Math 3313: Differential Equations
Second-order ordinary differential equations

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Outline

Outline

2nd order ODE

We've considered 1st order

$$\frac{dx}{dt} = f(x,t)$$

2nd order

$$\frac{d^2x}{dt^2} = f(x, \frac{dx}{dt}, t)$$

LInear, 2nd order, non-homogeneous

$$a_2(t)\frac{d^2x}{dt^2} + a_1(t)\frac{dx}{dt} + a_0(t)x = f(t)$$

Linear, 2nd order, constant-coefficient, non-homogeneous

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = f(t)$$

If $f(t) = 0$, ODE is homogeous.

Newton's 2nd law

$$\begin{aligned}
\frac{d}{dt}\text{Momemtum} &= \sum \text{forces} \\
\frac{d}{dt}(\text{mass times velocity}) &= \sum \text{forces} \\
\frac{d}{dt}\left(m(t)\frac{dx}{dt}\right) &= \sum \text{forces} \\
\frac{dm}{dt}\frac{dx}{dt} + m(t)\frac{d^2x}{dt^2} &= \sum \text{forces}
\end{aligned}$$

If $m(t) = \text{constant}$, then $\frac{dm}{dt} = 0$.

$$\begin{aligned}
m\frac{d^2x}{dt^2} &= \sum \text{forces} \\
\text{mass times acceleration} &= \sum \text{forces}
\end{aligned}$$

If forces depend only on time, we can simply integrate twice

$$m\frac{d^2x}{dt^2} = f(t) \quad (\text{"phys 101"})$$

In general, forces depend on position x and speed $\frac{dx}{dt}$.

Modeling the mass-spring

Outline

2 initial conditions

2nd order linear ODE (' = prime means derivative)

$$x'' + p(t)x' + q(t)x = f(t)$$

If constant-coeficient

$$ax'' + bx' + cx = f(t)$$

To find $x(t)$ we have to "integrate" twice.

- Expect 2 unknown constants.
- Need 2 ICs

$$x(t_0) = x_0, \quad x'(t_0) = v_0$$

Form of the solution

Homogeneous: $f(t) = 0$

$$x_h'' + p(t)x_h' + q(t)x_h = 0, \quad \text{where } x_h(t) = c_1 x_1(t) + c_2 x_2(t)$$

x_h is the sum of two *linearly independent solutions*, x_1 and x_2.

$$\text{Fundamental set } = \{x_1, x_2\}, \quad x_2 \neq c x_1$$

Nonhomogeneous with particular solution $x_p(t)$ due to forcing $f(t)$.

$$x_p'' + p(t)x_p' + q(t)x_p = f(t)$$

The complete solution is the sum of homogeneous and particular

$$x(t) = x_h(t) + x_p(t) = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$$

Linear Independence

A set of functions $\{x_1(t), x_2(t), \ldots, x_n(t)\}$ are linearly **dependent** if $k_1 x_1 + k_2 x_2 + \ldots + k_n x_n. = 0$ for some set of constants $k_n \neq 0$, for all n. Otherwise, **independent**.

Wronskian:

$$W(t_0) = x_1(t_0)x_2'(t_0) - x_1'(t_0)x_2(t_0) \tag{1}$$

Outline

2nd order ODEs as a system

We've considered

$$\frac{d^2x}{dt^2} = f\left(t, x, \frac{dx}{dt}\right), \quad x(0) = x_0, \;\; x'(0) = v_0.$$

Let the first derivative be a new variable:

$$\text{Let} \quad \frac{dx}{dt} = y$$

$$\text{Then} \quad \frac{d^2x}{dt^2} = \frac{dy}{dt} = f(t, x, y)$$

So instead of a single 2nd order ODE, we have two first order ODEs.

$$\begin{aligned} \frac{dx}{dt} &= y, \quad x(0) = x_0 \\ \frac{dy}{dt} &= f(t, x, y), \quad y(0) = x'(0) = v_0 \end{aligned}$$

Examples

ex.

$$x'' + p(t)x' + q(t)x = f(t) \tag{2}$$

Write as a system.

ex.

$$x'' + xx' + x^2 = 0 \tag{3}$$

Write as a system.

ex.

$$x''' + (1 - x^2)x' + x = 0 \tag{4}$$

Write as a system of 3 first order ODEs

Numerical solvers need ODEs as systems. –> matlab demo

Outline

Exponential solutions

Linear, constant coefficient differential equations have exponential solutions.

$$ax'' + bx' + cx = 0$$

Recall 1st order, linear, constant-coefficient

$$x' = kx \quad \Rightarrow \quad x \sim e^{kt}$$

Apply to higher-order: let $x = e^{rt}$.
Substitute and solve for r.
Note:

$$\frac{d^n}{dt^n} e^{rt} = r^n e^{rt}$$

Apply to 2nd order L-CC

$$a(r^2 e^{rt}) + b(re^{rt}) + c(e^{rt}) = 0$$
$$(ar^2 + br + c)e^{rt} = 0$$

The characteristic equation determines the values of r

$$ar^2 + br + c = 0 \tag{5}$$

- Turned ODE into algebra.
- Quadratic for $r \Rightarrow$ two values for r: r_1 and r_2.

