Dynamic Games.

Most economic interactions:

agents choose actions over time where some information over actions chosen in previous periods is available.

If previous actions unobserved or unknown to players that move later, the game is effectively a simultaneous move game.

Otherwise, its a true dynamic game.

Extensive forms capture the dynamic structure of moves.

Every extensive form game can be reduced to a normal form game where players simultaneously choose *strategies*.

Why not just use the theory of games developed for normal form games - for example, the concept of NE and apply it to the reduced normal form associated with the extensive form of a dynamic game?

Problem: Some NE strategies may not be *credible* under the sequential structure captured by the extensive form (information suppressed in the normal form).

Need refinement of Nash equilibrium.

Dynamic Finite Games of Complete and Perfect Information

Players not only have common knowledge of payoffs, but also know every choice made by players whose moves precede them.

Every information set consists of a single node.

Finite number of nodes.

Example.

• Maria

Y$\swarrow$ $\searrow$N

$\begin{bmatrix} 100 \\ 1000 \end{bmatrix}$ •Dwain

L$\swarrow$ $\searrow$D

$\begin{bmatrix} 1000 \\ 100 \end{bmatrix}$ $\begin{bmatrix} 25 \\ 0 \end{bmatrix}$

Strategy Sets:

Maria: {Y,N}

Dwain: {L if N, D if N}

Normal form:

$$\begin{bmatrix} & \text{L if N} & \text{D if N} \\ \text{Y} & 100, 1000 & 100, 1000 \\ \text{N} & 1000, 100 & 25, 0 \end{bmatrix}$$

Two pure strategy NE:

NE1: (N, L if N)

NE2: (Y, D if N).

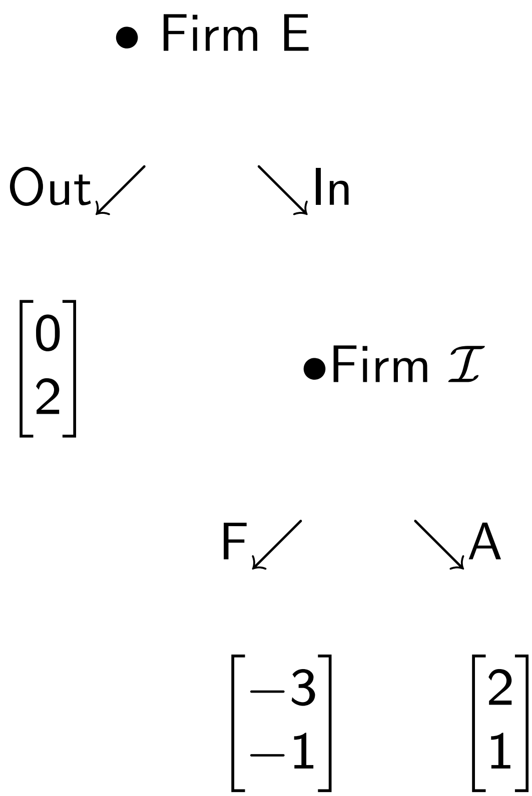
NE2 is not credible because if we look at the extensive form we immediately know that Dwain would never choose D if Maria chose N.

NE2: based on Dwain playing a strategy where he threatens to play an action (D) that he would never play if he actually had to choose an action in the real play of the game - his strategy is a bluff.

The reason why the outcome in NE2 remains a Nash equilibrium in the normal form is because in the actual play of this NE, Dwain will never have to actually choose between L and D - what he says he will do in that node does not affect his payoff - his bluff will never be called.

The normal form of the game does not allow us to see this credibility problem (the strategies can be re-labelled as A,B,C, D and the underlying story is not visible) - one needs the extensive form to discover it.

Example.



Strategy Sets:

Firm E: {Out,In}

Firm $\mathcal{I}$: {F if In, A if In}

Normal form:

$$\begin{bmatrix} & \text{F if In} & \text{A if In} \\ \text{Out} & 0,2 & 0,2 \\ \text{In} & -3,-1 & 2,1 \end{bmatrix}$$

Two pure strategy NE:

NE1: (In, A if In)

NE2: (Out, F if In).

NE2 is not credible.

In this Nash equilibrium, what firm I's strategy says it will do at the unreached node can actually ensure that firm E, taking firm I's strategy as given, wants to play "out" (even though, given firm E's strategy, firm I's strategy choice does not really make a difference to firm I's payoff).

* *Principle of Sequential Rationality*: A player's strategy should specify <u>optimal</u> actions at every point in the game tree.

At each decision node in the tree, a player should choose an action that is optimal *from that point on*, given the strategies of other players.

Sequential rationality violated by NE2 in both examples above.

NE1 is sequentially rational in both examples: credible.

Work backwards from the last stage of the game i.e., decision nodes whose only successor nodes are terminal nodes).

Solve for optimal behavior at each such decision node.

