Math 3337 Homework

Instructions:

- *Staple or bind* all pages together. *DO NOT* dog ear pages as a method to bind.
- Hand-drawn sketches should be neat, clear, of reasonable size, with axis and tick marks appropriately labeled. All figures should have a short caption explaining what they show and describe.
- Please label each problem and write clearly. If your work can not be read, or the problem not found, it will not be graded.
- *Mathews* = *Vector Calculus* by P.C. Mathews.
  Stewart = *Essential Calculus, Early Transcendentals*, J. Stewart.

1 Vectors

- Reading: P.C. Mathews, Secs. 1.1-1.4.
- Additional problems: Stewart, Secs. 10.1-10.5.

1. Find all three angles of the triangle with vertices at $A : (0, 0, 0)$, $B : (4, 2, 1)$, $C : (1, 2, 4)$.
2. Find the work done by the force $\mathbf{F} = (2, 6, 6)$N as it moves a mass of $m = 40$kg between the points $A : (3, 4, 0)$ and $B : (5, 8, 0)$, where distance is measured in meters.
3. Find the component of $\mathbf{a} = (4, 0, -3)$, in the direction of $\mathbf{b} = (1, 1, 1)$.
4. Find the area of the triangle with vertices given in problem 1).
5. Find the torque (moment vector) created by a force $\mathbf{F} = (2, 1, 0)$N acting at a point $A : (1, 3, 0)$, to rotate a lever of mass $m = 89.328$ about point $B : (4, -1, 0)$,where distance is measured in meters.
6. Are the vectors $(4, 2, 9)$, $(3, 2, 1)$ and $(-4, 6, 9)$ linearly independent? (All vectors may not lie in the same plane.)

2 Scaler and vector functions and fields.

- Reading: Mathews, Sec. 1.6
- Additional problems: Stewart, Sec. 10.7 and 13.1

1. (a) Find the equation of the line between the two points $A : (1, 2)$ and $B : (3, 4)$, i.e, $y = f(x)$.
   (b) Find a parametric space-curve representation for this line using the parameter t, i.e., $\mathbf{r}(t) = \langle x(t), y(t)\rangle$.
2. For the segment of the line $y = 2x + 1$, $0 \le x \le 2$, find two different parameterizations of the form $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, $a \le t \le b$.
3. Find the equation for the plane determined by the points $A : (4, -3, 1)$, $B : (6, -4, 7)$ and $C : (1, 2, 2)$.
4. Given $\mathbf{r}(t) = 2t\mathbf{i} + (8 - 2t^2)\mathbf{j}$ for $-2 \le t \le 2$:

(a) Sketch the curve C determined by $\mathbf{r}(t)$ and indicate the orientation.

(b) Sketch $\mathbf{r}(t)$ for $t = -2,\ -1,\ 0,\ 1,\ 2$.

5. Sketch the curve C determined by $\mathbf{r}(t)$ and indicate its orientation.

   (a) $\mathbf{r}(t) = \langle \cos t, t + \pi \rangle, \quad -\pi \leq t \leq \pi$

   (b) $\mathbf{r}(t) = e^t \cos t\mathbf{i} + e^t \sin t\mathbf{j}, \quad 0 \leq t \leq \pi$

6. (a) Sketch the scaler field $f(x, y) = x^2 - y$.

   (b) Sketch the vector field $\mathbf{f}(x, y) = (x + y, -x)$.

3 Line integrals

- Reading: Mathews, Secs. 2.1 and 2.2.
- Additional problems: Stewart, Secs. 13.2 and 13.3.

1. Evaluate the line integrals
$$\int_C f(x, y)ds, \text{ where:}$$
   (a) $f(x, y) = x^3 + y, \quad x = 3t, \quad y = t^3, \quad 0 \leq t \leq 1.$

   (b) $f(x, y) = xy^{2/5}, \quad x = \frac{t}{2}, \quad y = t^{5/2}, \quad 0 \leq t \leq 1.$

2. Evaluate the line integral below on the curve C given by the line segment from $(0, 0, 0)$ to $(1, 2, 3)$.
$$\int_C xyz\ ds$$

3. Evaluate the line integrals
$$\int_C \mathbf{F} \cdot \mathbf{dr}, \text{ where:}$$
   (a) $F = (xy, x^2y^2)$, where C is the quarter circle from $(2, 0)$ to $(0, 2)$ with center at the origin. *For this problem only, set up the integral but you do not have to evaluate it.*

   (b) Same F as above but with C the straight line segment from $(2, 0)$ to $(0, 2)$.

