MATH 4325
LINEAR SYSTEMS

(5.2.1)

$$\dot{x} = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} \cdot x$$

$$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = (4-\lambda)(1-\lambda) + 2 = 0$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$\lambda = 3: \begin{pmatrix} 4-3 & -1 \\ 2 & 1-3 \end{pmatrix} \cdot \underline{v} = 0$$

$$\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} \cdot v = 0$$

$$v^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda = 2: \begin{pmatrix} 4-2 & -1 \\ 2 & 1-2 \end{pmatrix} \cdot v = 0$$

$$\begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix} \cdot v = 0$$

$$v^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$x = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t}$$

Unstable node

$$x(0) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$c_1 + c_2 = 3$$
$$c_1 + 2c_2 = 4$$

$$\Rightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$x_1 = 2e^{3t} + e^{2t}$$
$$x_2 = 2e^{3t} + 2e^{2t}$$

(5.2.2)

$$\dot{x} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \cdot x$$

$$(1-\lambda)^2 + 1 = 0$$

$$\lambda^2 - 2\lambda + 2 = 0$$

$$\lambda = 1 \pm i$$

$$\lambda = 1+i: \begin{pmatrix} 1-(1+i) & -1 \\ 1 & 1-(1+i) \end{pmatrix} \cdot v = 0$$

$$\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \cdot v = 0$$

$$v = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$x = c \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{(1+i)t} + \text{c.c.}$$

let $c = Ae^{i\phi}$

$$x = A \begin{pmatrix} 1 \\ -i \end{pmatrix} e^t e^{i(t+\phi)} + \text{c.c}$$

$$x = A \begin{pmatrix} 1 \\ -i \end{pmatrix} e^t e^{i(t+\phi)}$$

$$+ A \begin{pmatrix} 1 \\ +i \end{pmatrix} e^t e^{-i(t+\phi)}$$

Now set

$$e^{\pm i(t+\phi)} = \cos(t+\phi) \pm i \sin(t+\phi)$$

Sub and expand.

Book has $v = \begin{pmatrix} i \\ 1 \end{pmatrix}$ which is simply $i \begin{pmatrix} 1 \\ -i \end{pmatrix}$. Just different by a constant.

(5.2.3)

$\dot{x} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \cdot x$

$-\lambda(-3-\lambda) + 2 = 0$

$\lambda^2 + 3\lambda + 2 = 0$

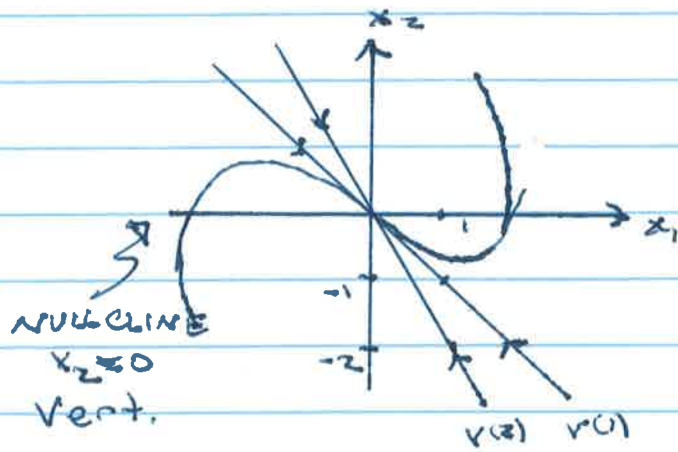
$\lambda = -1, -2$ A.S. NODE

$\lambda = -1$: $\begin{pmatrix} +1 & 1 \\ -2 & -2 \end{pmatrix} \cdot v = 0$

$v^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda = -2$: $\begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix} \cdot v = 0$

$v^{(2)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$



$x = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-2t}$

slow direction

(5.2.4)

$\dot{x} = \begin{pmatrix} 5 & 10 \\ -1 & -1 \end{pmatrix} \cdot x$

$(5-\lambda)(-1-\lambda) + 10 = 0$

$\lambda^2 - 4\lambda + 5 = 0$

$\lambda = 2 \pm i$ U. FOCUS

If $x_2 = 0$

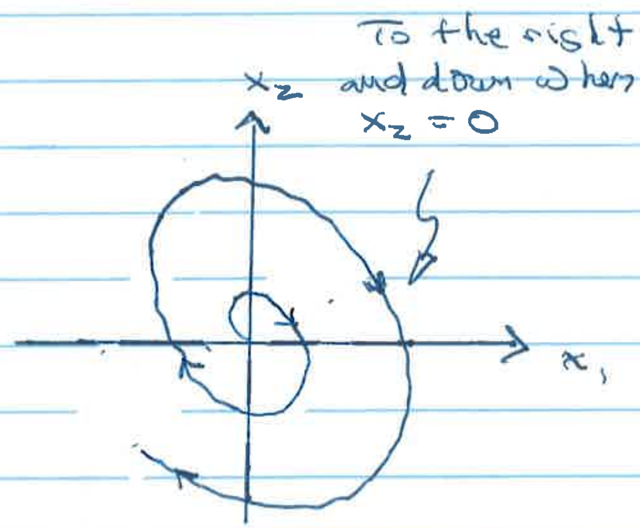
$\dot{x}_1 = 5x_1$

$\dot{x}_2 = -x_1$

So if $x_1 > 0$

$\Rightarrow$ Right & down

$\Rightarrow$ Clockwise



(5.25)