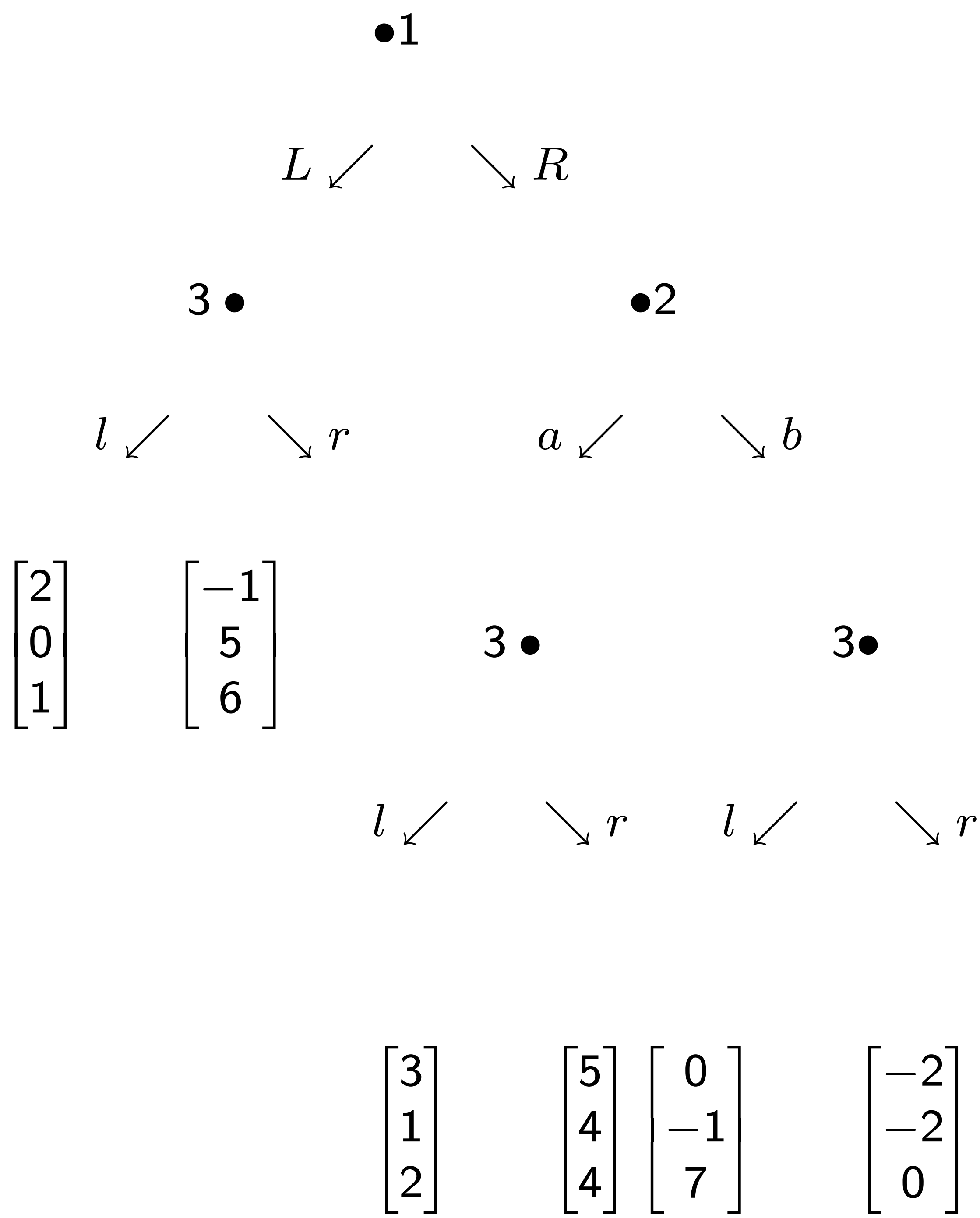
Then go to previous stage (i.e., nodes preceding the above) and figure out optimal action of decision maker

(while fixing continuation play in the successor nodes to the optimal actions derived previously)

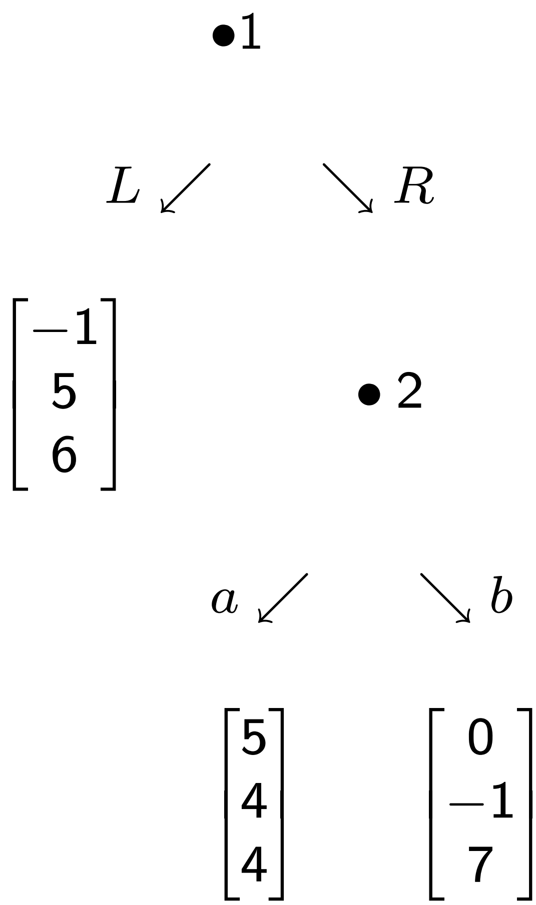
& so on until one reaches the beginning of the game.