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} \quad \text{.... sort of... depends...}$$

Details and three different cases.

Summary

Given

$$ax'' + bx' + cx = 0$$

Let $x = e^{rt}$.

- $r = r_1, r_2$: real & distinct

$$x = c_1 e^{r_1 t} + c_2 e^{r_2 t} \quad \text{exponential decay and/or growth}$$

- $r = r_0, r_0$: real repeated.

$$x = c_1 e^{r_0 t} + c_2 t e^{r_0 t} \quad t \exp r_0 t \text{ is second solution}$$

- $r = \alpha \pm i\beta$: complex conjugate

$$x = c_1 e^{\alpha t} \cos(\beta t) + c_2 t e^{\alpha t} \sin(\beta t) \quad \text{exponentials with oscillations}$$

- Important!! $e^{i\beta t} \Leftrightarrow \cos(\beta t)$ and $\sin(\beta t)$.

Outline

Char. equation with *n* roots

nth order, linear, constant coefficient

$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \ldots + a_1 \frac{dx}{dt} + a_0 x = 0$$

Let $x = e^{rt}$.

$$a_n r^n + a_{n-1} r^{n-1} + \ldots + a_1 r + a_0 = 0 \tag{6}$$

Characteristic equation for r = nth degree polynomial.
Can we finds it's n roots?

(i) Some are real and distinct:

$$(r - r_1)(r - r_2) \ldots (r - r_j)(\text{rest of poly}) = 0$$

$$r = r_1,\ r_2, \ldots, r_j,\ \text{the rest}$$

$$x = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \ldots + c_j e^{r_j t} + \text{ the rest}$$

n roots (continued)

ii) Some have multiplicity m. Suppose r_0 is a root m times.

$$(r - r_0)^m(\text{rest of poly}) = 0$$

Via *Reduction or Order* m times:

$$x = c_1 e^{r_0 t} + c_2 t e^{r_0 t} + \ldots + c_{m-1} t^{m-1} e^{r_0 t} + \text{ the rest}$$

ii) Some are complex-conjugate . . . with multiplicity m. $r = \alpha \pm i\beta$ are each a root m-times.

$$\begin{aligned} (r - (\alpha + i\beta))^m (r - (\alpha - i\beta))^m (\text{rest of poly}) &= 0 \\ (r^2 - 2\alpha r + (\alpha^2 + \beta^2))^m (\text{rest of poly}) &= 0 \end{aligned}$$

Number of roots is $2m$.

$$\begin{aligned} x = \; & e^{\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t) + \\ & e^{\alpha t} t (c_3 \cos \beta t + c_4 \sin \beta t) + \ldots + \\ & e^{\alpha t} t^{m-1} (c_* \cos \beta t + c_* \sin \beta t) + \text{ the rest} \end{aligned}$$

How do we factor the polynomial?

- "Obvious"
- Recognize standard form (Pascal's triangle).
- Find one root and the factor with synthetic division. (Optional and esoteric case.)

$$ODE + a_0 x = 0$$

$$\text{Let } x = e^{rt} \Rightarrow Poly(r) + a_0 = 0.$$

$$\text{Factor: } (r - r_1)(r - r_2)\dots(r - r_n) = 0$$

$$a_0 = r_1 r_2 \dots r_n \Rightarrow \text{ Try integer roots of } a_0.$$

- Numerical (Newton's method)

Examples

Solve the following:

ex.

$$x''' - x'' - 6x' = 0 \tag{7}$$

ex.

$$x'''' + 3x''' + 3x'' + x' = 0 \tag{8}$$

ex.

$$x'''' + 8x'' + 16x = 0 \tag{9}$$

ex.

$$x''' + x'' - 6x' + 4x = 0 \tag{10}$$

$r_1 = 1$. Factor... continue.

Outline

Terms to know

Simple harmonic motion
Polor coordinates
Amplitude & phase
Free response $\Rightarrow$ no friction/resistance/damping
Underdamped vs. critically damped vs. overdamped.