$$\dot{x} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \cdot x$$

U. IMPROPER NODE

$(3-\lambda)(-1-\lambda)+4=0$

$\lambda^2 - 2\lambda + 1 = 0$

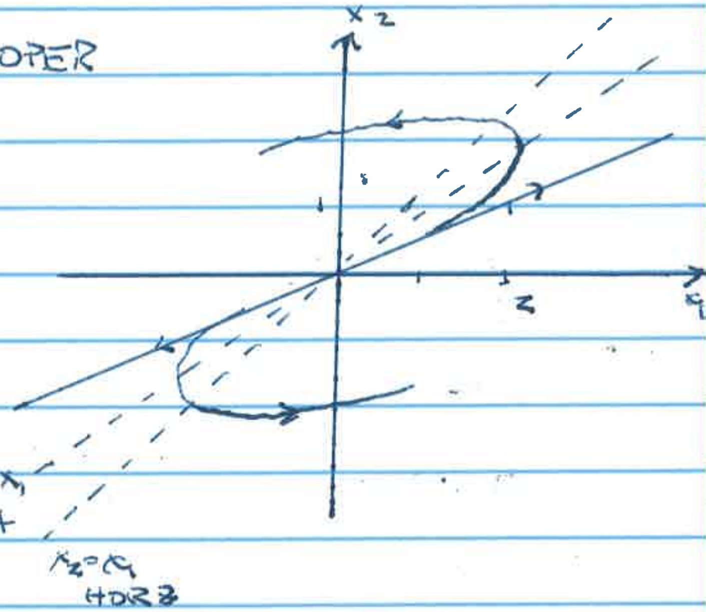
$(\lambda-1)^2 = 0$

$\lambda = 1, 1$

$\lambda = 1$: $\begin{pmatrix} 3-1 & -4 \\ 1 & -1-1 \end{pmatrix} \cdot v = 0$

$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \cdot v = 0$

$v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$



NC: $\dot{x}_1 = 0$ $\quad 3x_1 - 4x_2 = 0$ $\quad x_2 = \frac{3}{4}x_1$

$\dot{x}_2 = 0$ $\quad x_1 - x_2 = 0$ $\quad x_2 = x_1$

(5.2.7)

$$\dot{x} = \begin{pmatrix} 5 & 2 \\ -17 & -5 \end{pmatrix} \cdot x$$

$(5-\lambda)(-5-\lambda) + 34 = 0$

$\lambda^2 + 9 = 0 \Rightarrow \lambda = \pm 3i$

$\lambda = +3i$: $\begin{pmatrix} 5-3i & 2 \\ -17 & -5-3i \end{pmatrix} \cdot v = 0$

$v = \begin{pmatrix} 2 \\ -(5-3i) \end{pmatrix}$

or some multiple

$x = c \begin{pmatrix} 2 \\ -(5-3i) \end{pmatrix} e^{i3t} + c.c$

Let $c = Ae^{i\phi}$ and expand to get sin & cos

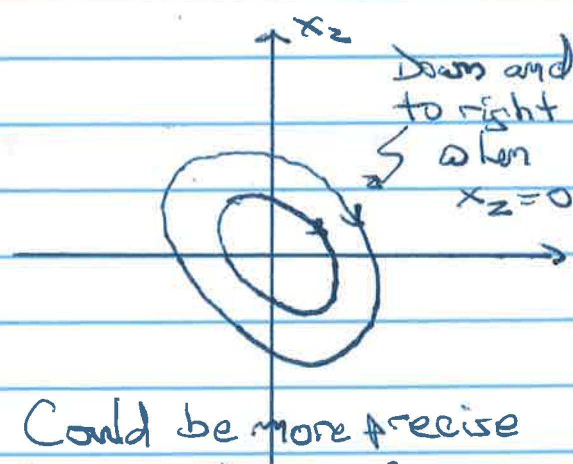
CENTER

If $x_2 = 0$ $\dot{x}_2 = -17x_1$

If $x_1 > 0$ $\dot{x}_2 < 0$ so clockwise.

Note $\dot{x}_1 > 0$ to the right.

$\Rightarrow$ Elliptical



Could be more precise by computing N.C.

(5.2.8)

$$\dot{x} = \begin{pmatrix} -3 & 4 \\ -2 & 3 \end{pmatrix} \cdot x$$

$$(-3-\lambda)(3-\lambda) + 8 = 0$$

$$\lambda^2 - 1 = 0 \quad \lambda = \pm 1 \text{ SADDLE}$$

$\lambda = +1$:
$$\begin{pmatrix} -3-1 & 4 \\ -2 & 3-1 \end{pmatrix} \cdot v = 0$$
$$v^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\lambda = -1$:
$$\begin{pmatrix} -3+1 & 4 \\ -2 & 3+1 \end{pmatrix} \cdot v = 0$$
$$v^{(2)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$x = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-3t}$$