total consumption of amount  $y$  :

$$B(y) = \max_{x_i, i=1, \dots, I} [\phi_i(x_i)]$$
$$s.t. \sum_{i=1}^I x_i = y, x_i \geq 0, i = 1, \dots, I.$$

Define Lagrangean

$$L(x_1, \dots, x_I, \mu) = \sum_{i=1}^I (\phi_i(x_i)) + \mu(y - \sum_{i=1}^I x_i)$$

First order necessary and sufficient conditions:

$$\begin{aligned}\phi'_i(\hat{x}_i) &= \mu, \hat{x}_i > 0 \\ &\leq \mu, \hat{x}_i = 0\end{aligned}$$

so that for all consumers that consume strictly positive amount of good  $l$ , marginal utility is equalized to  $\mu$  and for all consumers that consume zero amount, marginal utility at zero does not exceed  $\mu$ .

For any  $p$ , letting  $\mu = p$ , we can see that  $\hat{x}_i = x_i(p)$ ,  $i = 1, \dots, I$ , must maximize society's benefit from consuming

$$y = \sum_{j=1}^I x_j(p).$$

Further, using envelope theorem:

$$\begin{aligned} B'(y) &= \mu \\ &= \phi'_i(\hat{x}_i), \forall i \text{ such that } \hat{x}_i > 0. \end{aligned}$$

Thus, society's marginal benefit from consuming  $y = \sum_{j=1}^I x_j(p)$  is given by  $\phi'_i(x_i(p))$ ,  $\forall i$  such that  $x_i(p) > 0$  (the height of the individual demand curves at  $x_i(p)$ ) i.e.,  $P(y)$ , the inverse aggregate demand curve.

Therefore, (using  $B(0) = 0$ )

$$\begin{aligned} B(x) &= \int_0^x B'(y) dy \\ &= \int_0^x P(y) dy \end{aligned}$$

The area under the aggregate demand curve is equal to the (maximum) social benefit from consumption of good  $l$ .

The area under individual demand curve is the total benefit to the individual consumer.

To sum up:

1. Profit maximization by price taking firms ensures that the total output produced by the industry at any price minimizes the industry's cost of producing this amount (i.e., market distributes total output across firms optimally).
2. Utility maximization by price taking consumers ensures that the total consumption in society is distributed across consumers so as to maximize the total benefit to society (optimal distribution of consumption).

3. The height of the aggregate supply curve indicates the industry's marginal cost of production.

The area under the aggregate (inverse) supply curve indicates the industry's total variable cost.

4. The height of the aggregate demand curve indicates the society's marginal benefit from consumption.

The area under the aggregate (inverse) demand curve indicates society's total benefit from consumption of good  $l$ .



## Pareto Optimality & Competitive Equilibrium.

Fix the consumption and the production levels of good  $l$  at  $(\bar{x}_1, \dots, \bar{x}_I, \bar{q}_1, \dots, \bar{q}_J)$  where

$$\sum_{i=1}^I \bar{x}_i = \sum_{j=1}^J \bar{q}_j.$$

The total amount of the numeraire available for distribution amount consumers is

$$\omega_m - \sum_{j=1}^J c_j(\bar{q}_j).$$

Quasilinear utility: transferable utility.

Transferring numeraire good across consumers in various ways one can generate a utility possibility set:

$$\{(u_1, u_2, \dots, u_I) : \sum_{i=1}^I u_i \leq \sum_{i=1}^I \phi_i(\bar{x}_i) + \omega_m - \sum_{j=1}^J c_j(\bar{q}_j)\}$$

The right hand side of the inequality defining the set is a constant (given  $(\bar{x}_1, \dots, \bar{x}_I, \bar{q}_1, \dots, \bar{q}_J)$ ).

So, the frontier of this utility possibility set is a hyperplane with normal vector  $(1, 1, \dots, 1)$ .

Changing the consumption and production levels of good  $l$  i.e., the vector  $(\bar{x}_1, \dots, \bar{x}_I, \bar{q}_1, \dots, \bar{q}_J)$  shifts the frontier of the utility possibility set in a parallel fashion.

The frontier moves outward or inward according to whether

$$\sum_{i=1}^I \phi_i(\bar{x}_i) + \omega_m - \sum_{j=1}^J c_j(\bar{q}_j)$$

increases or decreases when we change  $(\bar{x}_1, \dots, \bar{x}_I, \bar{q}_1, \dots, \bar{q}_J)$ .

