MATH 5331

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1a) $f = \dfrac{z^3+1}{z^2(z+1)}$

ISP at $z=0$ & $z=-1$

$\lim_{z\to 0} f = \dfrac{1}{0} = \infty$ Pole (suggested)

$\lim_{z=-1} f = \dfrac{0}{0}$: bounded? Removable (suggested)

$f = \dfrac{1}{z^2}\left(\dfrac{z^3+1}{z+1}\right) = \dfrac{1}{z^2}g(z)$ $g(0)\neq 0$ g analytic at 0.

$z=0$ Pole of order 2.

$$\begin{array}{r} z^2-z+1 \\ z+1\,\overline{)\,z^3+1} \\ z^3+z^2 \\ -z^2 \end{array}$$

$f = \dfrac{1}{z^2\cancel{(z+1)}}\cancel{(z+1)}(z^2-z+1) = \dfrac{z^2-z+1}{z^2}$ $\lim_{z\to -1} \dfrac{1+1+1}{1} = 3$

Can refine $f = \begin{cases} f \text{ orig} & z\neq -1 \\ 3 & z=-1 \end{cases}$ $z=-1$ REMOVABLE

1b) $f = z^3 e^{1/z}$

by example 3 $e^{1/z}$ has an ESSENTIAL S.P.

$\therefore$ f has an E.S.P.

c) $f = \dfrac{\cos z + 4z^3+4z}{z^2+1}$ I.S.P at $z=\pm i$

$\cos(\pm i) = \frac{1}{2}(e^{i(i)} + e^{-i(i)}) = \frac{1}{2}(e^{-1}+e^{+1}) = \cosh(1) \neq 0$

$\therefore f = \dfrac{1}{z-i}g(z)$ where $g(i)\neq 0$ g analytic.

AND $f = \dfrac{1}{z+i}g(z)$ " $g(-i)\neq 0$ "

$\therefore$ $z=\pm i$ are BOTH SIMPLE POLES (ORDER 1)

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d) $f = \frac{1}{e^z - 1}$

Consider $\frac{1}{f} = e^z - 1 = 1 + z + \frac{1}{2}z^2 + \cdots - 1$

$= z + \frac{1}{2}z^2 + \cdots$

$= z(1 + \frac{1}{2}z + \cdots) = z\,g(z)$

$g(0) \neq 0$ & g analytic $\Rightarrow$ SIMPLE ZERO

$\Rightarrow f$ has simple POLE.

e) $f = \tan z = \frac{\sin z}{\cos z}$ IS.P when $\cos z = 0$

Consider $\frac{1}{f} = \frac{\cos z}{\sin z}$. Has zeros when $\cos z = 0$, ie, $z = z_0$

Further $\cos(z_0) = (\pm 1)(z - z_0) + \frac{1}{6}(\pm 1)(z - z_0)^3 + \cdots$

$= (z - z_0)\,g(z)$ $\quad g(z_0) \neq 0$, g anal.

$\therefore \frac{1}{f}$ has simple zeros.

f has simple poles.

See also ~~E~~xample 5.

f) $\cos(1 - \frac{1}{z}) = 1 - \frac{1}{2}(1 - \frac{1}{z})^2 + \frac{1}{4!}(1 - \frac{1}{z})^3 + \cdots$

ESSENTIAL S.P. at $z = 0$

g) $f = \frac{\sin 3z - 3z}{z^2}$ ISP at $z = 0$

$= \frac{1}{z^2}(\cancel{3z} - \frac{1}{6}(3z)^3 + \cdots - \cancel{3z}) = -\frac{1}{6}9z + \cdots$

$\Rightarrow$ REMOVABLE since $\lim_{z \to 0} f = 0$ bounded, etc

h) $\cot(\frac{1}{z})$ ISP at $z = 0$ & when $\frac{1}{z} = n\pi$ $n \in \mathbb{Z}$

- By series expansion in terms of $\frac{1}{z}$, $z = 0$ is E.S.P.
- Consider $\frac{1}{f} = \tan(\frac{1}{z}) = \frac{\sin \frac{1}{z}}{\cos \frac{1}{z}} \Rightarrow$ zero at $\frac{1}{z} = n\pi$

However, $\frac{d}{dz}\tan(\frac{1}{z}) = -\frac{1}{z^2}\frac{1}{\cos^2 \frac{1}{z}} \neq 0$ at $\frac{1}{z} = n\pi$

$\therefore$ SIMPLE zero of $\frac{1}{f} \Rightarrow$ SIMPLE POLE of f.

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3a) $\dfrac{(z-i)^2}{(z-(2-3i))^5}$

b) $z\,e^{\frac{1}{z-1}}$

c) $\dfrac{\sin z}{z}\,\dfrac{1}{(z-i)^6}\,e^{\frac{1}{z-i}}$

d) $\dfrac{1}{(z-(1+i))^2}\,e^{\frac{1}{z}}\,e^{\frac{1}{z-1}}$

5a) False. e.g. $\frac{1}{z-1} - \frac{1}{z-1} = 0$

b) True

c) True

d) False. Mult L.S. to see

e) True $f \cdot g = (z-z_0)^m \phi_1 \cdot \dfrac{\phi_2}{(z-z_0)^n}$

$= (z-z_0)^{m-n} \underbrace{\phi_1 \phi_2}_{\text{Analytic}}$

Removable for $m \geq n$