Outline

Quick review

2nd order, linear, variable coefficient, **non-homogeneous**

$$x'' + p(t)x' + q(t)x = f(t)$$

Constant coefficient

$$ax'' + bx' + cx = f(t)$$

Homogeneous problem:

$$ax_h'' + bx_h' + cx = 0, \quad \text{let } x = e^{rt} \quad \Rightarrow x_h = c_1 x_1 + c_2 x_2$$

Non-homogeneous problem:

$$ax_p'' + bx_p' + cx_p = f(t) \quad \Rightarrow \text{Find } x_p \text{ somehow.}$$

Complete solution

$$x = x_h + x_p = c_1 x_1 + c_2 x_2 + x_p$$

Apply ICs LAST to x. Do not apply ICs to x_h.

When to use MUC

IF the ODE is **constant coefficient**...
AND if $f(t) = \ldots$

- polynomial
- exponential
- sin or cos
- combinations of the above

Then find x_p with MUC.

Theory and examples.

Summary MUC

Given a linear ODE

$$L(x) = F_{1m}(t)e^{a_1 t} + F_{2n}(t)e^{a_2 t} + \dots$$

- Solve for x_h.

$$L(x_h) = 0 \quad \text{(note the roots } r \text{ and their multiplicities.)}$$

- Solve for x_p.
  - Superposition: consider each part of f separately. Find solutions and add.
  - $f_{jm} = F_{jm}(t)e^{a_j t}$
  - Guess $x_{pj} \sim P_{jm}(t)e^{a_j t}t^k$
  - $k =$ number of times r is a root if $a = r$.
  - If $f_j \sim \sin\beta t$ or $\cos\beta t$ always guess both.
- $x_p = x_{p1} + x_{p2} + \dots$.
  Substitute into ODE and find the coefficients.
- $x = x_h + x_p$
  Apply ICs last.

Outline

No damping & harmonic forcing

How does the mass-spring respond to a force $f(t)$?

$$mx'' + bx' + kx = f(t)$$

- No damping (no resistance, no dissipation): $b = 0$.
- Harmonic forcing: $f(t) \sim \sin \omega_f t$ or $\cos \omega_f t$.

$$mx'' + kx = F_0 \cos \omega_f t$$

- F_0 is the forcing amplitude and ω_f is the forcing frequency.

Solve using MUC.
Resonance: if $b = 0$, then when $\sqrt{k/m} = \omega_f$.

WITH damping & harmonic forcing

$$mx'' + bx' + kx = F_0 \cos \omega_f t$$

Solve using MUC.

- How does damping change the solution?
- How does damping change resonance?

Outline

Introduction

General, 2nd order, linear, homogeneous, ODE

$$a_2(t)x'' + a_1(t)x' + a_0(t)x = 0$$

- Coefficients depend on t
- Problem if $a_2(t = t_s) = 0$.
  ODE: 2nd order "becomes" 1st order near $t = t_s$.
- If $a_2(t_s) = 0$ or if either $a_1(t_s)$ or $a_0(t_s)$ is undefined, then $t = t_s$ is a singular point.

There is no general method to solve ODEs with variable coefficients.

- Exact solutions: only for specific cases (e.q. Euler).
- Approximate solutions: require series.

Euler Equation (2nd order)

$$at^2x'' + btx' + cx = 0 \tag{11}$$

The power of t matches the number of derivatives.

Solution: let $x = t^r$
Substitute and find equation for r.

Derive the Indicial equation for r;

$$ar^2 + (b - a)r + c = 0 \tag{12}$$

Are the values of r . . .

- Real and distinct.
- Real and repeated.
- Complex conjusgate.

What do the solutions look like in each case?

Euler examples

ex.

$$2t^2x'' + tx' - 15x = 0. \tag{13}$$

Solve.

ex.

$$t^2x'' + 7tx' + 13x = 0 \tag{14}$$

Solve.

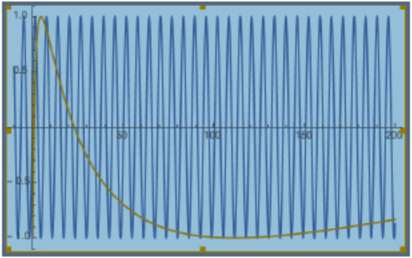
Higher order Euler

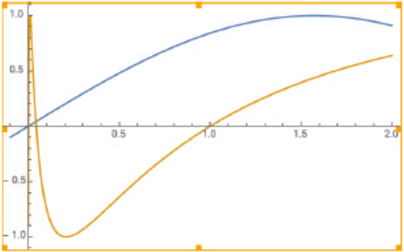
$$a_n t^n \frac{d^n x}{dt^n} + a_{n-1} t^{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \ldots + a_1 t \frac{dx}{dt} + a_0 x = 0 \tag{15}$$

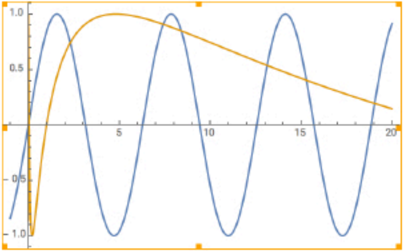
let $x = t^r$.
Higher order polynomial for r.