This is called *backward induction*.

Example.



Solving for optimal action in last stage leads to the following reduced form two-stage game :



Reduced form first stage game:

$$\bullet 1$$

$$L \swarrow \qquad \searrow R$$

$$\begin{bmatrix} -1 \\ 5 \\ 6 \end{bmatrix} \qquad \begin{bmatrix} 5 \\ 4 \\ 4 \end{bmatrix}$$

Solution by backward induction:

$[R, a\ if\ R, (r\ if\ L, r\ if\ R$ and $a, l\ if\ R$ and $b]$.

Can check that this is a NE.

There are two other NE that are not sequentially rational.

Proposition: *Every finite game of perfect information has a pure strategy NE that can be derived through backward induction. Moreover, if no player has the same payoffs at any two distinct terminal nodes, then there is a unique NE that can be derived in this manner.*

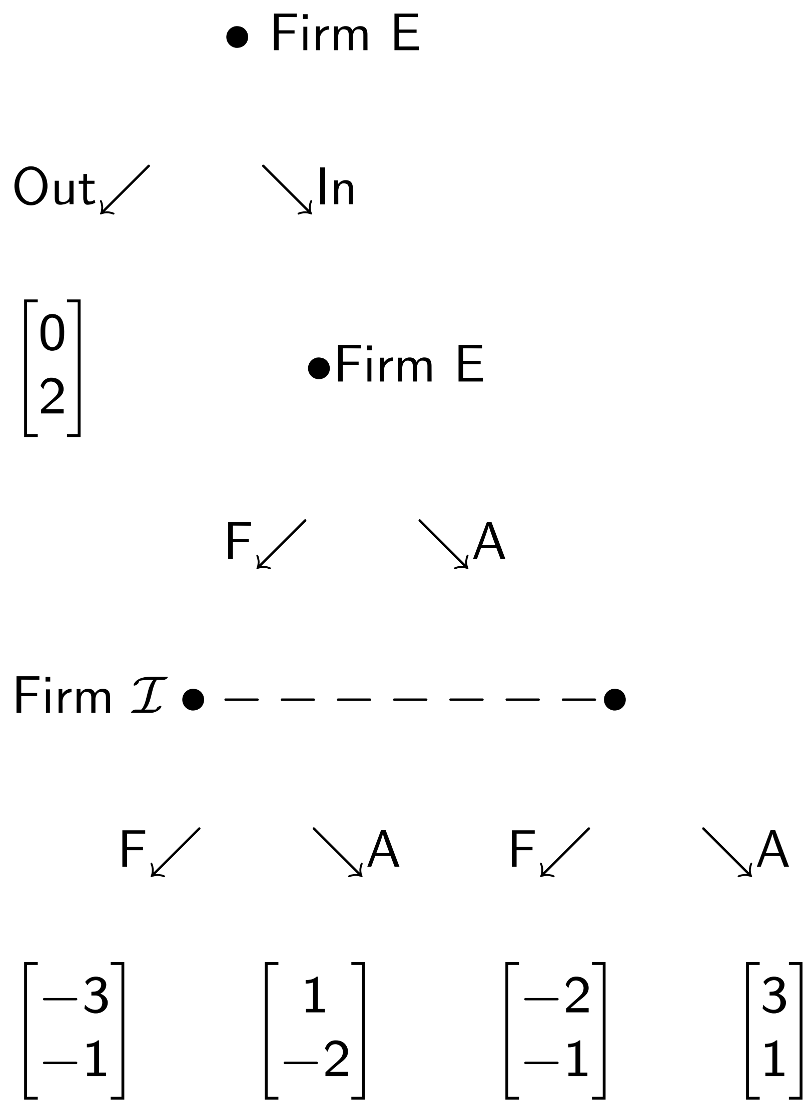
Games of Complete but possibly Imperfect Information.

How to apply sequential rationality if decision nodes are not necessarily singletons i.e., players do not necessarily observe all predecessor moves.

Example:

Consider the entrant-incumbent game with one modification.

If entrant form decides to play "In" (i.e., enter), entrant and incumbent play a simultaneous move game where they both choose whether to fight or accommodate.