As long as the frontier can be shifted outwards by change in the vector  $(\bar{x}_1, \dots, \bar{x}_I, \bar{q}_1, \dots, \bar{q}_J)$  the original situation is not Pareto optimal.

Thus, every Pareto optimal allocation must involve consumption and production profile  $(\tilde{x}_1, \dots, \tilde{x}_I, \tilde{y}_1, \dots, \tilde{y}_J)$  for good  $l$  so as to shift the frontier as far out as possible i.e.,  $(\tilde{x}_1, \dots, \tilde{x}_I, \tilde{y}_1, \dots, \tilde{y}_J)$  solves

$$\begin{aligned} & \max_{\substack{(x_1, \dots, x_I) \geq 0 \\ (q_1, \dots, q_J) \geq 0}} [\sum_{i=1}^I \phi_i(x_i) - \sum_{j=1}^J c_j(q_j)] \\ & \text{s.t.} \\ & \sum_{i=1}^I x_i = \sum_{j=1}^J q_j \end{aligned}$$

The maximand

$$\left[ \sum_{i=1}^I \phi_i(x_i) - \sum_{j=1}^J c_j(q_j) \right]$$

is often called the Marshallian aggregate surplus (or total surplus).

It measures the net benefit to society from producing and consuming good  $l$ .

There exists a solution to the surplus maximization problem.

While there are a continuum of Pareto optimal allocations corresponding to points on the highest utility possibility frontier, they all must involve a consumption and production vector for good  $l$  such as  $(\tilde{x}_1, \dots, \tilde{x}_I, \tilde{y}_1, \dots, \tilde{y}_J)$  which solves the surplus max problem.

In particular, if the solution to the surplus max problem is unique (for example, when  $c_j$  is strictly convex):

- all Pareto optimal allocations must involve exactly the same production and consumption vector for good  $l$
- the only difference in various Pareto optimal allocations would arise from differences in distribution of the numeraire good

(that can transfer utility from one agent to another unit for unit).

Lagrangian:

$$L = \sum_{i=1}^I \phi_i(x_i) - \sum_{j=1}^J c_j(q_j) + \mu \left[ \sum_{j=1}^J q_j - \sum_{i=1}^I x_i \right]$$

First order necessary and sufficient condition (maximand is concave, feasible set is convex):

$$\begin{aligned} \mu &\leq c'_j(\tilde{q}_j), \text{ if } \tilde{q}_j = 0 \\ &= c'_j(\tilde{q}_j), \text{ if } \tilde{q}_j > 0, j = 1, \dots, J. \end{aligned}$$

$$\begin{aligned} \phi'_i(\tilde{x}_i) &\leq \mu, \text{ if } \tilde{x}_i = 0, \\ &= \mu, \text{ if } \tilde{x}_i > 0, i = 1, \dots, I \end{aligned}$$

$$\sum_{j=1}^J \tilde{q}_j = \sum_{i=1}^I \tilde{x}_i$$

Setting  $\mu = p^*$ , we see that these conditions are satisfied by the production and consumption profile for good  $l$  in any competitive equilibrium allocation.



## The First Fundamental Theorem of welfare Economics.

### Proposition.

*If the price  $p^*$  and the allocation  $(x_1^*, \dots, x_I^*, q_1^*, \dots, q_J^*)$  constitutes a competitive equilibrium, then this allocation is Pareto optimal.*

Conversely, consider any Pareto optimal allocation.

The production and consumption levels of good  $l$  in any such allocation must solve the surplus max problem and setting  $p = \mu$  we can check that the solution to surplus max  $(\tilde{x}_1, \dots, \tilde{x}_I, \tilde{y}_1, \dots, \tilde{y}_J)$  is a competitive equilibrium.

Consider this competitive equilibrium.

The price, the equilibrium consumption and production of good  $l$  and the profits of firms are unaffected by any transfer of the numeraire good from one agent to another.

Transferring the numeraire good from one agent to another changes the utility of the agents by exactly the amount of the transfer.

Therefore, can always generate the exact profile of utilities and numeraire good consumption in the candidate Pareto optimal allocation by transferring numeraire good from one agent to another.

