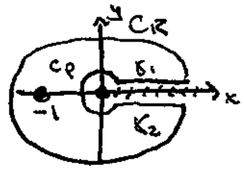
MATH 5331

SS: 6.6

2) $I = \int_0^\infty \frac{x^{\alpha-1}}{x+1}\,dx$

$\int_\gamma \frac{z^{\alpha-1}}{z+1}\,dz \quad \gamma:$



$= \int_{\gamma_1} + \underbrace{\cancel{\int_{\gamma_2}}}_{-e^{i2\pi\alpha}\int_{\gamma_1}} + \underbrace{\cancel{\int_{C_\rho}}}_{0} + \underbrace{\cancel{\int_{C_R}}}_{0} = 2\pi i \operatorname{Res}(f;-1)$

$(1-e^{i2\pi\alpha})I = 2\pi i\,(-1)^{\alpha-1}$

$I = \frac{-2\pi i\, e^{i\pi\alpha}}{1-e^{i2\pi\alpha}} = \frac{-2i\;\pi}{(e^{i\pi\alpha}-e^{i\pi\alpha})}\cancel{\frac{e^{i\pi\alpha}}{e^{i\pi\alpha}}}$

$I = \frac{\pi}{\sin(\pi\alpha)}$

3) $I = \int_0^\infty \frac{x^\alpha}{(x+q)^2}\,dx$

$= \frac{2\pi i \operatorname{Res}(f;-q)}{(1-e^{i2\pi\alpha})}$

$-q$ is pole of order 2.

$\operatorname{Res} = \frac{d}{dz}(\cancel{z}^\alpha)\Big|_{z=-q} = \alpha(-q)^{\alpha-1}$

$I = \frac{2\pi i\,\alpha\,(-q)^{\alpha-1}}{(1-e^{i2\pi\alpha})}$

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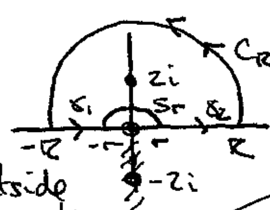
8) P.V. $\int_{-\infty}^{\infty} \frac{\operatorname{Log}|x|}{x^2+4}\,dx = I$

$\int_\gamma \frac{\operatorname{Log}|z|}{z^2+4}\,dz \quad \gamma:$



Move B Cut of Log outside the domain of integration.

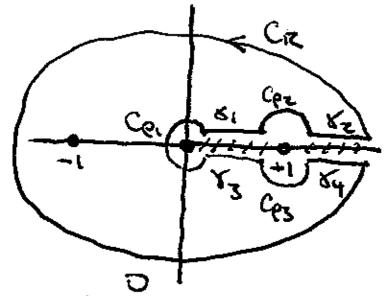
$= \underbrace{\int_{\gamma_1} + \int_{\gamma_2}}_{I} + \underbrace{\cancel{\int_{C_R}}}_{0} + \underbrace{\cancel{\int_{S_r}}}_{0} = 2\pi i \operatorname{Res}(f;2i)$

$\operatorname{Res}(f;2i) = \frac{\operatorname{Log}2i}{2i+2i} = \frac{\operatorname{Log}2}{4i}$

6) P.V. $\int_0^\infty \frac{x^\alpha}{x^2-1}\,dx = I$

$\int_\gamma \frac{z^\alpha}{z^2-1}\,dz \quad \gamma:$

Poles at ± 1



$\underbrace{\int_{\gamma_1} + \int_{\gamma_2}}_{I} + \underbrace{\int_{\gamma_3} + \int_{\gamma_4}}_{-e^{2\pi i\alpha}I} + \underbrace{\cancel{\int_{C_{\rho_1}}}}_{0} + \cancel{\int_{C_{\rho_2}}} + \cancel{\int_{C_{\rho_3}}} + \underbrace{\cancel{\int_{C_R}}}_{0} = 2\pi i \operatorname{Res}(f;-1)$

From indented contours.

