MATH 5331

EXAM 3

1a) $\left| \frac{nz}{n+1} + \frac{3z}{n} - z \right| < \epsilon$

$\left| \frac{n^2 z + 3z(n+1) - z(n)(n+1)}{n(n+1)} \right| < \epsilon$

$\left| \frac{3zn + 3z - zn}{n(n+1)} \right| < \epsilon$

$\left| \frac{z(2n+3)}{n(n+1)} \right| < \epsilon$

For n large and $|z| \le R$

$\frac{2R}{n} < \epsilon$

This will be true for $n > \frac{2R}{\epsilon}$

$\therefore$ Set $\frac{2R}{\epsilon} = N_\epsilon$, which is independent of z

1b) $f = \frac{1+z}{1-z} = \frac{(1+i) + (z-i)}{(1-i) - (z-i)}$:

$= \frac{(1+i)+(z-i)}{(1-i)} \frac{1}{1 - \frac{(z-i)}{(1-i)}}$

$= \frac{(1+i) + (z-i)}{(1-i)} \sum_{n=0}^{\infty} \left(\frac{z-i}{1-i}\right)^n$

Distance from $z=1$ to $z=i$?

c) $f(z) = \frac{4}{(3-z)} \frac{1}{z^2}$

$= \frac{4}{3} \frac{1}{1 - z/3} \frac{1}{z^2} = \frac{4}{3} \frac{1}{z^2} \sum_{n=0}^{\infty} \left(\frac{z}{3}\right)^n \quad \left|\frac{z}{3}\right| < 1$

$\Rightarrow 0 < |z| < 3$

$= \frac{4}{z} \frac{1}{\frac{3}{z} - 1} \frac{1}{z^2} = \frac{-4}{z^3} \frac{1}{1 - 3/z}$

$= \frac{-4}{z^3} \sum_{n=0}^{\infty} \left(\frac{3}{z}\right)^n \quad \left|\frac{3}{z}\right| < 1$

$\Rightarrow 3 < |z|$

MATH 5321

EXAM 3

2a) $(z^2+z)\sin\left(\frac{1}{z}\right) = (z^2+z)\left(\frac{1}{z} - \frac{1}{6}\frac{1}{z^3} + \dots\right)$

$= z + 1 - \frac{1}{6}\frac{1}{z} - \frac{1}{6}\frac{1}{z^2} + \dots$

ESP w/ RESIDUE $= -\frac{1}{6}$ at $z=0$

2b) $\frac{(z+2)(z-1)}{(z+2)} = z-1 \quad z \neq -2$

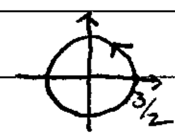
REMOVABLE SP at $z=-2$ w/ Res $=0$

2c) $\frac{e^z}{(z-i)^3} = \frac{e^i e^{z-i}}{(z-i)^3}$

$= \frac{e^i}{(z-i)^3}\left(1 + (z-i) + \frac{1}{2}(z-i)^2 + \dots\right)$

Pole of order 3 at $z = i$

w/ Residue $e^i\left(\frac{1}{2}\right)$

3a) 

a) S.Poles at $z = -2, -2i$

Outside C

By Cauchy Goursat $\oint_C f\,dz = 0$

b) S.Pole at $z = -i$ inside C.

$\oint f\,dz = 2\pi i \, \mathrm{Res}(f, -i)$

$= 2\pi i \left(\frac{z+1}{z+2}\right)\Big|_{-i} = 2\pi i \frac{1-i}{2-i}$

c) S Poles at $z = -i$

Pole of O-2 at $z=0$.

$= 2\pi i\left[\mathrm{Res}(f;0) + \mathrm{Res}(f,-i)\right]$

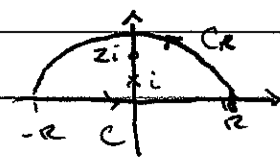
$= 2\pi i\left(\frac{d}{dz}\left(\frac{1}{z+i}\right)\Big|_{z=0} + \frac{1}{z^2}\Big|_{z=-i}\right)$

$= 2\pi i\left(-\frac{1}{i^2} + \frac{1}{i^2}\right) = 0$

4) $I = \int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}\,dx$

$= \frac{1}{2} P.V. \int_{-\infty}^{\infty} f(x)\,dx$

Consider $\int_{\gamma} f(z)\,dz \quad \gamma = C + C_R$

γ: 

$\int_C f(z)\,dz + \int_{C_R} f(z)\,dz = 2\pi i\left(\mathrm{Res}(f;i) + \mathrm{Res}(f;2i)\right)$

$\mathrm{Res}(f;i) = \frac{z^2}{(z+i)(z^2+4)}\Big|_{z=i} = \frac{-1}{2i(3)} = \frac{i}{6}$

$\mathrm{Res}(f;2i) = \frac{z^2}{(z^2+1)(z+2i)}\Big|_{z=2i} = \frac{-4}{-3(4i)} = \frac{-i}{3}$

$\left|\int_{C_R} f\,dz\right| = M \cdot L$

$L = \frac{1}{2} 2\pi R = \pi R$

$M \le \frac{R^2}{(R^2-1)(R^2-4)}$

$\left|\int_{C_R} f\,dz\right| \le \frac{\pi R^3}{(R^2-1)(R^2-4)} \to 0$ As $R \to \infty$

$I = \frac{1}{2} 2\pi i\left(\frac{i}{6} - \frac{i}{3}\right) + 0$

$= \frac{\pi}{6}$