8.7.9

1) $\dot{r} = r(1-r) \quad \dot{\theta} = 1$

a) $\theta = t + \theta_0$

$\theta(0) = 0 + \theta_0 = \pi/2$

$\theta = t + \pi/2$

$\frac{dr}{r(1-r)} = dt$

Partial Fractions

$\left(\frac{1}{r} + \frac{1}{1-r}\right)dr = dt$

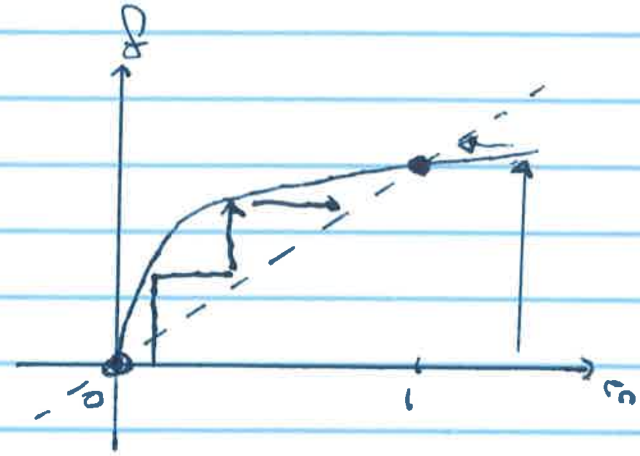
$\ln r - \ln(1-r) = t + c$

$\frac{r}{1-r} = ce^t$

$\vdots$

$r(t) = \frac{r_0 e^t}{(1-r_0) + r_0 e^t}$

b) $r(2\pi) = \frac{r_0 e^{2\pi}}{(1-r_0) + r_0 e^{2\pi}}$



$\Rightarrow r_{n+1} = \frac{r_n e^{2\pi}}{(1-r_n) + r_n e^{2\pi}}$

(Note, that $\theta_0 = \pi/2$ doesn't make a difference)

Note, in general we should consider r neg as well. However, in this case r corresponds to the radius of a L.C., which is positive.