Firm E
Out
In
0
2
Firm E
F
A
Firm $\mathcal{I}$
F
A
F
A
−3
−1
1
−2
−2
−1
3
1

Normal form:

Row player: E

Column Player: $\mathcal{I}$

	A if In	F if In
Out,A if In	$0, 2$	$0, 2$
Out,F if In	$0, 2$	$0, 2$
In, A if In	$3, 1$	$-2, -1$
In, F if In	$1, -2$	$-3, -1$

Three pure strategy NE in the normal form game:

NE1: [(Out, A if In), (F if In)]

NE2: [(Out, F if In),(F if In)]

NE3: [(In,A if In),(A if In)]

Extensive form equivalent to:

$$\bullet \text{ Firm E}$$

$$\text{Out} \swarrow \qquad\qquad \searrow \text{In}$$

$$\begin{bmatrix} 0 \\ 2 \end{bmatrix} \qquad \begin{bmatrix} & A & F \\ A & 3,1 & -2,-1 \\ F & 1,-2 & -3,-1 \end{bmatrix}$$

In the simultaneous move (sub)game that follows "In" (captured in the matrix), unique NE: (A,A).

Thus, firm E should expect that if it enters, they will both play (A,A).

So, firm E should choose 'In".

Only NE3 is a reasonable prediction of the game.

Subgame Perfect Nash Equilibrium. [Selten, 1965].

Definition. A subgame of an extensive form game is a subset of the game having the following properties:

(i) It begins with an information set containing a single decision node, contains all the decision nodes that are successors (both immediate and later) of this node, and contains only these nodes;

(ii) If a decision node x is in the subgame, then every other node contained in the information set containing x is also in the subgame (no broken information sets).

The entire game is also a subgame.

A subgame is an extensive form game in its own right and one can apply all of the equilibrium/solution concepts - including Nash equilibrium to it.

* A strategy profile σ in extensive form game Γ_E is said to induce a Nash equilibrium in a particular subgame of Γ_E if the moves specified by σ for information sets within the subgame constitute a Nash equilibrium when this subgame is considered in isolation.

Definition. *A profile of strategies* $\sigma = (\sigma_1, ..., \sigma_I)$ *in an* I*-player extensive form game* Γ_E *is a subgame perfect Nash equilibrium (SPNE) if it induces a Nash equilibrium in every subgame of* Γ_E.

By definition, every SPNE is a NE but the converse is not true.

SPNE: the most widely used refinement of NE in economic applications.* If the only subgame of a game is the game as a whole, then every NE is subgame perfect.

* A SPNE induces a SPNE in every subgame of the game.

* In finite games of perfect information, the set of SPNE coincides with the set of NE that can be derived through backward induction procedure.

[Why? Every decision node initiates a subgame.

Single decision maker at each level.

Consider decision nodes at the last stage of game: NE is simply the optimal action.

Backward induction: decision nodes before end of game, best response to optimal actions in the end decision nodes.

So, backward induction leads to NE in the subgames beginning from the decision nodes preceding the end decision nodes.

And so on...]

Proposition: *Every finite game of (complete and) perfect information has a pure strategy SPNE. Moreover, if no player has the same payoffs at any two terminal nodes, there is a unique SPNE.*

Generalized backward induction to solve for SPNE in more general *finite* dynamic games (not necessarily perfect information):

1. Look at the final subgames at the end of the game tree (no further nested subgame) and solve for NE.

2. Select one NE for each of them and replace the final subgames in the game tree by terminal payoffs equal to the NE payoffs of the players (in the relevant final subgames).

This is called the *reduced game*.

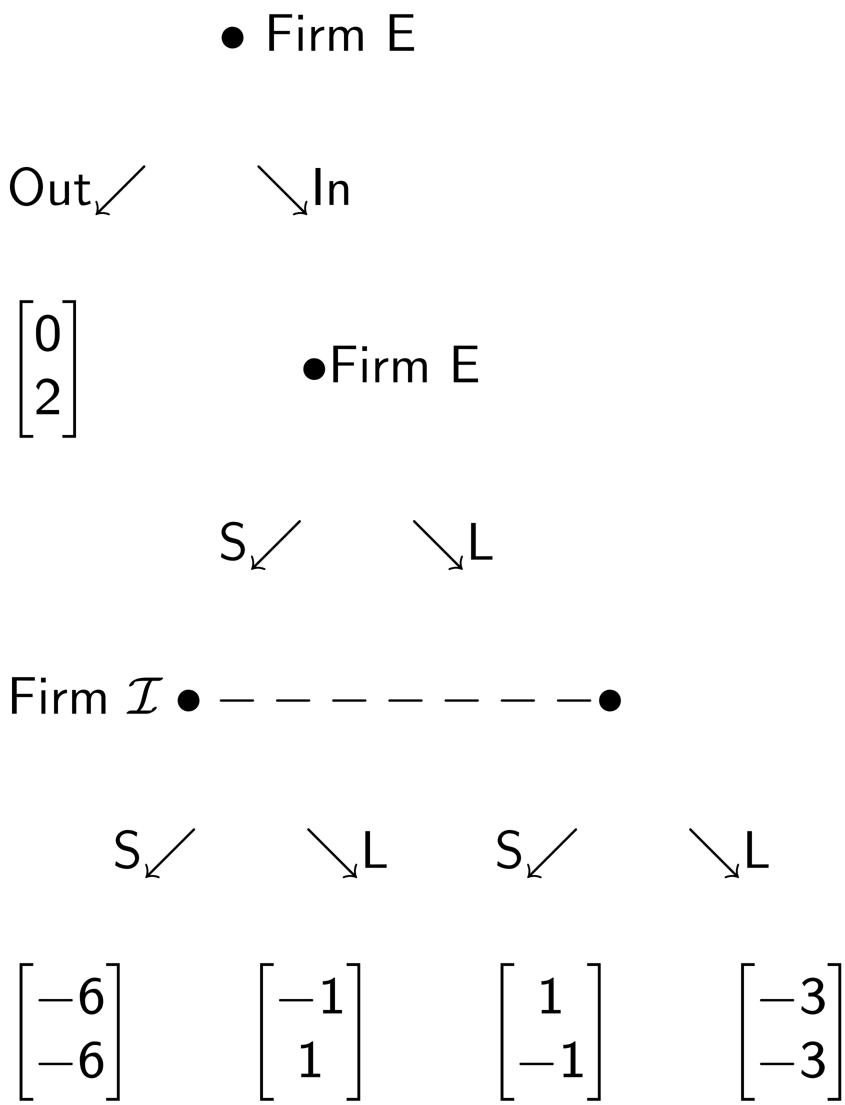
3. Now, repeat this for the reduced game & continue doing this until the moves at all information sets of the original game have been determined.