Can attain any point on the Pareto optimal boundary of the utility possibility set by transferring numeraire good across consumers.

## The Second Fundamental Theorem of Welfare Economics.

### Proposition.

*For any Pareto optimal levels of utility  $(u_1^*, \dots, u_I^*)$ , there are transfers of the numeraire commodity  $(T_1, \dots, T_I)$  satisfying  $\sum_{i=1}^I T_i = 0$  such that a competitive equilibrium reached from the endowments  $(\omega_{m1} + T_1, \dots, \omega_{mI} + T_i)$  yields precisely the utilities  $(u_1^*, \dots, u_I^*)$ .*

Convex structure important for this result.

## **Welfare Analysis in Partial Equilibrium.**

Social welfare function: assigns social welfare value (real number) to each profile of utility levels  $(u_1, u_2, \dots, u_I)$  :

$$W(u_1, u_2, \dots, u_I)$$

(Utilitarian welfare).

Assume:  $W$  is strongly monotonic in its arguments.

For any *given* consumption and production levels of good  $l$ ,  $(x_1, \dots, x_I, q_1, \dots, q_J)$ , where

$$\sum_{i=1}^I x_i = \sum_{j=1}^J q_j,$$

the utility vectors that are attainable are given by:

$$\{(u_1, u_2, \dots, u_I) : \sum_{i=1}^I u_i \leq \sum_{i=1}^I \phi_i(x_i) + \omega_m - \sum_{j=1}^J c_j(q_j)\}.$$

As the boundary of this set expands, the maximum social welfare  $W$  attainable on this set (through redistribution of the numeraire good) increases (strictly).

Thus,

*\*For any strongly monotonic social welfare function  $W$ , a change in the consumption and production of good  $l$  leads to an increase in (the maximum attainable) social welfare if and only if it increases the Marshallian surplus:*

$$S(x_1, \dots, x_I, q_1, \dots, q_J) = \left[ \sum_{i=1}^I \phi_i(x_i) - \sum_{j=1}^J c_j(q_j) \right].$$



Thus, social welfare analysis of changes in the consumption and production of good  $l$  can be carried out exclusively in terms of the Marshallian surplus.

Indeed, as we have seen, Pareto efficiency also requires that the consumption and production of good  $l$  must satisfy

$$\begin{aligned} & \max_{\substack{(x_1, \dots, x_I) \geq 0 \\ (q_1, \dots, q_J) \geq 0}} S(x_1, \dots, x_I, q_1, \dots, q_J) \\ & s.t. \sum_{i=1}^I x_i = \sum_{j=1}^J q_j. \end{aligned}$$

Consider a consumption and production vector of good

$l$ ,  $(\hat{x}_1, \dots, \hat{x}_I, \hat{q}_1, \dots, \hat{q}_J)$  such that for  $\hat{y} = \sum_{i=1}^I \hat{x}_i$

(i)  $(\hat{x}_1, \dots, \hat{x}_I)$  solves:

$$\begin{aligned} & \max_{x_i, i=1, \dots, I} \left[ \sum_{i=1}^I \phi_i(x_i) \right] \\ & \text{s.t. } \sum_{i=1}^I x_i = \hat{y}, x_i \geq 0, i = 1, \dots, I. \end{aligned}$$

(ii)  $(\hat{q}_1, \dots, \hat{q}_J)$  solves

$$\begin{aligned} & \min_{q_j, j=1, \dots, J} \sum_{j=1}^J c_j(q_j) \\ & \text{s.t. } \sum_{j=1}^J q_j = \hat{y}, q_j \geq 0, j = 1, \dots, J. \end{aligned}$$

We have seen that:

$$\phi'_i(\hat{x}_i) = P(\hat{y}) = B'(\hat{y}), \quad \forall i \text{ such that } \hat{x}_i > 0$$

$$c'_j(\hat{q}_j) = C'(\hat{y}), \quad \forall j \text{ such that } \hat{q}_j > 0,$$

where  $P$  is the inverse aggregate demand function,  $B'(\cdot)$  is the industry marginal benefit and  $C'(\cdot)$  is the industry marginal cost (or the aggregate inverse supply function).