c) $r_f = \frac{r_f e^{2\pi}}{(1-r_f) + r_f e^{2\pi}}$

$r_f(1-r_f)(1-e^{2\pi}) = 0$

$r_f = 0, 1$

d) $f = \frac{r e^{2\pi}}{(1-r) + r e^{2\pi}}$

$f' = \frac{e^{2\pi}}{[(1-r) + r e^{2\pi}]^2}$

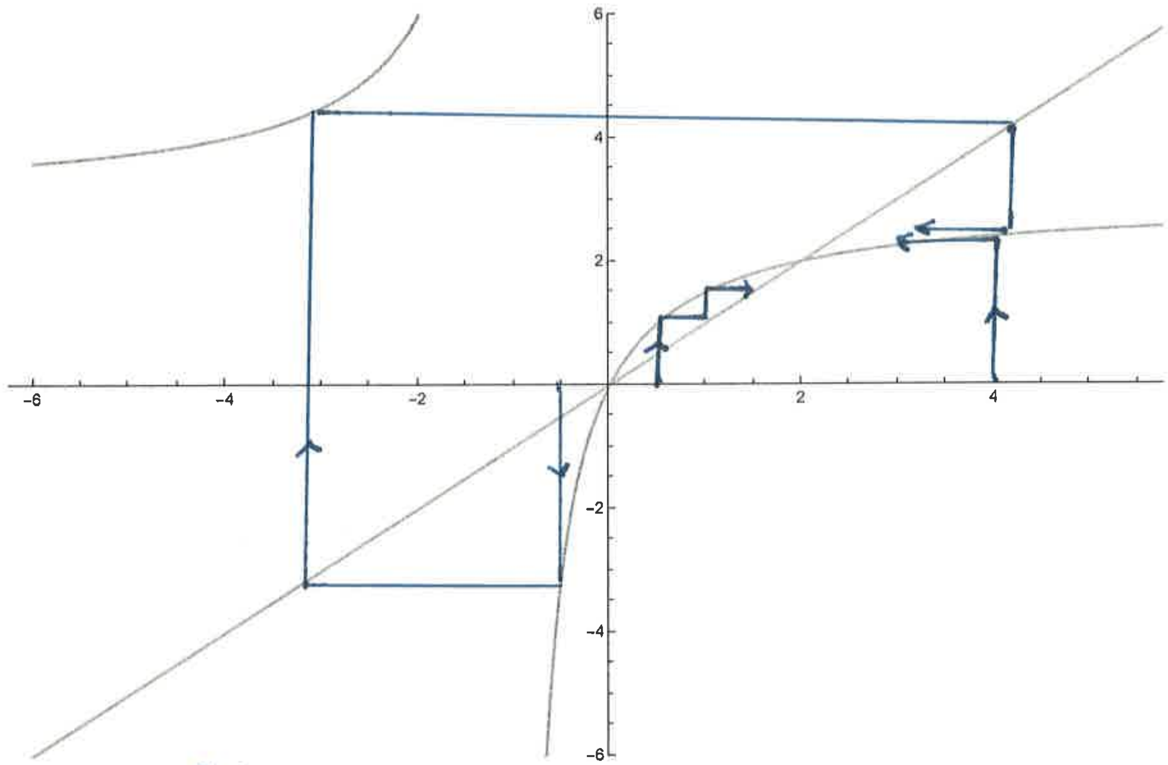
$f'(0) = e^{2\pi} > 1 \Rightarrow U$

$f'(1) = \frac{1}{e^{2\pi}} < 1 \Rightarrow S$

10.1.9

```
In[11]:= Plot[{3 * x / (1 + x), x}, {x, -6, 6}, PlotRange → {-6, 6}]
```