The strategies that specify the collection of moves obtained through this process constitute a SPNE.

If multiple Nash equilibria are never encountered in this generalized backward induction process, then this profile of strategies is the unique SPNE.

If multiple NE are encountered, the full set of SPNE is identifying the procedure for each possible equilibrium that could occur at the subgames.

Example.



• Firm E

Out $\swarrow$ $\qquad\qquad$ $\searrow$ In

$$\begin{bmatrix} 0 \\ 2 \end{bmatrix} \qquad \begin{bmatrix} & S & L \\ S & -6,-6 & -1,1 \\ L & 1,-1 & -3,-3 \end{bmatrix}$$

Two NE in the last subgame: $(S, L), (L, S)$.

Two reduced games:

1) $\bullet$ Firm E

Out $\swarrow$ $\searrow$ In

$\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

2) $\bullet$ Firm E

Out $\swarrow$ $\searrow$ In

$\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Two SPNE:

{(In, L if In), S if In}

{(Out, S if In), L if In}.

Finitely repeated simultaneous move game.

Consider a normal form game (simultaneous move game) Γ_N which is played repeatedly for a finite (T) number of times.

The normal form game which is played repeatedly is called a "(one) stage game" (or "one shot game").

Allow players to play mixed "strategies" in the one stage game if they wish.

After each round, all players observe the (pure) strategies actually played in the previous round and then play the next round.

This entire T- stages game is a dynamic game of complete and imperfect information.

For $t > 1$,let h_t be the history of the game (i.e., what players played) as observed till the end of $(t-1)$ rounds of play and before the t^{th} round is played.

Each possible $t = 1, ...T$, and each possible h_t (for $t > 1$) defines a distinct decision node for each player. .

In this dynamic game, the strategy of a player specifies what she is going to choose for each $t = 1, ...T$, and each possible h_t (for $t > 1$).

Payoff in the dynamic game: sum of payoffs over the T-stages (can also look at discounted sum).

Proposition. *If the stage game* Γ_N *has a unique NE, then there is a unique SPNE of the game where for all* $t = 1, ...T$, *and all* h_t, *players play the NE of the stage game.*

More generally:

Proposition. *If the stage game* Γ_N *has multiple NE, then any strategy profile of the dynamic game where for each* $t = 1, ...T$, *players play one of the NE of the stage game* *independent of* h_t, *is a SPNE.*

However, if Γ_N has multiple NE, then there may be SPNE where players do not play any of the NE of the stage game Γ_N for some t.

Example.

Suppose the following normal form game is repeated twice.

Payoff : sum of payoffs in the two stages.

$$\begin{bmatrix} & L & C & R \\ T & 1,1 & 5.0 & 0,0 \\ M & 0,5 & 4,4 & 0,0 \\ B & 0,0 & 0,0 & 3,3 \end{bmatrix}$$

The stage game has two NE: $(T, L), (B, R)$.

SPNE1: $\{(T$ in stage 1, and in stage 2, play T whatever be the history$), (L$ in stage 1, and in stage 2, play L whatever be history$)\}$

SPNE2: $\{(B$ in stage 1, and in stage 2, play B whatever be the history), $(R$ in stage 1, and in stage 2, play R whatever be history)$\}$

SPNE3: $\{(T$ in stage 1, and in stage 2, play B whatever be the history), $(L$ in stage 1, and in stage 2, play R whatever be history)$\}$

SPNE4: $\{(B$ in stage 1, and in stage 2, play T whatever be the history), $(R$ in stage 1, and in stage 2, play L whatever be history)$\}$

These four SPNE correspond to playing some NE of the stage game in each period.

