Math 3313: Differential Equations
Laplace transforms

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Outline

Outline

Introduction

Inverse Laplace transform

Solving ODEs with Laplace transforms

Discontinuous forcing functions

Convolution

Dirac Delta

Definition

Solution process:

Differential equation	$\longrightarrow$ Laplace transform: $\mathcal{L}$ $\longrightarrow$	Algebraic equation
$\downarrow$ solve		$\downarrow$ solve
Solution to ODE	$\longleftarrow$ Inverse laplace: $\mathcal{L}^{-1}$ $\longleftarrow$	Algebraic solution

- Idea is that using $\mathcal{L}$ and $\mathcal{L}^{-1}$ allows for easier solution.
- Allows us to tackle discontinuous functions.

Definition of $\mathcal{L}$:
$F(s)$ is the $\mathcal{L}$-Transform of $f(t)$, $t \geq 0$:

$$F(s) = \mathcal{L}[f(t)] = \int_0^{\infty} e^{-st} f(t) dt \tag{1}$$

Write this down!

Definition (cont.)

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t) dt \tag{2}$$

Write this down!

- Integration in t leaves a function of s.
- $\int_0^\infty \Rightarrow$ Improper integral.
  Must make sure the limit exists.

$$\int_0^\infty g(t)\, dt = \lim_{b\to\infty} \int_0^b g(t)\, dt$$

If the limit exists, *convergence*, otherwise, divergence

Laplace of e^{at}

ex. Use the integral definition to find the Laplace transform of e^{at}.

Substitute $f(t) = e^{at}$ and integrate.

$$F(s) = \frac{1}{s-a}, \quad s > a \tag{3}$$

Given $f(t) \overset{\mathcal{L}}{\to} F(s)$, there is an inverse Laplace operator so that we can take $F(s)$ back to $f(t)$.

$$f(t) \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} F(s) \quad \text{and} \quad \mathcal{L}^{-1}\left[\mathcal{L}\left[f(t)\right]\right] = f(t) \tag{4}$$

$$e^{at} \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} \frac{1}{s-a} \tag{5}$$

Laplace is a linear operator

$$\begin{aligned}\mathcal{L}[c_1 f_1 + c_2 f_2] &= c_1 \mathcal{L}[f_1] + c_2 \mathcal{L}[f_2] \\ &= c_1 F_1 + c_2 F_2\end{aligned}$$

$$\begin{aligned}\mathcal{L}^{-1}[c_1 F_1 + c_2 F_2] &= c_1 \mathcal{L}^{-1}[F_1] + \mathcal{L}^{-1}[F_2] \\ &= c_1 f_1(t) + c_2 f_2(t)\end{aligned}$$

ex.

$$F(s) = \frac{5}{s-2} + \frac{8}{3}\frac{1}{s+3} \tag{6}$$

Find $f(t)$.

Examples

ex. $f(t) = 1$.
Find $F(s)$.

ex. $f(t) = \cos(bt)$
Find $F(s)$.

ex. $F(s) = \frac{3+3s}{s^2+10}$
Find $f(t)$.

ex. $f(t) = t^n$.
Find $F(s)$.

ex. $f(t) = 2t^5$
Find $F(s)$.

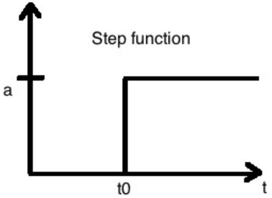
ex. $F(s) = \frac{6}{s^4}$
Find $f(t)$.

Step function

ex.

$$f(t) = \begin{cases} 0 & 0 \leq t \leq t_0 \\ a & t_0 \leq t \end{cases}$$

Find $F(s)$.



ex. Heaviside- or Unit-step function

$$H(t - t_0) = \begin{cases} 0 & 0 \leq t \leq t_0 \\ 1 & t_0 \leq t \end{cases}$$

$$H(t - t_0) \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} \frac{1}{s} e^{-st_0}$$

Some properties

Linearity: already done.

Shifting property:

$$e^{ct} f(t) \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} F(s-c)$$

$$\text{Mult by exp in } t \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} \text{Shift in } s.$$

Derive using the integral definition.

ex.

$$F(s) = \frac{2}{(s-2)^3} \tag{7}$$

Invert

Some properties (cont)

Derivative of $F(s)$:

$$-tf(t) \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} \frac{dF(s)}{ds}$$

Derive using the integral definition.

ex.

$$\mathcal{L}[t\cos(bt)] = ? \tag{8}$$

Use the derivative property.