Out[11]=



$$x_{n+1} = \frac{3x_n}{1+x_n}$$

$$x_f = \frac{3x_f}{1+x_f}$$

$$x_f(x_f - 2) = 0$$

$$x_f = 0, 2$$

$$f(x) = \frac{3x}{1+x}$$

$$f'(x) = \frac{3}{(1+x)^2}$$

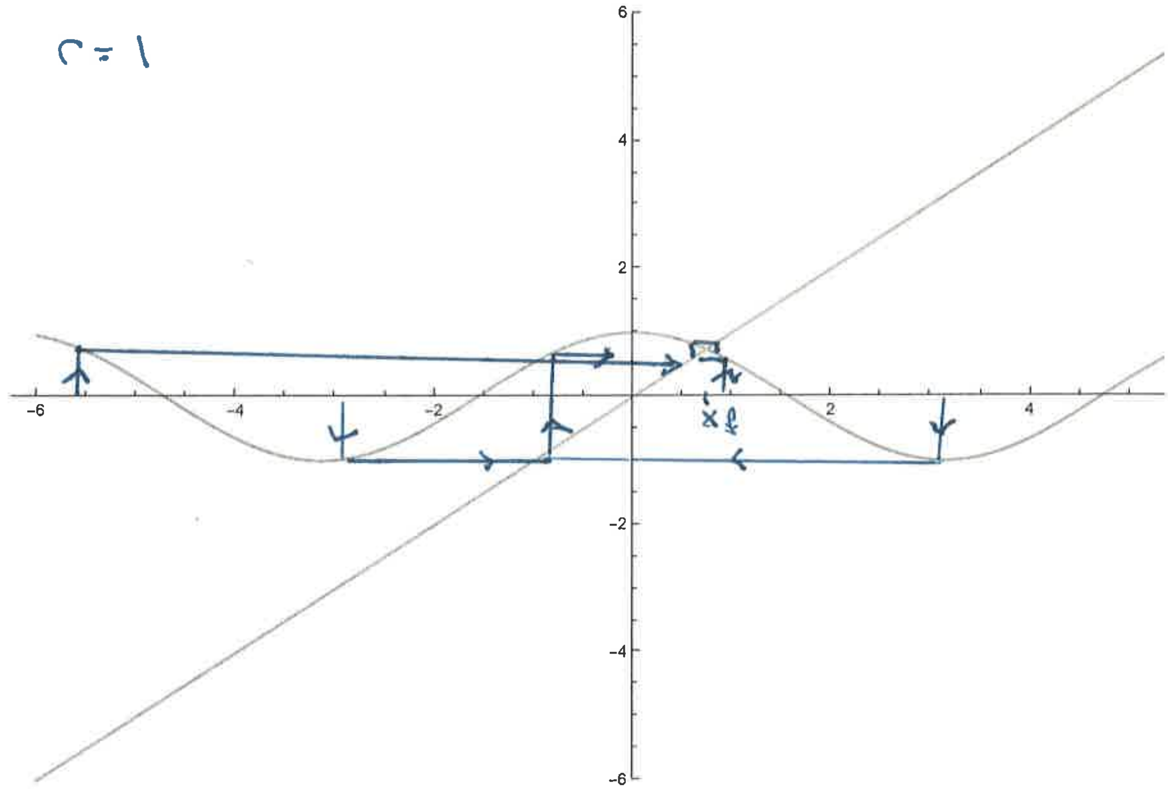
$$f'(0) = 3 > 1 \implies U$$

$$f'(2) = \frac{1}{3} < 1 \implies \text{A. Stable}$$

10.1.12

In[14]:= Plot[{Cos[x], x}, {x, -6, 6}, PlotRange → {-6, 6}]