   (c) $F = (x - y, y - z, z - x)$, $C : \mathbf{r}(t) = (2 \cos t, t, 2 \sin t)$ with t in $[0, 2\pi]$. Sketch the curve C.

   (d) $F = (e^x, e^{-y}, e^z)$, $C : \mathbf{r}(t) = (t, t^2, t)$ from $(0, 0, 0)$ to $(1, 1, 1)$

4. Determine the work done by a force $F = \langle 2, 5 \rangle$ moving a particle once around the ellipse $x^2/4 + y^2 = 4$. Does your answer make physical sense?

4 Surface Integrals

- Reading: Mathews, Secs. 2.3.
- Additional problems: Stewart, Secs. 13.6 and 13.7.

1. Compute
$$\int_S f(x, y)dS, \text{ where}$$

(a) $f(x,y) = \cos x + \sin y$ and $S:\ x+y+z=1$ in the first quadrant. Note, it is easy to find the normal vector to the plane without even parameterizing. You still need to parameterize to evaluate the integral, but you don't have to work on the cross-product of the derivative components.

(b) $f(x,y) = x^4 + y^4$, and $S:\ \mathbf{r}(u,v) = (5\cos u, 5\sin u, v)$ with $0 \le u \le \pi$ and $-0.2 \le v \le 0.2$. Also, make a sketch of the surface. Note, the integrals of $\sin^4 u$ and $\cos^4 u$ can be found on the inside cover of your calculus text.

2. Compute

$$\int_S \mathbf{f} \cdot \mathbf{n}\, dS, \text{ where}$$

(a) $\mathbf{f}(x,y,z) = (x^2, e^y, 1)$ and $S:\ x+y+z=1,\ x \ge 0,\ y \ge 0,\ z \ge 0$.

(b) $\mathbf{f}(x,y,z) = (\sinh(yz), 0, y^4)$ and $S:\ \mathbf{r} = (u, \cos v, \sin v)\ -4 \le u \le 4$, and $0 \le v \le \pi$. Also, make a sketch of the surface.

(c) $\mathbf{f}(x,y,z) = (y^3, x^3, z^3)$ and $S:\ x^2 + 4y^2 = 1,\ x \ge 0,\ y \ge 0$ and $0 \le z \le h$.

3. The surface integral

$$\int_S \mathbf{f} \cdot \mathbf{n}\, dS, \text{ where}$$

is also referred to as the flux integral. If $\mathbf{f}$ represents the flow velocity of a fluid, and $\mathbf{n}$ represents the unit outward normal to a surface S, then the flux integral computes the volume of fluid crossing the surface per unit time, i.e., the units are vol/time.

Suppose $\mathbf{F} = \langle 1, 0, 0 \rangle$ and S is a sphere centered at the origin with radius 3337. What will be the value of the flux integral, i.e., what is the volume of fluid cross the surface per time? Remember, direction is important. Note, no integral need be computed to answer this question?

5 Volume Integrals

- Reading: Mathews, Secs. 2.4.
- Additional problems: Stewart, Secs. 12.5.

Find the total mass of a distribution f in the region V.

$$m = \int\int\int_V f dV$$

1. $f = x^2 + y^2 + z^2$, $V:\ |x| \le 1,\ |y| \le 3,\ |z| \le 2$
2. $f = \frac{1}{3}(x^2+y^2)^2$, V : the cylinder $x^2 + y^2 \le 9$, $-3 \le z \le 3$.
3. $f = 4z$, V : region in first octant bound by $y = 1 - x^2$ and $z = x$.
4. Near sea level, the density f of the earth's atmosphere at a height of z meters can be approximated as $f(z) = 1.225 - 0.0000113z$ kg/m^3. Approximate the mass of a cubic kilometer of the atmospher that has one face on the surface of the earth.

6 Div, Grad, Curl and all that

- Reading: Mathews, Chap. 3.
- Additional problems: Stewart, Secs. 11.6 and 13.5.

1. The flow of heat in a temperature field is in the direction of maximum decrease of temperature. For the temperature fields $T(x, y)$ below:
(i) Find the direction of maximum heat flow in general.
(ii) Find the direction of heat flow at the specified point P.
(ii) Determine if the temperature field satisfies *Laplace's equation*, $\nabla^2 T = 0$. Laplace's equation is a partial differential equation for the steady-state temperature in a spatial region.