$-\pi i \operatorname{Res}(f;1) \quad -\pi i \operatorname{Res}(f;1)\,e^{2\pi i\alpha}$

Under the BC

$(1-e^{2\pi i\alpha})I = 2\pi i \operatorname{Res}(f;-1) + \pi i\,(1+e^{2\pi i\alpha})\operatorname{Res}(f;1)$

$\frac{(-1)^\alpha}{-2} \qquad \frac{1^\alpha}{2}$

$I = \frac{-\pi i\, e^{i\pi\alpha}}{(1-e^{2\pi i\alpha})} + \frac{\pi i}{2}\frac{(1+e^{2\pi i\alpha})}{(1-e^{2\pi i\alpha})}$

$\int_{S_r} \frac{\operatorname{Log}|z|}{z^2+4}\,dz \le \left|\frac{\operatorname{Log}|z|}{z^2+4}\right|_{\text{Max}} \cdot 2\pi r$

$\le \frac{\operatorname{Log} r}{r^2-4}\cdot 2\pi r$

$\lim_{r\to 0} \frac{r\operatorname{Log} r}{r^2-4} = -\frac{1}{4}\lim_{r\to 0}\frac{\operatorname{Log} r}{\frac{1}{r}} = \left(\frac{-1}{4}\right)\frac{-\infty}{\infty}$

$= -\frac{1}{4}\lim_{r\to 0}\frac{\frac{1}{r}}{-\frac{1}{r^2}} \to 0$

So $I = \frac{2\pi i \operatorname{Log}2}{4i} = \frac{\pi}{2}\operatorname{Log}2$

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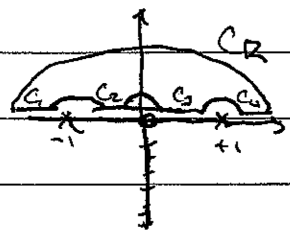
S&S #6 (Alternative)

$$\text{P.V.} \int_0^\infty \frac{x^\alpha}{x^2-1}\,dx$$



$$\int_{C_R} + \int_{C_{\rho_{-1}}} + \int_{C_{\rho_0}} + \int_{C_{\rho_{+1}}} + \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} = 2\pi i \sum \text{Res} \to 0$$

$R \to \infty$, 0 $\qquad$ $\rho \to 0$, 0

$-i\pi \text{Res}(f,-1)$ $\qquad$ $-i\pi \text{Res}(f,+1)$

$$\int_{C_1} + \int_{C_2};\quad z = re^{i\pi}\quad z' = e^{i\pi}\quad \int_\infty^0 \frac{r^\alpha e^{i\alpha\pi}\cancel{e^{i\pi}}^{-1}}{r^2-1}\,dr = +e^{i\alpha\pi}\int_0^\infty \frac{r^\alpha}{r^2-1}\,dr \to I$$

$$-i\pi\left(\text{Res}(f,-1) + \text{Res}(f,+1)\right) + (1+e^{i\alpha\pi})\,I = 0$$

$$\left.\frac{z^\alpha}{(z-1)}\right|_{z=-1} \qquad \left.\frac{z^\alpha}{z+1}\right|_{z=1}$$

$$\frac{(-1)^\alpha}{-2} \qquad \frac{1}{2}$$

$$\frac{e^{i\alpha\pi}}{-2}$$

$$-i\pi\frac{1}{2}\left(1 - e^{i\alpha\pi}\right) + \left(1+e^{i\alpha\pi}\right)I = 0$$

$$I = \frac{i\pi}{2}\frac{(1-e^{i\alpha\pi})}{(1+e^{i\alpha\pi})}$$

Same

Note, other solution is

$$I = \frac{i\pi}{2}\,\frac{1}{1-e^{i2\pi\alpha}}\left(1-2e^{i\pi\alpha}+e^{i2\pi\alpha}\right) = \frac{i\pi}{2}\,\frac{(1-e^{i\pi\alpha})^{\cancel{2}}}{\cancel{(1-e^{i\pi\alpha})}(1+e^{i\pi\alpha})}$$