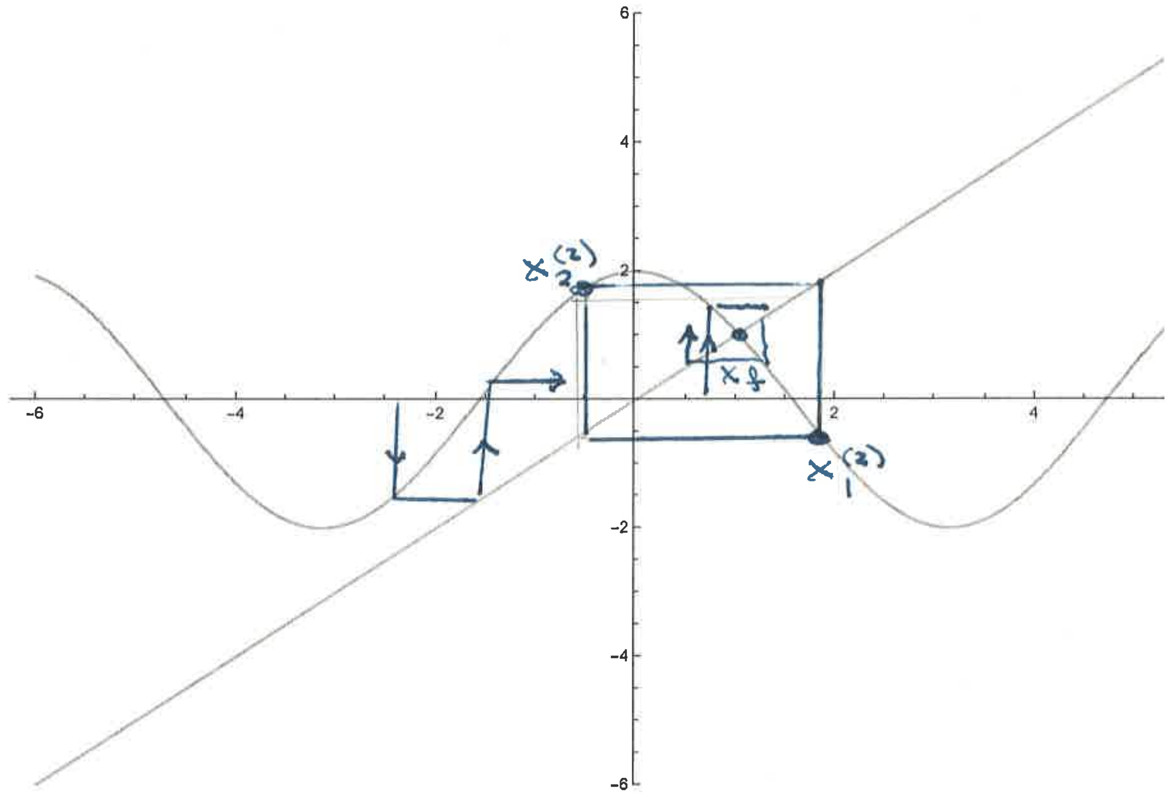
Out[14]=



All ICs go to xf

In[15]:= `Plot[{2 * Cos[x], x}, {x, -6, 6}, PlotRange → {-6, 6}]`

Out[15]=

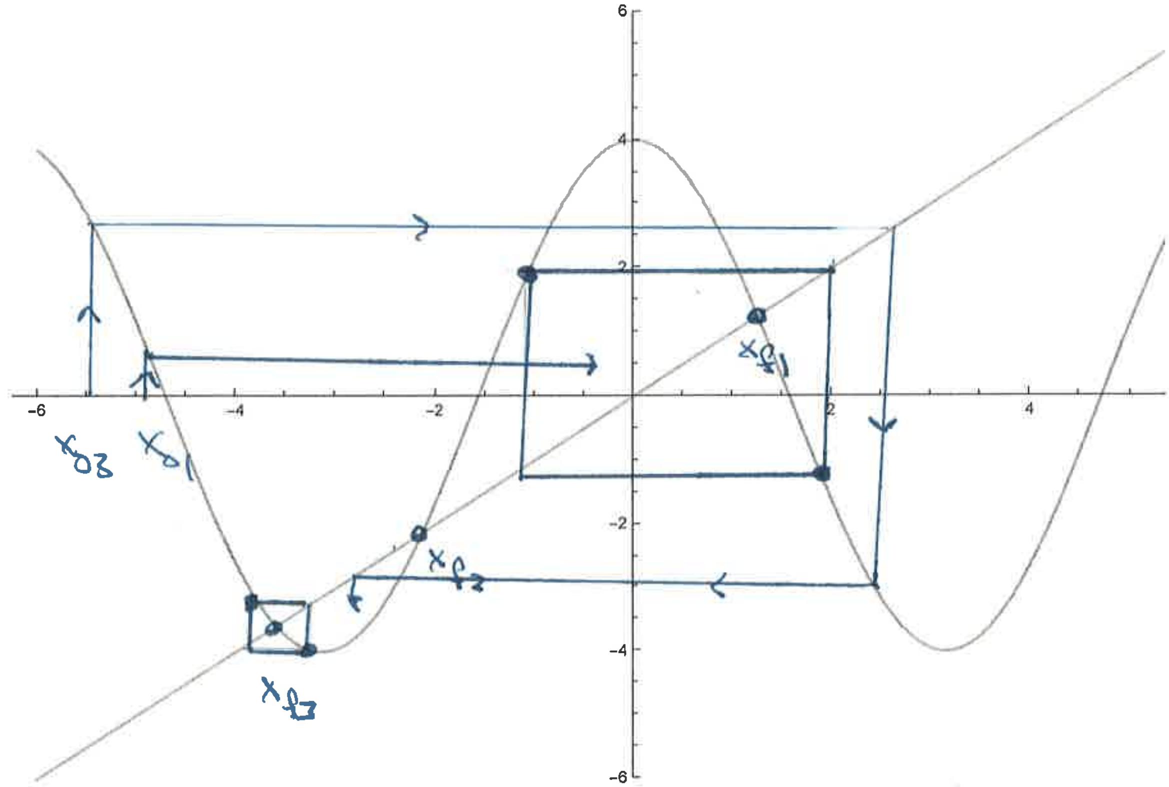


Separately, I plot $f^{(2)}(x) = 2\cos[2\cos(x)]$ and located F.P. x_f of $f^{(2)}$. These are period-2 pts of $f(x)$.