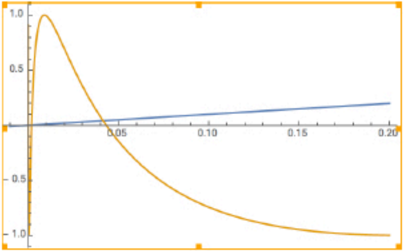
Compare $\sin(x)$ to $\sin(\ln(x))$

(Plots made using Wolfram Alpha)









Outline

Review of solutions

2nd-order, linear, ODE

$$x'' + p(t)x' + q(t)x = f(t)$$

We've consider:

$$\begin{aligned} ax'' + bx' + cx &= f(t) \sim F_m(t)e^{at}, \quad a \text{ possibly complex} \\ at^2x'' + btx' + cx &= 0 \end{aligned}$$

Now allow for arbitrary coefficients ($p(t)$ and $q(t)$) and forcing ($f(t)$).
Solution = homogeneous + particular

$$x = x_h + x_p = c_1x_1 + c_2x_2 + x_p$$

Var. of Par. method

IF we can find $\{x_1, x_2\}$ such that

$$x_h = c_1 x_1 + c_2 x_2$$

Then we can find x_p FOR ANY $f(t)$ using variation of parameters.

$$x_p = v_1(t)x_1(t) + v_2(t)x_2(t) \tag{16}$$

- v_1 and v_2 are the "parameters" that are now variable functions.
- v_1 and v_2 are unknown.
- x_1 and x_2 are from the homogeneous solution and known.
- Substitute proposed solution and find 2 conditions for the 2 unknowns v_1 and v_2.

Var. of Par. example

ex.

$$x'' + x = \tan t \tag{17}$$

Find x_h and then x_p.

See next slide for discussion of alternative var par formula.

ex.

$$x'' - x = t^{-2} e^t \tag{18}$$

Solve.

ex.

$$t^2 x'' + t x' - x = t^{1/2} \tag{19}$$

What is $f(t)$? Solve.

Alternative var par formula

Solve for v_1 and v_2 from the two conditions.

$$\begin{aligned} v_1' x_1 + v_2' x_2 &= 0 \\ v_1' x_1' + v_2' x_2' &= f. \end{aligned}$$

From the first equation

$$v_1' = -v_2' \frac{x_2}{x_1}$$

Substitute into the second equation

$$\begin{aligned} -v_2 \frac{x_2}{x_1} x_1' + v_2' x_2' &= f \\ v_2' (x_1 x_2' - x_1' x_2) &= f x_1 . \end{aligned}$$

Note that the term in parenthesis is the Wronskian.

$$v_2' = \frac{f x_1}{W(x_1, x_2)} \quad \Rightarrow \quad v_2 = \int \frac{f x_1}{W(x_1, x_2)} \, dt \tag{20}$$

$$v_1' = -\frac{f x_1}{W(x_1, x_2)} \frac{x_2}{x_1} = -\frac{f x_2}{W(x_1, x_2)} \frac{x_2}{x_1} \quad \Rightarrow \quad v_1 = -\int \frac{f x_2}{W(x_1, x_2)} \, dt \tag{21}$$

Alternative var par formula

Substitute results for v_1 and v_2 into x_p.

$$x_p(t) = -\int \frac{f(s)x_2(s)}{W(s)}\, ds\; x_1(t) + \int \frac{f(s)x_1(s)}{W(s)}\, ds\; x_2(t)$$

Bring x_1 and x_2 into the integral.

$$x_p(t) = \int \frac{[x_1(s)x_2(t) - x_1(t)x_2(s)]}{W(s)} f(s)\, ds \tag{22}$$

Call that big fraction $G(t, s)$ whose parts are all known. G is a known function in terms of the homogeneous solution.

$$x_p(t) = \int G(t, s) f(s)\, ds$$

G is referred to as the Green's function or Influence function or Impulse response.

The integral is a *solution machine*. Given $L(x) = f(t)$. Find G. Then for any f just plug into the integral and out pops x_p.