   (a) $T(x, y) = \cos x \cosh y, \quad P : (\frac{\pi}{2}, 1)$.

   (b) $T(x, y, z) = \sin(x + z), \quad P : (\frac{\pi}{8}, 1, \frac{\pi}{8})$.

2. For the vector fields $\mathbf{u}$ given below, find the corresponding potential (scaler) field f, where $\nabla f = \mathbf{u}$.

   (a) $\mathbf{u} = \langle yz, xz, xy \rangle$

   (b) $\mathbf{u} = \langle -3e^{-3x} y^2, 2ye^{-3y^2} \rangle$

3. Find a conservative vector field that has the potential function $f(x, y) = y^2 e^{-3x}$.

4. Calculate the divergence of the vector field $\mathbf{u} = e^x \mathbf{i} + ye^{-x} \mathbf{j} + 2z \sinh x \mathbf{k}$.

5. For the flow velocity vector $\mathbf{u} = x\mathbf{i}$, find the position vector for particles within the flow. Show that the flow is compressible.

6. For the flow velocity vector $\mathbf{u} = (x, y, -z)$, find the position vector for particles withing the flow. Is the flow irrotational?

7. Show that any vector field $\mathbf{u}$ that is *conservative* is also irrotational.

7 Green's Theorem

- Reading: Mathews, Sec. 5.2. Green's Thm. is actually the 2D version of the more general Stokes Thm., which will be the next topic covered. Thus, I would suggest waiting until we finish Stokes Thm. before reading Sec. 5.2.
- Additional problems: Stewart, Secs. 13.4.

In problems 1-4, evaluate the line integrals below using Green's Theorem. If the line integral around the specified closed contour is 0, then see if you can find a corresponding potential function U such that $\mathbf{u} = \nabla U$, thus proving that u is conservative.

$$\oint_C \mathbf{u} \cdot dr$$

1. $\mathbf{u} = (x^2 e^y, y^2 e^x)$, C : the rectangle with vertices $(0, 0),\ (2, 0),\ (2, 3),\ (0, 3)$.
2. $\mathbf{u} = (2xy^3, 3x^2 y^2)$, C : $\ x^4 + y^4 = 1$.
3. $\mathbf{u} = (\sin y, \cos x)$, C : the boundary of the triangle with vertices $(0, 0),\ (\pi, 0),\ (\pi, 1)$.
4. $\mathbf{u} = (\cosh y, -\sinh x)$, C : the boundary of the region given by $1 \leq x \leq 3,\ x \leq y \leq 3x$.

5. Find the area enclosed by the ellipse by following the steps indicated.

$$\frac{1}{16}x^2 + \frac{1}{9}y^2 = 1.$$

Finding the area inside the ellipse can be done using a complicated double integral. Alternatively, we can make use of Green's Theorem in reverse. Specifically, we want to compute

$$A = \int\int_R 1\, dx\, dy.$$

(i) Find a vector $\mathbf{u} = (u_1, u_2)$ such that $\partial u_2/\partial x - \partial u_1/\partial y = 1$.
(ii) Parameterize the ellipse.
(iii) Evaluate the line integral

$$\oint_C \mathbf{u} \cdot dr$$

8 Stoke's Theorem

- Reading: Mathews, Sec. 5.2.
- Additional problems: Stewart, Secs. 13.8.

1. Evaluate the line integral below using Stoke's theorem.

$$\oint_C \mathbf{u} \cdot d\mathbf{r}, \quad \mathbf{u} = (y, xz^3, -zy^3), \quad C: \; x^2 + y^2 = 4, z = -3.$$

2. Evaluate the line integral below by direct parameterization and by using Stoke's theorem.

$$\oint_C \mathbf{u} \cdot d\mathbf{r}, \quad \mathbf{u} = (e^z, e^z \sin y, e^z \cos y),$$

$$C: \text{ edge of surface } S \text{ formed by } z = y^2, \;\; 0 \le x \le 4, \;\; 0 \le y \le 2.$$

9 Divergence Theorem

- Reading: Mathews, Sec. 5.1.
- Additional problems: Stewart, Secs. 13.9.

