Exponential Growth & Decay.

1a) $\frac{dP}{dt} = (r_1 - r_2)P, \quad P(0) = 1$

$$P(t) = c e^{(r_1 - r_2)t} \Rightarrow P(0) = c e^0 = 1 \Rightarrow c = 1$$

$$P(t) = e^{(r_1 - r_2)t}$$

b) $P(t_T) = 3\cancel{P(0)}^{=1}\; 3$

$$P(t_T) = e^{(r_1 - r_2)t_T} = 3$$

$$t_T = \frac{\ln 3}{(r_1 - r_2)}$$

This should be k/1000.

c) $\frac{dP}{dt} = (\cancel{r_1 - r_2})^{r} P + k(100) \leftarrow 1\text{kg} = 1000\text{g}$

$$P' = rP + 1000k$$

$$P' - rP = 1000k \Rightarrow \text{IF: } u = e^{-rt}$$

$$\frac{d}{dt}(e^{-rt}P) = 1000ke^{-rt}$$

$$e^{-rt}P(t) - \underbrace{\cancel{e^{0}}}_{1}\underbrace{\cancel{P(0)}}_{1} = \frac{1000k}{-r}(e^{-rt} - \underbrace{\cancel{e^{0}}}_{1})$$

$$e^{-rt}P = \frac{1000k}{-r}(e^{-rt} - 1) + 1$$

$$P(t) = \frac{1000k}{-r} + \left(\frac{1000k}{r} + 1\right)e^{rt}$$

$$P(t_D) = 2\cancel{P(0)}^{=1} = 2$$

$$P(t_D) = \frac{1000k}{-r} + \left(\frac{1000k}{r} + 1\right)e^{rt_D} = 2$$

$$t_D = \frac{1}{r}\ln\left(\frac{2 + \frac{1000k}{r}}{1 + \frac{1000k}{r}}\right)$$

Note in c) we have to formulate and solve a new differential equation that account for the new process.

2a) Exp growth law $\Rightarrow$ $\frac{dP}{dt} = rP$ such that $P = P(0)e^{rt}$

Given doubling time $\Rightarrow$ $P(t_D) = 2P(0)$ with $t_D = 0.5$

Can use to solve for r:

$$P(t_D) = P(0)\, e^{r t_D} = 2P(0)$$

$$r = \frac{\ln 2}{t_D} = 2\ln 2$$

Now include new process

$$\frac{dP}{dt} = rP - 5, \quad P(0) = 10$$

Question: $P(2) = ?$

Solve ODE:

$$\frac{d}{dt}\left(e^{-rt}P\right) = -5e^{-rt}$$

$$e^{-rt}P(t) - \underbrace{\cancel{e^{0}}}_{1}\underbrace{\cancel{P(0)}}_{10} = \frac{5}{r}\left(e^{-rt} - \cancel{e^{0}}^{\,1}\right)$$

$$P(t) = \frac{5}{r} + (10 - 5/r)e^{rt} \quad \text{where } r = 2\ln 2$$

$$P(2) = \frac{5}{r} + (10 - 5/r)\, e^{r2} = \#$$

2b) Want $\frac{dP}{dt} = 0$ at equilibrium.

$$rP - K = 0$$

$$K = rP = (2\ln 2)(10) = 20\ln 2$$

3a) $\frac{dP}{dt} = -kP$ ← ODE for radioactive decay

$P(t) = P(0)e^{-kt}$

In 10 yrs, $P(10) = 0.8P(0)$. Use to find k.

$P(10) = P(0)e^{-k10} = 0.8P(0)$

$k = -\frac{1}{10}\ln(0.8)$

When do we have ½ of what we started with?

$P(t_{1/2}) = 0.5P(0)$

$P(t_{1/2}) = P(0)e^{-kt_{1/2}}$ from ODE solution

$0.5\cancel{P(0)} = \cancel{P(0)}e^{-kt_{1/2}}$

$t_{1/2} = -\frac{\ln(0.5)}{k} = +10\frac{\ln(0.5)}{\ln(0.8)} = \#$

b) when will $P(t_A) = 0.1P(0)$?

$P(t_A) = P(0)e^{-kt_A}$

$0.1\cancel{P(0)} = \cancel{P(0)}e^{-kt_A}$

$t_A = -\frac{\ln(0.1)}{k}$

Actually asks for ADDITIONAL YEARS.

t_A is the total years since $t=0$

ADD. YEARS $= -\frac{\ln(0.1)}{k} - 10 = \#$

4a) $\frac{dP}{dt} = rP$, $P(0) = 2000 - 500 = 1500$

$P(t) = 1500\, e^{rt}$ $\quad r = 0.06$

But this is the portion that earns interest.

You still have the other 500

$P_T(t) = 1500 e^{rt} + 500$

$P_T(10) = 1500\, e^{r10} + 500 \Big|_{r=0.06} = \#$

b) when do you have $2(2000) = 4000$

$P_T(t_D) = 1500\, e^{r t_D} + 500 = 4000$

$t_D = \frac{1}{r} \ln\left(\frac{3500}{1500}\right) = \frac{1}{r} \ln\left(\frac{7}{3}\right)$

c) $\frac{dP}{dt} = rP - 200$ (New process $\Rightarrow$ New ODE)

$\frac{d}{dt}\left(e^{-rt} P\right) = -200 e^{-rt}$

$e^{-rt} P(t) - \underbrace{\cancel{e^{0}}}_{1} \underbrace{\cancel{P(0)}}_{1500} = \frac{200}{r}\left(e^{-rt} - \underbrace{\cancel{e^{0}}}_{1}\right)$

$P(t) = \frac{200}{r} + \left(1500 - \frac{200}{r}\right) e^{rt}$

... plus the other 500

$P_T(t) = \left(\frac{200}{r} + 500\right) + \left(1500 - \frac{200}{r}\right) e^{rt}$

d) $\frac{dP}{dt} = rP - 200 + \left(1 + \cos 24\pi t\right)$

Solving by I.F. is possible

but involves I.B.P. (integration by parts)

Give it a try!