SPNE5:

$$\begin{bmatrix} & L & C & R \\ T & 1,1 & 5.0 & 0,0 \\ M & 0,5 & 4,4 & 0,0 \\ B & 0,0 & 0,0 & 3,3 \end{bmatrix}$$

Strategies:

Player 1: Play M in stage 1.

In stage 2, play B if (M, C) has been played in stage 1 and play T,otherwise.

Player 2: Play C in stage 1.

In stage 2, play R if (M, C) has been played in stage 1 and play L,otherwise.

Generalized backward induction.

Subgames in the second stage are of two types:

(i) The one following (M, C) being played in stage 1

(ii) The ones following (M, C) not being played in stage 1

The indicated strategies induce NE in both classes of subgames.

The reduced game in stage 1 (given the above strategies):

$$\begin{bmatrix} & L & C & R \\ T & 2,2 & 6.1 & 1,1 \\ M & 1,6 & 7,7 & 1,1 \\ B & 1,1 & 1,1 & 4,4 \end{bmatrix}$$

The specified strategies for the first stage clearly a NE in the reduced game.

Thus, this is a SPNE.

In SPNE5, players do better than the if they played the best (or Pareto efficient) NE of the stage game twice.

They behave cooperatively in the first round (even though playing cooperatively i.e., (M, C) is not a NE in the one stage game).

In stage 2 (last period), players must play one of the two NE of the stage game as it is essentially a one shot game.

However, multiplicity of NE here allows players to incorporate the threat of playing the bad NE rather than the good one in case they do not play cooperatively in the first round.

This is a credible threat (if we ignore renegotiation possibilities).

This illustrates: *finitely repeated interaction can induce "cooperation" in early periods when there are multiple NE in the stage game.*

In infinitely repeated games, we will see that there are SPNE that involve cooperative play even though there is a unique NE in the stage game.

A deep problem with sequential rationality.

SPNE: players should play an SPNE wherever they find themselves in the game tree, even after a sequence of events that is contrary to the prediction of the theory (i.e., the actual play that ought to be induced if the players played the SPNE strategies).

Example. (Centipede game).

Finite game of perfect information.

2 players 1 & 2.

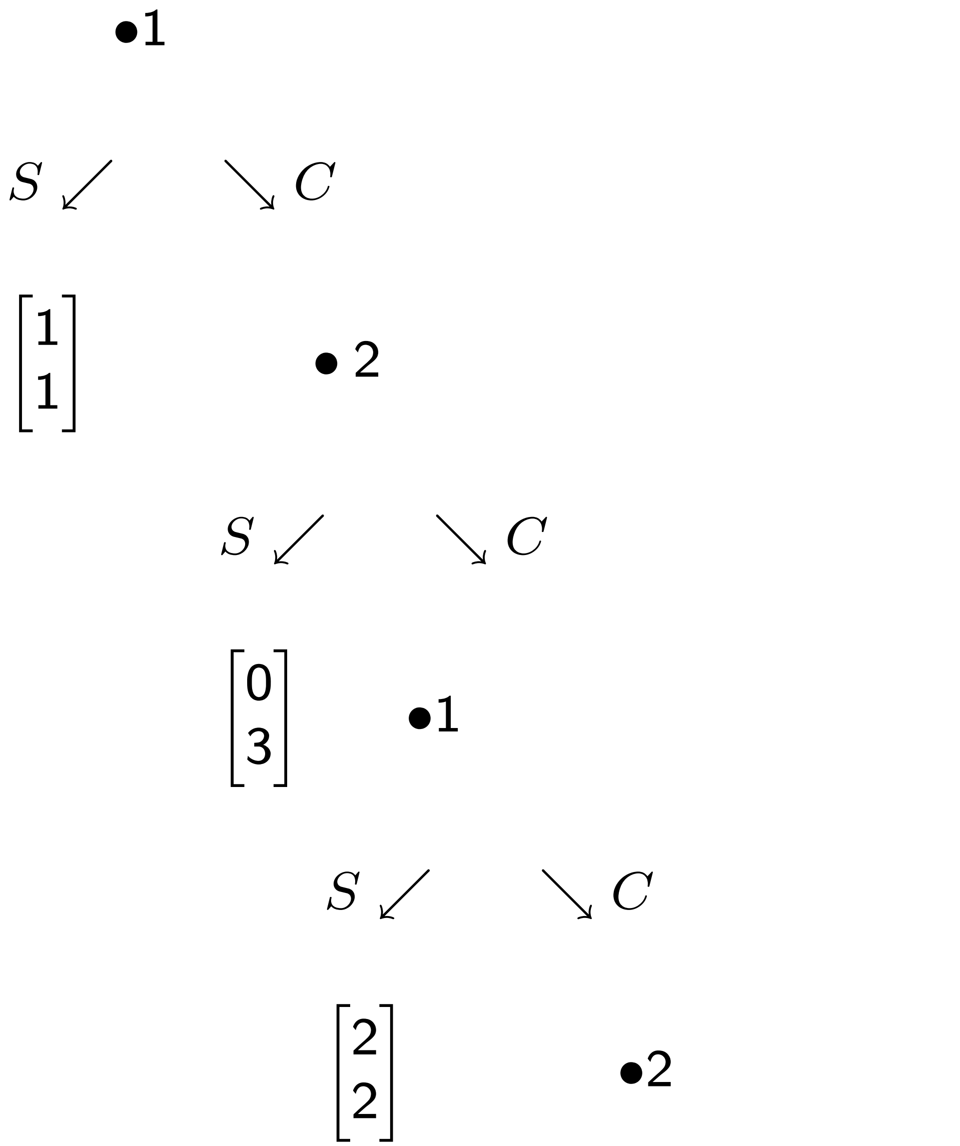
Each player starts with $1 in front.