Evaluate the surface integrals below using the divergence theorem.

$$\oiint_S \mathbf{u} \cdot \mathbf{n}\, dS = \iiint_V \nabla \cdot \mathbf{u}\, dV$$

1. $\mathbf{u} = (x^2, 0, z^2)$, S : the surface of the box formed by $|x| \le 1$, $|y| \le 3$, $|z| \le 2$.
2. $\mathbf{u} = (\cos y, \sin x, \cos z)$, S : the surface of the cylinder $x^2 + y^2 \le 4$, $0 \le z \le 2$.
3. $\mathbf{u} = (4x, x^2 y, -x^2 z)$,
   S : the surface of the tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$.
4. $\mathbf{u} = (x^3, y^3, z^3)$, S: surface of the sphere $x^2 + y^2 + z^2 = 9$.

10 Review of ODEs

- Review App. A and old notes.
- Practice problems.
- You should be able to solve:
  First-order, separable,
  First-order, linear, solve with integrating factor,
  Second-order, constant coefficient.

11 Intro. to PDEs

In Logan, Sec. 1.1

Problems 1, 3, 4, 5, 6, 7

Some notes:

- Problem 1 asks that you use a computer algebra program to plot solutions; this includes programs such as Maple, Matlab or Mathematica. Alternatively, a graphing calculator or other graphing software will work just as well. I have put links to three web based graphing programs in the "Course Materials" section of Blackboard.
- For 3: When you integrate the partials with respect to x, the "constants" will be functions of t.

12 Advection Equ. and Method of Characteristics

In Logan, Sec. 1.2

Problems 2, 5b (i.e., the second one), 7, 8.

Some notes:

- For 2: Note that we solved this problem in class for a general initial condition $u(x, 0) = F(x)$. Here it is given that $F(x) = e^{-x^2}$.
- For 7: When you solve the ODE for u, you will need to use an integrating factor. To do the integral on the right-hand-side you will need to integrate by parts.
- For 8: The PDE is similar to the Advection-Decay problem but with a different decay term on the right-hand side.
- Do not save these for the last minute or you will be up late!!

13 Acoustics and the wave equation

- In class we derived the pde for a taught string. This is also discussed in the first part of Logan, Sec. 1.5. You should also skim the section on how the wave equation describes acoustic waves.
- Sec. 1.5, problems 3 & 4.
- Our discussion of the solutions of the wave equation followed Logan, Sec. 2.2.

- Sec. 2.2, problems 1 & 4.
  For problem 1), start from the solution for u as
$$u(x,t) = F(x-ct) + G(x+ct).$$
  Apply the initial conditions to find F and G in terms of
$$u(x,0) = f(x), \;\; u_t(x,0) = g(x).$$
  Specifically,
$$\begin{aligned} u(x,0) = F(x) + G(x) &= f(x) \text{ and taking the derivative w/rt } x, \\ F'(x) + G'(x) &= f'(x). \; * \\ u_t(x,t) = F'(x-ct)(-c) + G'(x+ct)(+c) & \\ u_t(x,0) = -cF'(x) + cG'(x) &= g(x). \; * \end{aligned}$$
  Use both equations * to find F' and G', then integrate.

14 Diffusion Equation & Boundary Conditions

- Sec. 2.1, problems 1(a) & 2.
  - For 1(a): Consider the integral form of the solution given by Eq. (2.8). Simplify the integrand and limits for the give $\phi(x)$ (when inserted into the integral $\phi(x) \to \phi(y) = 1$. Then in the integral make the change of variable $y \to r$ using
$$r = \frac{x-y}{\sqrt{4kt}}.$$
    Rewrite the integral using the $\text{erf}(z)$ function given on page 61.
  - For 2: Again consider the integral form of the solution. Take the absolute value and use the hint.
- Sec. 1.3, problems 5 & 6.
  - For 5: "Steady-state" implies $u_t = 0$.
  - For 6: The steady-state problem is linear and constant coefficient. Thus, it will have exponential solutions. Rewrite your final result using hyperbolic sinh and cosh.
$$\cosh\theta = \frac{1}{2}(e^{\theta} + e^{-\theta}), \quad \sinh\theta = \frac{1}{2}(e^{\theta} - e^{-\theta}),$$

15 Laplace's Equation

- Sec. 1.8, problems 1, 2 & 4.
  - For 1: The domain is a disk but it is **not** cylindrically symmetric because the boundary condition depends on θ. However, the problem doesn't ask you to solve Laplace's equation. You only need to state the value of u at the origin and the origin is equidistant from all the points on the boundary.

16 Classification and Well Posedness

- Sec. 1.9, problems 2,3 & 5.
  - Just classify the PDE, that is, determine its type. You do NOT have to solve any of these problems or do any of the extra work requested in the text.
- Sec. 2.3. No problems due.

