MATH 5331

S&S: 4.5

3) a) $f = \sin 3z \Rightarrow 2\pi i \sin\left(\frac{3\pi}{2}\right) = -2\pi i$

b) $f = \frac{ze^z}{2} \Rightarrow 2\pi i \frac{\left(\frac{3}{2}\right)e^{3/2}}{2} = \frac{3\pi i}{2} e^{3/2}$

c) $f = \frac{\cos z}{z^2+9} \Rightarrow 2\pi i \frac{\cos 0}{0^2+9} = \frac{2\pi i}{9}$

d) $f(z) = 5z^2+2z+1$

$\Rightarrow 2\pi i \frac{1}{2!}(5z^2+2z+1)''\Big|_{z=i} = 10\pi i$

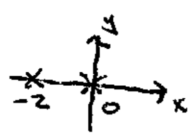
e) $f = e^{-z}$

$\Rightarrow 2\pi i \frac{1}{1!}(-e^{+1}) = -2\pi i e$

f) $f = \frac{\sin z}{(z-4)}$

$\Rightarrow 2\pi i \frac{1}{1!}\left(\frac{\sin z}{z-4}\right)'\Big|_{z=0} = -\frac{\pi i}{2}$

4) $\int_C \frac{z+i}{z^2(z+2)} dz$



a) $f = \frac{z+i}{z+2} \Rightarrow \underset{\text{CCW}}{\overset{\uparrow}{+}} 2\pi i \frac{1}{1!} f'(0) = \frac{\pi}{2} + i\pi$

b)  $f = \frac{z+i}{z^2}$

$\Rightarrow \underset{\text{CCW}}{\overset{\uparrow}{+}} 2\pi i f(-2) = -\frac{\pi}{2} - i\pi$

c) Both singularities are outside contour ∴ f is analytic in and on C. $\oint f\,dz = 0$.

6) Note $(z^2+1)^2 = (z+i)^2(z-i)^2$

$= \oint_{C_1} \frac{\frac{e^{iz}}{(z+i)^2}}{(z-i)^2} dz + \int_{C_2} \frac{\frac{e^{iz}}{(z-i)^2}}{(z+i)^2} dz$

$= 2\pi i \left(\frac{e^{iz}}{(z+i)^2}\right)'\Big|_{z=i} + 2\pi i \left(\frac{e^{iz}}{(z-i)^2}\right)'\Big|_{z=-i}$

$= 2\pi i\left(-\frac{e^{-1}i}{2}\right) + 2\pi i \, 0 = \frac{\pi}{e}$

7) Γ encloses 0 only.

$f = \frac{\cos z}{z-3} \Rightarrow 2\pi i \frac{1}{1!} f'(0) = -\frac{2\pi i}{9}$

15)

- Consider the function $G(z)$ defined by
$$G(z) = \frac{1}{2\pi i}\int_{|\xi|=1} \frac{f(\xi)}{\xi(\xi - z)} d\xi$$
- By Thm 15, because f is analytic (and thus continuous) $G(z)$ is analytic.
- If $z=0$
$$G(0) = \frac{1}{2\pi i}\int_{|\xi|=1} \frac{f(\xi)}{\xi^2} d\xi = f'(0)$$
by Thm 19.
- If $z \neq 0$
  - By partial fractions
$$\frac{1}{\xi}\cdot\frac{1}{\xi - z} = \frac{1}{z}\left(\frac{1}{\xi - z} - \frac{1}{\xi}\right)$$
$$G(z) = \frac{1}{2\pi i}\frac{1}{z}\Big[\underbrace{\int_{|\xi|=1} \frac{f}{\xi - z}d\xi}_{2\pi i f(z)} - \underbrace{\int_{|\xi|=1}\frac{f}{\xi}d\xi}_{\cancel{f(0)}\ 0 \text{ by def}}\Big]$$
$$G(z) = \frac{1}{2\pi i}\frac{1}{z} 2\pi i f(z) = \frac{f(z)}{z}$$
- So $G(z) = \begin{cases} f'(0) & z=0 \\ \frac{f(z)}{z} & z \neq 0 \end{cases} = F(z)$

Because we showed G is analytic, F is analytic. Yea!