What's is a *dynamical system*?

Phase Space / State Space

- Abstract space whose coordinates indicate the state of the system.

Dynamical System

- Phase space . . . plus . . .
- A *rule* that takes system from one state to a different state . . . plus . . .
- An *initial state* to seed the rule.

A differential equation is a dynamical system (continuous).

- All the possible values of $x(t)$ comprise the *phase space*. The "space" is a 1D line.
- The differential equation is the "rule", which indicates how $x(t)$ evolves.

$$\frac{dx}{dt} = f(x)$$

- The initial condition is the "seed" that indicates where to start.

$$x(t_0) = x_0$$

A *map* is a dynamical system (discrete).

- Phase space: x_n. (n is discrete time.)
- Rule: $x_{n+1} = f(x_n)$
- Seed: $x_{n_0} = x_0$.

Nonlinear differential equations

Nonlinear differential equations

$$\dot{x} = f(x), \quad \text{where } f(x) \text{ is nonlinear in } x.$$

e.g. $f(x) = x^{1/2}$ or x^3 or $\sin(x)$.
Nonlinearity is determined by x not t.
Nonlinear equations arise naturally in applications are so are unavoidable.

$$\text{Newton's laws for pendulum:} \quad \ddot{\theta} + \sin\theta = 0$$

Solving nonlinear differential equations is difficult if not impossible.
Can't necessarily find $x(t) =$ something.

Nonlinear DEs difficult?

Why?

- Linearity and superposition fail.
- Existence and uniqueness may fail.
- Long term predictability may fail = Chaos.
  Butterfly in Brazil $\Rightarrow$ hurricane in the US.
- Sensitive to parameter changes = Nonhyperbolic.
  Bifurcations.

We won't always be able to just "solve for $x(t)$.

- We will make approximaitons.
- We will use graphical analysis.
- We will use numerical simulations.
- We need a strong vocabulary. Have to learn the DS language.

Nonlinearity makes the world interesting!

Systems of ODEs

Any n^{th}-order ODE can be written as a system of n-first order ODEs.
ex.

$$\ddot{x} + x = 0$$

ex.

$$\ddot{x} + x\dot{x} + x^2 = 0$$

ex.

$$\dddot{x} + 5\ddot{x} + (1 - x^2)\dot{x} + x = 0$$

General n-D system

$$
\begin{aligned}
\dot{x}_1 &= f_1(x_1, x_2, \ldots, x_n, t), \quad x_1(t_0) = x_{10} \\
\dot{x}_2 &= f_2(x_1, x_2, \ldots, x_n, t), \quad x_2(t_0) = x_{20} \\
\ldots & \quad \ldots \\
\dot{x}_n &= f_n(x_1, x_2, \ldots, x_n, t), \quad x_n(t_0) = x_{n0}
\end{aligned}
$$

Introduce vector notation with:

$x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ and $f(x,t) = (f_1(x,t), f_2(x,t), \ldots, x_n(t))$

Then write as

$$x(\dot{t}) = f(x,t), \quad x(t_0) = x_0, \quad x \in \mathbb{R}, \quad f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$$

- x is a vector of n $\mathbb{R}$eal numbers x_j.
- f takes as argument the n $\mathbb{R}$eals x_j and a $\mathbb{R}$eal t and produces a vector of n $\mathbb{R}$eals f_j.

Nonautonomous ODEs

Given $\dot{x} = f \ldots$
If $f(x,t)$ depends on t, then nonautonomous.
If $f(x)$ depends only on x, then autonomous.
n-dimensional NONauto $\equiv (n+1)$-dimensional auto.
ex.

$$\dot{x} = f(x,t) = x\cos(t)$$

Let $y = t$ then $\dot{y} = 1$.

$$\begin{aligned} \dot{x} &= x\cos(y) \\ \dot{y} &= 1 \end{aligned}$$

Why important?
The n-D nonauto system will have properties similar to an $(n+1)$-D auto.
Phase space will be $(n+1)$-D.

If Autonomous set initial time to 0.

ex. ODE is the same, solution just shifted.

$$\dot{x}(t) = x^2(t), \quad x(1) = x_0$$

Shift time to $t + 1$.

$$x(\dot{t+1}) = x^2(t+1)$$

$$\text{Let } \; y(t) = x(t+1)$$

$$\dot{y}(t) = y^2(t), \quad y(0) = x(0+1) = x(1) = x_0$$

In other words

$$y(0) = x(1), \quad y(1) = x(2), \quad , y(2) = x(3), \ldots$$

ex. Can't shift the origin because the ODE (rule) depends on time

$$\dot{x}(t) = tx^2(t), \quad x(1) = x_0$$

$$x(\dot{t+1}) = (t+1)x^2(t+1)$$

$$\text{Let } \; y(t) = x(t+1)$$

$$\dot{y}(t) = (t+1)y^2(t) = ty^2(t) + y^2(t)$$

ODE has an extra term.