$$\begin{aligned}
S(\hat{x}_1, \dots, \hat{x}_I, \hat{q}_1, \dots, \hat{q}_J) &= \left[ \sum_{i=1}^I \phi_i(\hat{x}_i) - \sum_{j=1}^J c_j(\hat{q}_j) \right] \\
&= B(\hat{y}) - C(\hat{y}) \\
&= \int_0^{\hat{y}} B'(y) dy - C(0) - \int_0^{\hat{y}} C'(y) dy \\
&= \int_0^{\hat{y}} P(y) dy - \int_0^{\hat{y}} C'(y) dy - C(0) \\
&= \left[ \int_0^{\hat{y}} [P(y) - C'(y)] dy \right] - S(0)
\end{aligned}$$

Note:

$$\left[ \int_0^{\hat{y}} [P(y) - C'(y)] dy \right]$$

is the area between the aggregate demand and supply curves and can be written as :

$$\begin{aligned} & \left[ \int_0^{\hat{y}} [P(y) - C'(y)] dy \right] \\ &= \left[ \int_0^{\hat{y}} [P(y) - \hat{y}P(\hat{y})] + [\hat{y}P(\hat{y}) - (C(\hat{y}) - C(0))] \right] \\ &= CS(P(\hat{y})) + PS(P(\hat{y})) \end{aligned}$$

where  $CS(p)$  and  $PS(p)$  denote the aggregate consumer and producer surplus generated in a (hypothetical) market with price taking consumers and producers at price market price  $p$ .

Therefore, in partial equilibrium analysis, social welfare maximization, Marshallian surplus maximization and Pareto efficiency are roughly equivalent in their implication for the production and consumption of "the good" and eventually reduce to maximization of  $CS + PS$ .

It is easy to see that  $\left[ \int_0^{\hat{y}} [P(y) - C'(y)] dy \right]$  is maximized at the output where:

$$P(y^*) = C'(y^*)$$

i.e., social marginal benefit equates industry's marginal cost.

As  $C'(y)$  is inverse aggregate supply curve, this is also the aggregate output consumed and produced in a competitive equilibrium (supply=demand).

Thus, competitive equilibrium outcome is equivalent to Marshallian surplus maximization.



All of this assumes no externalities or other distortions (taxes, subsidies etc).

Welfare loss due to distortions is measured by the change in  $CS + PS$  i.e., the area between aggregate demand and the supply (or industry MC curve).

Sometimes, called deadweight loss.

Example. Welfare loss due to a distortionary tax (in a competitive market).

Sales tax on good  $l$ :  $t$  per unit paid by consumers.

Tax revenue returned to consumers through lump sum transfer (non distortionary spending).

Let  $(x_1^*(t), \dots, x_I^*(t), q_1^*(t), \dots, q_J^*(t))$  and  $p^*(t)$  be the competitive equilibrium allocation and price given tax rate  $t$ .

FOC:

$$\phi'_i(x_i^*(t)) = p^*(t) + t, \text{ for all } i \text{ such that } x_i^*(t) > 0.$$

$$c'_j(q_j^*(t)) = p^*(t), \text{ for all } j \text{ such that } x_j^*(t) > 0.$$

Let

$$x^*(t) = x(p^*(t) + t) = \sum_{i=1}^I x_i^*(t).$$

Market clearing:

$$x(p^*(t) + t) = q(p^*(t))$$

\* Easy to check that (over the range where a strictly positive quantity is traded)  $p^*(t)$  is strictly decreasing in  $t$  and that  $(p^*(t) + t)$  is strictly increasing in  $t$ .

$x^*(t)$  is strictly decreasing in  $t$  (as long as it is strictly positive) and

$$x^*(t) < x^*(0), t > 0.$$

Let  $S^*(t) = S(x_1^*(t), \dots, x_I^*(t), q_1^*(t), \dots, q_J^*(t))$ .

We have that

$$S^*(t) = \left[ \int_0^{x^*(t)} [P(y) - C'(y)] dy \right] - S(0)$$

$$\begin{aligned} & \text{Welfare change} \\ &= S^*(t) - S^*(0) \\ &= \int_{x^*(0)}^{x^*(t)} [P(y) - C'(y)] dy \end{aligned}$$

which is negative since  $x^*(t) < x^*(0)$  and  $P(y) > C'(y)$  for all  $y \in [0, x^*(0))$ .