

Radioactive decay as an exponential decay process

1 Radioactive waste*

**Unverified information from Wikipedia*

Radioactive wastes are usually by-products of nuclear power generation and other applications of nuclear fission or nuclear technology, such as research and medicine. Radioactive waste is hazardous to most forms of life and the environment, and is regulated by government agencies in order to protect human health and the environment.

Radioactivity naturally decays over time, so radioactive waste has to be isolated and confined in appropriate disposal facilities for a sufficient period of time until it no longer poses a hazard. The period of time waste must be stored depends on the type of waste and radioactive isotopes. It can range from a few days for very short-lived isotopes to millions of years for spent nuclear fuel. Current major approaches to managing radioactive waste have been segregation and storage for short-lived waste, near-surface disposal for low and some intermediate level waste, and deep burial or partitioning / transmutation for the high-level waste.

Radioactive medical waste tends to contain beta particle and gamma ray emitters. It can be divided into two main classes. In diagnostic nuclear medicine a number of short-lived gamma emitters such as technetium-99m are used. Many of these can be disposed of by leaving it to decay for a short time before disposal as normal waste. Other isotopes used in medicine, with half-lives in parentheses, include:

- Y-90 (Yttrium), used for treating lymphoma (2.7 days)
- I-131 (Iodine), used for thyroid function tests and for treating thyroid cancer (8.0 days)
- Sr-89 (Strontium), used for treating bone cancer, intravenous injection (52 days)
- Ir-192 (Iridium), used for brachytherapy (74 days)
- Co-60 (Cobalt), used for brachytherapy and external radiotherapy (5.3 years)
- Cs-137 (Caesium), used for brachytherapy, external radiotherapy (30 years)

2 Designing a storage facility

The design of a facility is going to depend on understanding how much radioactive material is being stored (and how much radiation is emitted). Suppose we start a 1 kg hunk of radioactive I-131.

- *How much I-131 will we have in 16 days?*
Half live is 8 days. So in 8 days we will have $(1 \text{ kg})/2 = 0.5 \text{ kg}$. In another 8 days we will have $(0.5 \text{ kg})/2 = 0.25 \text{ kg}$.
- *How much I-131 will we have at the end of a month, ie, 30 days?*
Mathematical model for exponential decay.

$$\frac{dP}{dt} = kP, \quad P(0) = P_0$$

We can solve this to get $P(t) = P_0 e^{kt}$ and then $P(30) = P_0 e^{k30}$
What is P_0 ? That's the initial amount of I-131, which is 1 kg.

What is the growth-rate k ? We can compute it from the 1/2-life.

$$\begin{aligned}\text{We know this: } P(8) &= \frac{1}{2}P_0 \\ \text{And this: } P(8) &= P_0 e^{k8} \\ \text{So: } P_0 e^{k8} &= \frac{1}{2}P_0 \\ \text{Giving: } k &= -\frac{1}{8} \ln 2 \quad \frac{1}{\text{days}}\end{aligned}$$

In general we have that $kt_{1/2} = -\ln 2$. It comes from solving the exponential decay ODE, without including sources or sinks, only exponential decay. Note that we have a growth rate that is negative. We could write $P' = -kP$, where k is a decay rate of $+(1/8) \ln 2$. So now we have that:

$$\begin{aligned}P(30) &= 1 \text{ (kg)} \exp\left(-\frac{1}{8} \ln 2 \left(\frac{1}{\text{days}}\right) 30 \text{ (days)}\right) \\ &\approx 0.0743 \text{ kg} \quad \text{or} \quad \approx 74 \text{ g}\end{aligned}$$

Is it safe to enter the room with 74 g of radioactive I-131? We should note that we still have a 1 kg hunk. It's just that after emitting a beta particle and some gamma radiation it is now mostly Xenon-131 with only 74 g still I-131.

Now suppose that in addition to the initial 1 kg hunk, another 10 g of I-131 is deposited each day (that's 10 g/day... note the units!).