x_f appears to be unstable while the period-2 cycle $\{x_1^{(2)}, x_2^{(2)}\}$ is stable.

```
In[17]:= Plot[{4 * Cos[x], x}, {x, -6, 6}, PlotRange → {-6, 6}]
```

Out[17]=



Again, I found the period-2 cycle by examining $f^{(2)}$. The larger constant (4 insted of 2) has produced two new F.P. x_{f2} & x_{f3}.

Both x_{f1} & x_{f3} have period-2 cycles around them. The slope at x_{f2} is >1 and so unstable. However, the slope is pos. and so there won't be a flip saddle or higher period cycles

Notice, x_{01} & x_{03} start very close. However x_{01} eventually goes to the P2 cycle around x_{f1}, whereas x_{03} goes to the P2 cycle around x_{f3}.

10.1.12

$$x_{n+1} = f(x_n) = x_n + \frac{g(x_n)}{g'(x_n)}$$

If $g(x) = x^2+4$, $g'(x) = 2x$

$$x_{n+1} = x_n - \frac{(x_n^2-4)}{2x_n} = \frac{x_n^2+4}{2x_n}$$

$$x_f = \frac{x_f^2+4}{2x_f} \Rightarrow x_f = \pm 2$$

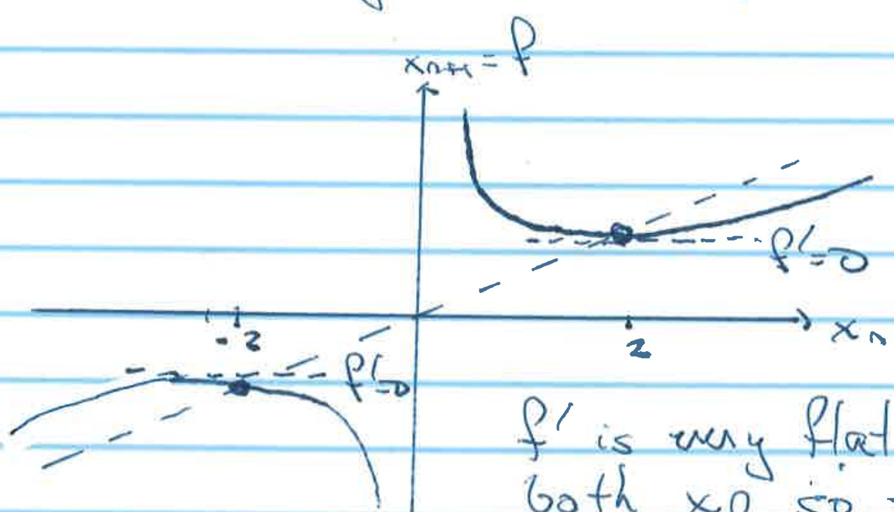
$$f'(x) = \frac{x^2-4}{2x^2} \qquad f'(\pm 2) = 0 \leftarrow \text{Superstable}$$

Suppose $x_0 = 1$

$$x_1 = \frac{1+4}{2} = 5/2 = 2.5$$

$$x_2 = \frac{41}{20} = 2.05$$

After just 2 iterates, already very close to $x_f = 2$



f' is very flat near both x_f so the cobweb will show ICs rapidly "moving" horizontally to x_f

10.3.4

$x_{n+1} = x_n^2 + c$

F.P.: $x_f = x_f^2 + c$

$x_f^2 - x_f + c = 0 \quad x_f = \frac{1}{2}(1 \pm \sqrt{1-4c})$

If $c < 1/4$, there exist 2 solutions $x_{\oplus}$ & $x_{\ominus}$

If $c > 1/4$, there are no real solutions.