Definition

CHAOS: Aperiodic long-term behavior in a deterministic system that exhibits sensitive dependence on initial conditions.

APERIODIC: Never repeats.

- Not an equilibrium point, periodic orbit, quasiperiodic, etc.
- Yet not unbounded. Trajectories are "attracted" to a "strange" limit set where it never repeats.

Definition

CHAOS: Aperiodic long-term behavior in a deterministic system that exhibits sensitive dependence on initial conditions.

DETERMINISTIC: Not due to noise.

- Stochastic: random kicks/noise. For a fixed IC don't know what will happen.
- Given **exact** knowledge of the IC, the asymptotic ($t \to \infty$) behavior is determined exactly. Just "solve."

$$\dot{x} = f(x), \quad x(t_0) = x_0 \quad \Rightarrow x(t) = \Phi(t, t_{,0}, x_0)$$

Definition

CHAOS: Aperiodic long-term behavior in a deterministic system that exhibits sensitive dependence on initial conditions.

Sensitive Dependence: (The main issue.)

- Given same system as above but slight change in IC.

$$\dot{x} = f(x), \quad x(t_0) = x_0 + \delta_0, \quad \delta_0 \ll 1.$$

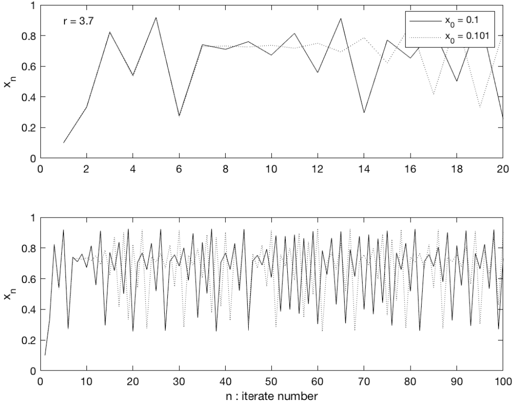
- Find that
$$|\phi(t, t_0, x_0 + \delta_0) - \phi(t, t_0, x_0)| \neq O(\delta_0)$$
- In non-chaotic systems small errors not likely to be a big concern.
- If chaotic, a small error in the IC can lead to total unpredictability.
- In experiments: measurement error.
  In numerics: round off error.
  Both impossible to eliminate.

Example with Logistic Map with $r = 3.7$

- ICs differ as follows:

$$\frac{0.101 - 0.1}{0.1} = 1\%$$

- For small n, error is small. By $n = 20$, uncorrelated.



Liapunov Exponent
Measuring sensitive dependence

Given

$$\begin{aligned} x_{n+1} &= f(x_n), \quad x|_{n=0} = x_0 \\ x_{n+1} &= f(x_n), \quad x|_{n=0} = x_0 + \delta_0 \end{aligned}$$

Initial error: $|(x_0 + \delta_0) - x_0| = \delta 0$
After n iterates: $|f^{n)}(x_0) + \delta_0) - f^{(n)}(x_0)| = \delta_n$
At each iterate the error changes $\delta_0, \delta_1, \delta_2, \ldots$.
Define the Liapunov Exponent λ:

$$|\delta_n| = |\delta_0| e^{\lambda}$$

Can show that as $n \to \infty$ that

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f'(x_i)$$

Liapunov Exponent

Consider the "local Liapunov Exponent"

$$|\delta_1| = |\delta_0| e^{\lambda_L}$$

$$\lambda_L = \ln |\frac{\delta_1}{\delta_0}| = \ln |f'(x_0)|$$

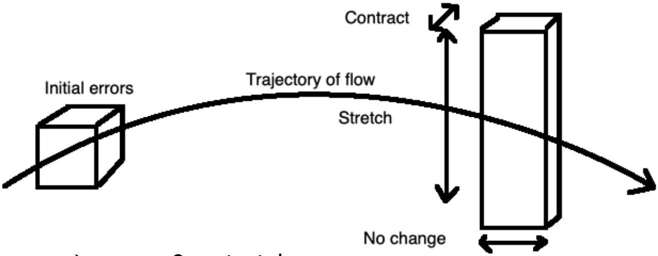
Measures the "stretching" after 1 iterate.

The Lyapunov Exponent λ measures the MEAN stretching of all of the local errors.

- It is not the measure of the growth of a single error.
- Instead, at each iterate remeasure the local "stretching" and average.

CHAOS $\Rightarrow \lambda > 0$.

Condition for chaos



$$\begin{aligned} \lambda_s &> 0 \quad \text{stretch} \\ \lambda_f &= 0 \quad \text{No change in direction of flow} \\ \lambda_c &< 0 \quad \text{contract} \end{aligned}$$

However,

$$|\lambda_c| > |\lambda_s|$$

Net contraction is greater than net stretching.

Strange Attractor

- There are many vague definitions.
- Attractor:

$$\lim_{n \to \infty} \in \text{strange set}$$

- What is strange?
  Stretch, contract AND FOLD.
  Fractal structure
  Non-integer dimension
  Self-simularity across scales