Derivative of $x(t)$

We want to solve ODEs

$$ax'' + bx' + cx = f(t)$$

We will need to know the Laplace transform of x' and x''.

$$\frac{dx}{dt} \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} s\mathcal{L}[x] - x(0) \tag{9}$$

Derive using the integral definition.

Using integration by parts twice, we can show that

$$\frac{d^2x}{dt^2} \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} s^2\mathcal{L}[x] - sx(0) - \frac{dx(0)}{dt} \tag{10}$$

The ICs are part of the result for Laplace of derivatives.

Table of Laplace Transform Pairs

Given on quizzes and exams

$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$$
$$\sin(a \pm b) = \sin a \cos b \pm \sin b \cos a$$

Table of Laplace Transforms

$f(t)$	$\mathcal{L}[f(t)] = F(s) = \int_0^\infty e^{-st} f(t) dt$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
$\sin bt$	$\frac{b}{s^2+b^2}$
$\cos bt$	$\frac{s}{s^2+b^2}$
t^n	$\frac{n!}{s^{n+1}}$
$f(t-a)u(t-a)$	$e^{-as}F(s) \quad a > 0$
$g(t)u(t-a)$	$e^{-as}\mathcal{L}[g(t+a)] \quad a > 0$
$e^{ct}f(t)$	$F(s-c)$
$\frac{df}{dt} = f'(t)$	$sF(s) - f(0)$
$\frac{d^2f}{dt^2} = f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$tf(t)$	$-F'(s)$
$\int_0^t f(\tau)d\tau$	$\frac{1}{s}F(s)$
$\int_0^t f(\tau)g(t-\tau)d\tau$	$F(s)G(s)$

Outline

Inverse Laplace integral operator

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi i} \int_c e^{st} F(s)\, ds$$

where c is a Bromwich contour in the complex s plane.

For any given $F(s)$, substitute into the integral definition for the inverse Laplace and compute the line integral.

Ack! Instead, use the table of transform pairs whenever possible.

Partial fractions

Goal: break $F(s)$ into simpler functions each invertible using the table of transform pairs.

Partial fractions: the thing that breaks $F(s)$ into pieces if $F(s)$ is a rational polynomial the the degree of the denomination greater than the numerator.

$$F(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_a s + b_0}{s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}, \quad n > m.$$

ex.

$$F(s) = \frac{2s}{s^2 - 5s + 6} \tag{11}$$

Partial fractions then use table.

- Factor the denominator (find roots).
- Expand using partial fractions.
- Multiply by the denominator.
- Equate powers of s.
- Solve for the coefficients.
- Use the table

Outline

Solution process

Differential equation	$\longrightarrow$ Laplace transform: $\mathcal{L}$ $\longrightarrow$	Algebraic equation
		$\downarrow$ solve
Solution to ODE $x(t)$	$\longleftarrow$ Inverse laplace: $\mathcal{L}^{-1}$ $\longleftarrow$	Algebraic solution $X(s)$

Consider a constant-coefficient ODE

$$ax'' + bx' + cx = f(t), \quad x(0) = x_0, \quad x'(0) = v_0$$

- Apply the Laplace operator.
- Use the ICs
- Solve for $X(s)$.
- Invert

Challenge is typically $\mathcal{L}^{-1}$.

Examples

ex.

$$x'' - x' - 6x = 0, \quad x(0) = 2, \quad x'(0) = -1 \tag{12}$$

Solve using Laplace transforms.

ex.

$$x'' + 2x' + 5x = \cos t, \quad x(0) = 0, \quad x'(0) = 1 \tag{13}$$

Solve using Laplace transforms.

ex.

$$x'' + x = \cos t, \quad x(0) = 0, x'(0) = 0 \tag{14}$$

Solve using Laplace transforms.

Outline

Not so nice forcing

For

$$ax'' + bx' + cx = f(t),$$

if f is "nice" we can use MUC and/or perhaps Var of Par.

Suppose $f(t)$ is not so nice, specifically, a piece-wise continuous or discontinuous function. The other methods may be possible treating each piece separately and then patching the solutions together. However, Laplace transforms can often find the answer in a straightforward way.