Suggest a Saddle-Node type bif. at $c = 1/4$

L.S. $f'(x) = 2x$

$x_{\oplus}$: $-1 \overset{?}{<} 2(\frac{1}{2} + \sqrt{1-4c}) \overset{?}{<} 1$

$-2 < \sqrt{1-4c} < 0$

Assuming $\sqrt{}$ is always pos so neither is ever satisfied. In other words $\sqrt{1-4c} > 0$ always, which means $f'(x_{\oplus})$ is > 1 always. ∴ always Unstable.

$x_{\ominus}$: $-1 \overset{?}{<} 2(\frac{1}{2} - \sqrt{1-4c}) \overset{?}{<} 1$

$-2 < -\sqrt{1-4c} < 0$

$2 > \sqrt{1-4c} > 0$

$4 > 1-4c > 0$

$3 > -4c > -1$

$-3/4 < c < 1/4$

$x_{\ominus}$ is stable if $-3/4 < c < 1/4$.

If $c > 1/4$ $x_{\ominus}$ doesn't exist.

If $c = -3/4$ there is a FLIP BIF

Expect a period 2 cycle for $c < -3/4$

10.3.11

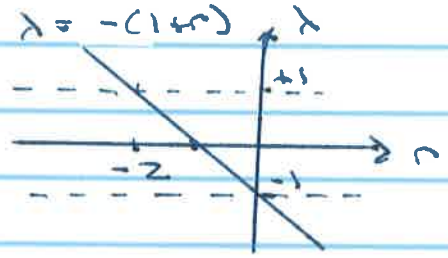
$$x_{n+1} = f(x_n) = -(1+r)x_n - x_n^2 - 2x_n^3$$

a) $x_f = 0$ is an F.P.

b) $y_{n+1} = f'(0)\, y_n = \left(-(1+r) - 2x - 6x^2\right)\Big|_{x=0} y_n$

$= -(1+r)\, y_n$

$= \lambda\, y_n$



$x_f = 0$ is A. Stable for $-2 < r < 0$

At $r = -2$ there is a Bif. (steady)

At $r = 0$ there is a FLIP BIF

c) $f^{(2)} = -(1+r)\left[-(1+r)x - x^2 - 2x^3\right]$
$\quad - \left[-(1+r)x - x^2 - 2x^3\right]^2$
$\quad - 2\left[-(1+r)x - x^2 - 2x^3\right]^3$

For $x \ll 1$ and $r \ll 1$ (x near $x_f = 0$ and r near $r_c = 0$)

Keep only the biggest terms.

$$f^{(2)}(x) \approx (1+2r)x - x^2 + O(r^2x, rx^2, x^3)$$

$\therefore$ 2^{ND} iterate map is approx

$$u_{n+1} = (1+2r)u_n - u_n^2$$

$$u_f = (1+2r)u_f - u_f^2$$

$$0 = u_f(2r - u_f) \qquad u_f = 0 \equiv x_f$$
$$u_f = 2r$$

When $x_f = 0$ has a flip bif we expect $\underline{2}$ new fixed points of $f^{(2)}$. We only got one, $u_f = 2r$. To "see" the other we have to zoom out and keep some cubic terms.

Note $\lim_{r\to 0} (u_f = 2r) = (0 = x_f)$

$\therefore$ As $r \to 0$ the P2 cycle collides w/ x_f

d) For $r<0$ x_f is A. stable so
$\lim_{n\to\infty} x_n = x_f$

For $r>0$ x_f is Unstable. The orbit will converge to the P2 cycle w/ $u_f = 2r$ (and the other P2 point).