They alternate saying "stop" or "continue".

When a player (whose turn it is to move) says "continue", $1 is taken by a referee from her pile and $2 is added to rival's pile.

When a player says "stop", play is terminated and each player receives the money currently in her pile.

Play stops in any case if both players' pile reaches $100.


•1
S
C
[1, 1]
• 2
S
C
[0, 3]
•1
S
C
[2, 2]
•2
....

..........●2

$S \swarrow \qquad \searrow C$

$\begin{bmatrix} 97 \\ 100 \end{bmatrix} \qquad \bullet\ 1$

$S \swarrow \qquad \searrow C$

$\begin{bmatrix} 99 \\ 99 \end{bmatrix} \qquad \bullet\ 2$

$S \swarrow \qquad \qquad \searrow C$

$\begin{bmatrix} 98 \\ 101 \end{bmatrix} \qquad \begin{bmatrix} 100 \\ 100 \end{bmatrix}$

Backward induction: unique SPNE is that players choose stop at each decision node where they are asked to move.

In actual play of this SPNE, play will end in the first move with player 1 stopping the game and both players getting \$1 each.

Really bad outcome, considering that they could get \$100 each if they played to continue every time.

Is the SPNE a reasonable prediction?

Player 1 says stop in stage 1, because she thinks player 2 will choose stop at her first turn.

But if player 1 *thinks that* either :

(i) player 2 is not fully rational and therefore does not compute the SPNE by backward induction

or (ii) player 2 is rational but does not know whether player 1 is rational and thus, observing player 1 choose to continue (when SPNE says player 1 should play stop), she supposes that player 1 is not rational (in the sense of playing SPNE) and therefore, if she chose to continue, player 1 would not stop the game in the next stage but would actually continue it further allowing player 2 to move again

- then it may be optimal for an actually rational player 1 to continue in the first stage.

Note that arguments like (i) or (ii) involve some contradiction to common knowledge of rationality.

SPNE denies this possibility - however, it then leaves open the question of how players think about the game and other players when they find themselves at decision nodes that ought not be reached if players played by sequential rationality.

One resolution: treat deviations from SPNE as mistakes that occur with extremely small probability and unlikely to be repeated again.

This is the approach taken by a somewhat different refinement concept called "trembling hand perfection".

Application of SPNE: Bilateral Bargaining with Alternating Offers.

Finite time horizon: $T > 1, t = 1, ...T.$

Two player 1 & 2 bargain to determine how to split a dollar.

They make offers in alternating time periods.

First, player 1 offers a split $\in [0, 1]$.

Then, player 2 either accepts or rejects.

If she accepts, the split is immediately implemented and the game ends.

If she rejects, nothing happens till period 2.

In period 2, player 2 makes an offer.

Then, player 1 accepts or rejects.

If she accepts, the split is immediately implemented and the game ends.

If she rejects, nothing happens till period 3.

In period 3, player 1 makes an offer.

And so on...

If by the end of period T, no offer is accepted, the bargaining is terminated and both players get 0.

Each player has a time discount factor $\delta \in (0, 1)$: getting \$$x$ in period t yields a (present value) payoff of $\delta^{t-1}x$ to the player.

Unique SPNE through backward induction.

In this equilibrium, players accept an offer if they are indifferent between accepting and rejecting.

Suppose T is odd.

Then, player 1 offers in period T.

In the last subgame, player 2 decides whether to accept or reject.

Optimal choice: accept any split (as the alternative is getting 0).

Seeing this, player 1 offers 0 to player 2 in period T.

So, the payoffs from equilibrium play in the subgame beginning in period T is $(\delta^{T-1}, 0)$.

Now, consider period $(T-1)$ where player 2 makes an offer.

In period $(T-1)$, player 1 knows that she can get a payoff of δ^{T-1} by rejecting and moving to period T.

So, she will accept an offer if and only if it gives her a payoff $\geq \delta^{T-1}$.

On the other hand, player 2 knows that she will get zero if the bargaining moves to period T.

So, her optimal offer to player 1 at the beginning of period $T-1$ is \$$\delta$ which will be accepted (yielding player 1 a payoff equal to δ^{T-1}).

The payoffs arising if the game reaches period $T-1$ is $(\delta^{T-1}, \delta^{T-2}(1-\delta))$.