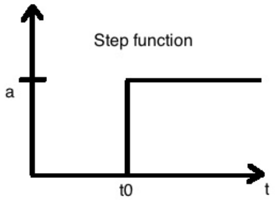
Representing piecewise continuous $f(t)$

Heaviside- or unit-step function.

$$H(t) = \begin{cases} 0 & 0 \leq t < t_0 \\ 1 & t_0 \leq t \end{cases}$$

$$H(t - t_0) \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} \frac{1}{s} e^{-st_0}$$

- H is "off" for $t < t_0$ then "on" for $t \geq t_0$.
- "Switching" time is t_0.



ex.

$$f(t) = H(t-a) - H(t-b), \quad a < b. \quad \text{Sketch it.} \tag{15}$$

ex.

$$f(t) = \sin(t-a)[H(t-a) - H(t-b)], \quad a < b. \quad \text{Sketch it.} \tag{16}$$

ex.

$$f(t) = \textit{sketch}. \quad \text{Construct function} \tag{17}$$

ex.

$$f(t) = 3H(t) + H(t-2) + 4\left(e^{-(t-4)} - 1\right) H(t-4) \quad \text{Sketch it.} \tag{18}$$

Laplace of piecewise continuous $f(t)$

In general, we can construct piecewise continuous $f(t)$ by adding together the separate pieces:

$$f(t) = f_1(t - c_1)H(t - c_1) + f_2(t - c_2)H(t - c_2) + \dots .$$

To find $\mathcal{L}[f(t)]$ we need to find

$$\mathcal{L}[f(t-a)H(t-a)] = \text{Use the integral definition to compute.} \tag{19}$$

$$f(t-a)H(t-a) \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} e^{-sa}F(s) \tag{20}$$

ex.

$$\mathcal{L}[e^{3t}H(t-4)] = \text{Sketch and transform.} \tag{21}$$

Derive the alternative formula

$$g(t)H(t-a) \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} e^{-sa}\mathcal{L}[g(t+a)] \tag{22}$$

Examples

ex.

$$\mathcal{L}[\sin(t)H(t-\frac{\pi}{2})] = \text{transform} \tag{23}$$

ex.

$$X(s) = \frac{e^{-s}}{s^2+1} - \frac{e^{-2s}}{s^2+2} \tag{24}$$

Invert

ODEs with discontinuous forcing

$$\begin{array}{ccc} \text{Differential equation} & \longrightarrow \text{Laplace transform: } \mathcal{L} \longrightarrow & \text{Algebraic equation} \\ \downarrow \text{difficult} & & \downarrow \text{solve} \\ \text{Solution to ODE } x(t) & \longleftarrow \text{Inverse laplace: } \mathcal{L}^{-1} \longleftarrow & \text{Algebraic solution } X(s) \end{array}$$

Process with Laplace remains the same, just a bit more work with $\mathcal{L}$ and $\mathcal{L}^{-1}$.

ex.

$$x'' - 3x' + 2x = g(t) = \begin{cases} 0 & t < 1 \\ 3 & 1 \leq t < 2 \\ 0 & 2 \leq t \end{cases} = 3[H(t-1) - H(t-2)] \tag{25}$$

$$x(0) = 0, \quad x'(0) = 0$$

Solve

ex.

Solve the LC-circuit problem with cosine forcing that turns on at $t = 0$ and off at $t = 3\pi/2$.

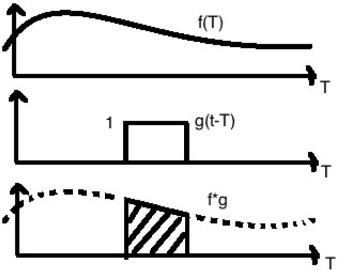
Outline

Convolution: definition

How much do f and g have in common and when?

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)\, d\tau$$

- Multiply $f(\tau)$ ...
- by a shifted version of $g(\tau)$ ...
- t is the amount of the shift ...
- Determine the resulting area.



$$f * g = \int_0^t f(\tau)g(t-\tau)\, d\tau = \int_0^t g(\tau)f(t-\tau)\, d\tau = g * f$$

fix f and shift g = fix g and shift f

$$\mathcal{L}[f * g] = \mathcal{L}\left[\int_0^t f(\tau)g(t-\tau)\, d\tau\right] = F(s)G(s)$$

Examples

ex.

$$X(s) = \frac{1}{s(s^2+1)} \tag{26}$$

Invert

ex.

$$X(s) = \frac{1}{(s^2+1)^2} \tag{27}$$

Invert

ODEs and the convolution

Consider

$$ax'' + bx' + cx = f(t), \quad x(0) = x_0, \quad x'(0) = v_0 \tag{28}$$

Apply the Laplace transform.

For simplicity assume $x_0 = 0$ and $v_0 = 0$.

$$X(s) = F(s)\frac{1}{as^2 + bs + c} = F(s)G(s) \quad \text{where } G(s) = \frac{1}{as^2 + bs + c}$$

$G(s)$ contains info from the ODE. Called the Transfer Function.