17 Fourier Series

- Sec. 3.2, problems 1,2,5.
  - For 1: Simply check that $\cos(nx)$ and $\cos(mx)$ are orthogonal by computing the integral as we did in class. Note that the case n and m equal to zero must be check separately. You do NOT have to do the second part of the problem with $\cos(n\pi x/l)$ because we did that in class.
  - For 2: Goal is to show that the set of g_n constructed by Gramm-Schmidt are orthogonal. The proof is the same as in linear algebra with vectors. For functions, we have the notation below

$$(f,g) = \int_a^b f(x)g(x)dx \text{ and } \|g\|^2 = \int_a^b g(x)^2 dx.$$

    However, evaluating integrals is not needed. We will use proof by induction.

    1. Given that
    $$g_2 = f_2 - \frac{(f_2, g_1)}{\|g_1\|^2} g_1,$$
    show that $(g_1, g_2) = 0$.
    2. Assume it is true for g_n. That is, given the formula/definition for
    $$g_n = f_n - \sum_{i=1}^{n-1} \frac{(f_n, g_i)}{\|g_i\|^2} g_i$$
    assume that $(g_n, g_i) = 0$ for $i = 1, 2, \ldots n-1$. Thus, g_n is assumed orthogonal to all the previous functions.
    3. Show that it is true for g_{n+1}. That is, given that
    $$g_{n+1} = f_{n+1} - \sum_{i=1}^{n} \frac{(f_{n+1}, g_i)}{\|g_i\|^2} g_i,$$
    show that $(g_{n+1}, g_n) = 0$.

    Thus, you showed it is true for g_1 and g_2. Then you showed that if it is true for g_n it is true for g_{n+1}. Therefore, it must also be true for g_3. And if true for g_3, also g_4. And so on. The final point being that the sine and cosine functions are not the only valid sets of orthogonal functions that can be used as basis functions. Depending on the application and geometry, there are many different sets of basis functions $\phi_n(x)$. A brief glance at the exercises in the book mentions others.

- Sec. 3.3, problem 5.
- Optional (will not be graded): In class we discussed the Fourier Sine series for the function x and observed "ringing" or the Gibbs Phenomena. Calculate by hand the Fourier coefficients for x on the interval $[0, 2\pi]$. That is, calculate the b_n in

$$x = \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi}{l} x), \qquad l = 2\pi.$$

18 Separation of Variables – Dirchlet BCs

- Warm up problem. The following will not be collected and the solutions are provided. Work it through on your own then compare against the solutions. Being neat and organized really helps to prevent getting lost.

$$u_t = 3u_{xx}$$
$$u(0,t) = u(2,t) = 0, \quad u(x,0) = 1 - (x-1)^2$$

- Sec. 4.1, problems 1, 3 and 4.
  - For 1: Because $f(x)$ equals 0 are part of the interval the integrals for determining the Fourier coefficients will simplify a bit. Then approximate u based on only the first four terms, i.e.,

    $$u \approx u_1 + u_2 + u_3 + u_4.$$

    Pick a value of t and then plot as a function of x using a graphing calculator, maple, mathematica, or try the "Equations/plot" tool at http://www.quickmath.com/. Now pick a higher value of t and repeat. You do not need to print out your results; a quick/neat/labelled sketch will do. Finally, do not worry about "estimating the error"; you can skip that last part.
  - For 3: The Eigenvalue/function problem should be in the x direction; this is the direction with the homogeneous (zero) BC. Treat the y direction as you did t in the heat equation, i.e., do it second and evaluate the BCs in y using orthogonality. Your solutions in the y direction will be of the form

    $$G = c_1 e^{\frac{n\pi}{l}y} + c_2 e^{-\frac{n\pi}{l}y}, \text{ or } G = c_1 \cosh(\frac{n\pi}{l}y) + c_2 \sinh(\frac{n\pi}{l}y)$$
  - For 4: Just a different ODE for $G(t)$. Otherwise the method is the same. The extra condition will determine the type of roots obtained when solving the characteristic equation for $G(t)$. You should find that for all values of n the roots are complex. Thus, G will be of the form

    $$G = e^{-\frac{k}{2}t}(c_1 \cos \beta_n t + c_2 \sin \beta_n t)$$

19 Separation of Variables – Neuman & Robin BCs

- Sec. 4.2, problems 1 & 2.