Working backwards, in period 1, an offer will be made by player 1 that is accepted by player 2 (leaving the latter indifferent between accepting and rejecting) and player 1's payoff:

$$\begin{aligned}
v_1^*(T) &= 1 - \delta + \delta^2 - ..... + \delta^{T-1} \\
&= (1-\delta)\frac{1-\delta^{T-1}}{1-\delta^2} + \delta^{T-1} \\
&= 1 - \frac{\delta}{1+\delta}(1-\delta^{T-1})
\end{aligned}$$

and player 2's payoff

$$\begin{aligned}
v_2^*(T) &= 1 - v_1^*(T) \\
&= \frac{\delta}{1+\delta}(1-\delta^{T-1}).
\end{aligned}$$

To derive the above expressions: need to use induction on T.

Observe the first mover's advantage ($v_2^*(T) < \frac{1}{2}$) in the division of the dollar & that it shrinks as T increases.

If T is even, player 2 will make the first offer in the subgame beginning in period 2 (with $T-1$) and as the latter is a bargaining game with odd number of periods, her payoff in this subgame is $\delta v_1^*(T-1)$.

Hence, in any SPNE, player 1 makes an offer of exactly this amount to her and player 1's payoff is $[1- \delta v_1^*(T-1)]$.

As $T \to \infty$, player 1's payoff converges to $\frac{1}{1+\delta}$ while player 2's payoff converges to $\frac{\delta}{1+\delta}$.

The asymptotic share of the dollar for player 2 is increasing in δ.

As $\delta \rightarrow 1$, the asymptotic division converges to $(\frac{1}{2}, \frac{1}{2})$.

Infinite horizon: Rubinstein (1982).

Same game but no longer terminated in any finite T.

If the game goes on forever (no player accepts in any time period), both players receive zero payoff .

Unique SPNE: immediate agreement in period 1 with payoffs $(\frac{1}{1+\delta}, \frac{\delta}{1+\delta})$.

Observe: Time Stationarity.

For the proposer and the responder, the subgame beginning period 1 is exactly identical to that beginning in any odd period $t > 1$.

The subgame beginning period 2 is exactly identical to that beginning in any even period $t > 1$.

Further, the subgame starting in any odd period is identical to that starting in any even period but with the players' roles reversed.

This stationarity is used in establishing the SPNE payoffs and its uniqueness.

Let $\overline{v}_1$ denote the largest payoff that player 1 gets in any SPNE.

This is also the largest payoff that player 1 gets in any SPNE of any subgame beginning in an odd period (when evaluated in terms of the present value at commencement of the subgame).

The largest payoff that player 2 can expect in any SPNE beginning in any even period (when evaluated in terms of the present value at the commencement of the subgame) is also $\overline{v}_1$.

Claim: Player 1's payoff in any SPNE $\geq 1 - \delta\overline{v}_1$.

[Why?

If player 1's payoff in any SPNE $< 1 - \delta\overline{v}_1$, then player 1 can deviate and offer $\delta\overline{v}_1 + \epsilon$ in period 1 to player 2 and this will be accepted as player 2 can make at most $\overline{v}_1$ in any SPNE of the subgame beginning next period].

Define:

$$\underline{v}_1 = 1 - \delta\overline{v}_1.$$

Next, we claim that

$$\overline{v}_1 \leq 1 - \delta \underline{v}_1.$$

To see this, note that in any SPNE, player 2 will reject an offer in period 1 which is less than $\delta \underline{v}_1$ because she can always get at least that much payoff in any SPNE of the continuation game beginning next period.

So, if player 1 makes an offer in period 1 which is accepted, he can do no better than $1 - \delta \underline{v}_1$.

Further, if player 1 makes an offer that is rejected by player 2, then since player 2 can get a payoff of at least $\delta \underline{v}_1$ by moving to the subgame beginning in period 2, player 1 can earn no more than $\delta(1 - \underline{v}_1)$ by making an offer that is rejected.

Since $\delta(1 - \underline{v}_1) < 1 - \delta \underline{v}_1$, we have the above inequality.

Thus,

$$\begin{aligned} \overline{v}_1 &\leq 1 - \delta \underline{v}_1 \\ &= \underline{v}_1 + \delta \overline{v}_1 - \delta \underline{v}_1 \end{aligned}$$

so that

$$\overline{v}_1(1-\delta) \leq \underline{v}_1(1-\delta)$$

i.e.,

$$\overline{v}_1 \leq \underline{v}_1$$

which implies

$$\overline{v}_1 = \underline{v}_1 = v_1^{e.}.$$

Thus, SPNE payoff is uniquely determined.

Using $\underline{v}_1 = 1 - \delta \overline{v}_1$, we have

$$v_1^{e.} = \frac{1}{1+\delta}$$

So, player 2's payoff is

$$v_2^{e.} = \frac{\delta}{1+\delta}.$$

Finally, agreement is reached in period 1 in any SPNE. It is optimal for player 1 to make an offer that is exactly equal to $v_2^{e.} = \delta v_1^{e.}$.