- How much I-131 do we have after 30 days?
Is it 1 kg + 30 days (10 kg/day) = 1.3kg? No, because that simple addition ignores the decay. Is it 0.0743 kg + 30days (10 kg/day)? No, that accounts for the decay of the original chunk but not the decay each deposit.
- The solutions cannot be added! We must start with a new differential equation that accounts for both processes simultaneously!

$$\frac{dP}{dt} = kP + l, \quad k = -\frac{1}{8} \ln 2 \frac{1}{\text{days}}, \quad l = 0.010 \frac{\text{kg}}{\text{day}}.$$

Let's rewrite again noting the consistency of units.

$$\frac{dP}{dt} \frac{\text{kg}}{\text{day}} = k \frac{1}{\text{days}} P \text{ kg} + l \frac{\text{kg}}{\text{day}}$$

- Note $+l$ implies $l = 0.01$. We could have written $-l$ with $l = -0.1$. Also, do not put in the numerical values. That just makes things messy and introduces the risk error due to mistakes or rounding. Save numerical values until the end.
- Solve by separation or via and integrating factor with $P(0) = P_0 = 1 \text{ kg}$.

$$P(t) = \left(P_0 + \frac{l}{k}\right) e^{-kt} - \frac{l}{k}$$

- How much do we have after 30 days?

$$P(30) = \left(1 - \frac{0.01}{-(1/8) \ln 2}\right) e^{-(30/8) \ln 2} + \frac{0.01}{-(1/8) \ln 2} \approx 0.1812 \text{ kg}.$$

- Is it more or less than when there were no deposits? Is it safe to enter the storage room with that much radioactive I-131? Ask your doctor.
- Suppose the safe level to enter the room is $P(30) = 0.15 \text{ kg}$. What is the maximum amount of additional I-131 that we can add each day? Is there an l such that $P(30) \geq P_0$?

$$\left(P_0 + \frac{l}{k}\right) e^{-kt} - \frac{l}{k} = 0.15$$

Substitute for P_0 and k , and solve for l .

3 Compound Interest

The ideas presented above are applicable to considering compound interest problems.

1. 10,000 is invested in a bond mutual fund that has an annual interest rate of 5 %. When will the balance in the account double?
2. Suppose that in addition to the original deposit that \$ 500 per year is added to the fund. Now when does the balance double?
3. When using differential equations we have made a very fundamental assumption regarding the behavior of the system as a function of time. What is that assumption and how applicable is that to the compound interest problem posed?

Please work the above problem before looking at the solution!

1. We only have exponential growth of the initial amount.

$$P' = kP, \quad P(0) = P_0 \quad \Rightarrow \quad P(t) = P_0 e^{kt}.$$

$$t_2 \text{ is such that } P(t_2) = 2P_0$$

$$\begin{aligned} 2P_0 &= P_0 e^{kt_2} \\ t_2 &= \frac{1}{k} \ln 2 \\ &= \frac{1}{0.05} \ln 2 \\ &\approx 13.86 \text{ yrs (Save numbers for last)} \end{aligned}$$

Note, the result does not depend at all on the initial amount.

2. Now we have exponential growth and deposits.

$$P' = kP + l \quad P(0) = P_0 \quad \Rightarrow \quad P(t) = \left(P_0 + \frac{l}{k}\right)e^{kt} - \frac{l}{k}$$

$$t_2 \text{ is such that } P(t_2) = 2P_0$$

$$\begin{aligned} 2P_0 &= \left(P_0 + \frac{l}{k}\right)e^{kt_2} - \frac{l}{k} \\ t_2 &= \frac{1}{k} \ln \left(\frac{2P_0 + \frac{l}{k}}{P_0 + \frac{l}{k}} \right) \\ &\approx 8.11 \text{ yrs (Save numbers for last)} \end{aligned}$$

3. A differential equation assumes that there is differentiable solution, which implies that it is a continuous function of time. If compounding occurs “continuously” (daily), then over the course of years, that can be considered approximately continuous. However, one must then be careful of what is used for the interest rate. For example, most often an *Annual Percentage Rate* (APR) is quoted. Suppose we use the APR, compounded yearly, ie, t is in years.

$$\begin{aligned} t = 0 & \quad P(t) = P_0 \\ t = 1 & \quad P(t) = P_0 + rP_0 \text{ (Principal + total interest earned.)} \\ & \quad P(t) = P_0(1 + r) \\ t = 2 & \quad P(t) = (P_0 + rP_0) + r(P_0 + rP_0) \\ & \quad P(t) = P_0(1 + r)^2 \\ & \quad \vdots \\ t = n & \quad P(t) = P_0(1 + r)^n \end{aligned}$$

When would the principal double?

$$\begin{aligned} 2P_0 &= P_0(1 + r)^n \\ n &= \ln \left(\frac{2}{1 + r} \right) \end{aligned}$$

Suppose the APR is 5%.

$$n = \left(\frac{2}{1.05} \right) \approx 14.2 \text{ yrs.}$$

Does it make sense that this is greater than “continuous” compounding?