MATH 5331

SS: 4p6

1) On $|z| = R$

$|(1-z)^2| \geq (1-|z|)^2 = (1-R)^2$

$\therefore \max|f(z)| = \max\left|\frac{1}{(1-z)^2}\right| \leq \frac{1}{(1-R)^2}$

If $f = \frac{1}{(1-z)^2}$

$f' = -2\frac{1}{(1-z)^3}(-1) = \frac{2}{(1-z)^3}$

$\vdots$

$f^{(n)}(z) = \frac{2\cdot3\cdot4\cdot\cdots\cdot(n+1)}{(1-z)^{n+2}}$

$f^{(n)}(0) = \frac{(n+1)!}{1}$

By Thm 20

$|f^{(n)}(z_0)|\Big|_{z_0=0} = (n+1)! \leq \frac{n!}{R^n}\cancel{M}\frac{1}{(1-R)^2}$

2). $|f^{(n)}(0)| \leq \frac{n!}{R^n}M$

$M = \max|f(z)| = \frac{1}{1-|z|}\Big|_{|z|=R} = \frac{1}{1-R}$

$|f^{(n)}(0)| \leq \frac{n!}{R^n(1-R)} \equiv U(R)$

upper bound of $|f|$

$\frac{dU}{dR} = \frac{-n!(-R^n + nR^{n-1}(1-R))}{(R^n(1-R))^2}$

$\frac{dU}{dR} = 0$ if $-R^n + nR^{n-1}(1-R) = 0$

$\Rightarrow R = \frac{n}{n+1}$

This is crit pt $\Rightarrow$ possible max or min.
Because $\lim_{R\to0 \text{ or } 1} U = \infty$ this must be a MIN.

13) $\frac{f(z)}{g(z)}$ is analytic in D.

$\therefore$ Its max must be on the boundary

$|f(z)| \leq |g(z)|$ on B.

$\frac{|f(z)|}{|g(z)|} \leq 1$ on B.

$\therefore$

$\frac{f(z)}{g(z)} < 1$ in D since max of $\frac{f}{g}$ is on boundary.

24) Let $P(z)$ be factored and written as

$P(z) = a_n(z-z_1)(z-z_2)\cdots(z-z_n)$

By product rule

$$\begin{aligned}P'(z) &= a_n(z-z_2)\cdots(z-z_n)\\ &+ a_n(z-z_1)(z-z_3)\cdots(z-z_n)\\ &+ \cdots a_n(z-z_1)(z-z_2)\cdots(z-z_{n-1})\\ &= a_n\sum_{k=1}^{n}(z-z_1)\cdots(z-z_{k-1})(z-z_{k+1})\cdots(z-z_n)\end{aligned}$$

ie the $(z-z_k)$ term is missing.

Then

$$\frac{P'(z)}{P(z)} = \frac{1}{(z-z_1)} + \frac{1}{(z-z_2)} + \cdots + \frac{1}{z-z_n} = \sum_{k=1}^{n}\frac{1}{(z-z_k)}$$

Now $\frac{1}{2\pi i}\int_\Gamma \frac{1}{(z-z_k)}dz = \begin{cases}1 & \text{if } z_k \text{ in } \Gamma\\ 0 & \text{otherwise}\end{cases}$

$\therefore \frac{1}{2\pi i}\int_\Gamma \frac{P'}{P}dz = \sum_{k=1}^{n}\underbrace{\frac{1}{2\pi i}\int_\Gamma\frac{1}{z-z_k}dz}_{= \begin{cases}1 & \text{if } z_k \text{ in } \Gamma\\ 0 & \text{otherwise}\end{cases}}$

So count the # inside Γ

$= N$