Use convolution to invert.

$$\mathcal{L}^{-1}[F(s)] = f(t) \qquad \mathcal{L}^{-1}[G(s)] = g(t)$$

$$\mathcal{L}^{-1}[\text{Transfer function}] = \text{ Impulse response}$$

$$x(t) = \int_0^t f(\tau)g(t-\tau)\, d\tau$$

Solution machine

- Given the ODE: $L[x(t)] = f(t)$
  $L[x]$ represents the left-hand side with all the $x's$.
- The ODE operator L determines $G(s)$ and hence $g(t)$. KNOWN!
- The right hand side is the forcing $f(t)$. KNOWN.
- The solution for ANY forcing f can be found by using the convolution.
- Plug in a new f and integrate.
- Same idea as variation of parameters.

ex.

$$x'' - 16x = f(t), \quad x(0) = 0, \quad x'(0) = 1. \tag{29}$$

Solve and express the result using a convolution integral.

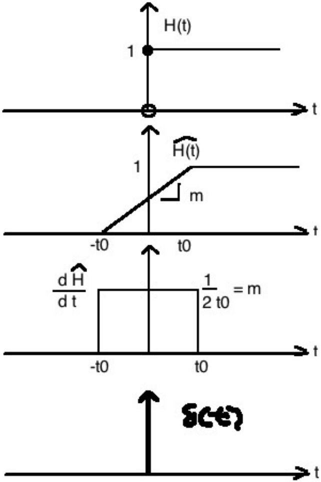
ex. $f(t) = e^t$. Substitute into the integral and integrate.

ex. $f(t) = e^t[H(t-1) - H(t-2)]$. Substitute into the integral and integrate.

Outline

Definition (lazy) of Dirac Delta

- Recall unit step $H(t)$
  Jumps instantaneously from 0 to 1.
- Consider gradual change with $\hat{H}(t)$.
  Increases over interval $-t_0$ to t_0.
- Consider the derivative of $\hat{H}(t)$.
  Slow is 0, then m, then 0.
- Take the limit as $\hat{H} \to H$.
  m (slope) $\to \infty$. $2t_0$ (width) $\to 0$.
- Then $\frac{d\hat{H}(t)}{dt} \to \frac{dH}{dt} = \delta(t)$



$\delta(t)$:a function with 0 width, infinite height, located at $t = 0$.

Properties

Dirac delta located at $t = a$ (instead of 0).

$$\begin{aligned} \delta(t-a) &= 0, \quad \text{for } t \neq a \\ \delta(0) &= \text{undefined (infinite) for} t = a. \end{aligned}$$

$$\begin{aligned} \int_{-\infty}^{t} \delta(\tau)\, d\tau &= \int_{-\infty}^{t} \frac{dH(\tau)}{d\tau}\, d\tau \\ &= H(t) - H(-\infty) \\ &= H(t) - 0 \\ &= 1 \text{ if } t > 0 \end{aligned}$$

$\delta(t-a)$: located at $t = a$, has 0 width, infinite height, and area of 1.

Sifting property and Laplace

Sifting property:

$$\int_{-\infty}^{\infty} g(t)\delta(t-a)\, dt = g(a) \tag{30}$$

Integral of g with $\delta(t-a)$ gives the value of g at $t=a$.
Derive

Laplace:

$$\delta(t-a) \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} e^{-sa} \tag{31}$$

Derive

Apply to ODEs

(Borrowing from the slide on Convolution)

Consider

$$ax'' + bx' + cx = f(t), \quad x(0) = 0, \quad x'(0) = 0. \tag{32}$$

Apply the Laplace transform.

$$X(s) = F(s)\frac{1}{as^2 + bs + c} = F(s)G(s) \quad \text{where } G(s) = \frac{1}{as^2 + bs + c}$$

$G(s)$ contains info from the ODE. Called the Transfer Function.

Consider

$$ax'' + bx' + cx = \delta(t), \quad x(0) = 0, \quad x'(0) = 0. \tag{33}$$

Apply the Laplace transform.

$$X(s) = \frac{1}{as^2 + bs + c} = G(s)$$

$G(s)$ is the Laplace transform of the Impulse response $g(t)$.

Some examples

ex. Consider a mass-spring system with mass of 1 kg, damping coefficient of 2 kg/s and spring constant of 2 kg/s^2. The mass is initially at rest. At $t = 3$ it is given a sharp impulse with a hammer. What is the resulting motion?
Model and solve.

ex. Marching soldiers have sometimes been told to break stride and march out of step when crossing a bridge. Why? Suppose the bridge can be modeled as a mass-spring system with $m = 1$ and $k = 1$ and the soldiers footsteps a sequence of delta-dirac functions. Thus,

$$x'' + x = \sum_{k=1}^{\infty} \delta(t - 2k\pi), \quad x(0) = x'(0) = 0. \tag{34}$$

Solve.