6333e2-1.

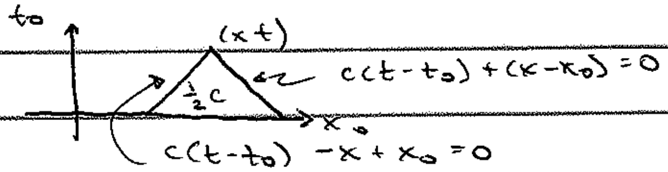
$$u(x,t) = \int_0^t \int_a^b G\,h\,u\,dx_0\,dt_0 + \int_a^b (G u_{t_0} - G_{t_0} u)\Big|_{t_0=0} dx_0$$
$$- c^2 \int_0^t (u G_{x_0} - u_{x_0} G)\Big|_{x_0=a}^{x_0=b} dt_0$$

- Assume u & G are 0 for x large enough
  ie as a & $b \to \infty$
- Have $u_t\big|_{t_0} = 0$

$$u = \int_0^t \int_{-\infty}^{\infty} G\,Q\,dx_0\,dt_0 - \int_a^b u\,G_{t_0}\Big|_{t_0=0}$$

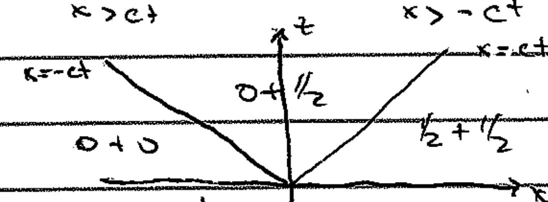
$$G = \tfrac{1}{2c} H\left(c(t-t_0) - |x-x_0|\right)$$

$$G_{t_0}\big|_{t_0=0} = \tfrac{1}{2}\,\delta(ct - |x-x_0|)$$



$$u = \tfrac{1}{2c}\int_0^t \int_{x-c(t-t_0)}^{x+c(t-t_0)} Q(t_0)\,dx_0\,dt_0 + \int_{-\infty}^{x} f(x_0)\tfrac{1}{2}\,\delta(ct-x+x_0)\,dx_0$$
$$+ \int_x^{\infty} f(x_0)\tfrac{1}{2}\,\delta(ct+x-x_0)\,dx_0$$

$$= \tfrac{1}{2c}\int_0^t Q(t_0) \underbrace{\int_{x-c(t-t_0)}^{x+c(t-t_0)} dx_0}_{\substack{x+c(t-t_0)-x+c(t-t_0)\\ 2c(t-t_0)}} + \tfrac{1}{2} \underbrace{f(x-ct)}_{\substack{1 \text{ if } x-ct>0\\ x>ct}} + \tfrac{1}{2} \underbrace{f(x+ct)}_{\substack{1 \text{ if } x+ct>0\\ x>-ct}}$$



$$u = \int_0^t Q(t_0)(t-t_0)\,dt_0 + \begin{cases} 0 & x<-ct \\ 1/2 & -ct<x<ct \\ 1 & ct<x \end{cases}$$

$$u = \begin{cases} \int_0^t q(t_0)(t-t_0)\,dt_0 & t<T_M \\ \int_0^{T_M} q(t_0)(t-t_0)\,dt_0 & T_M<t \end{cases} + \begin{cases} 0 & x<-ct \\ 1/2 & -ct<x<ct \\ 1 & ct<x \end{cases}$$

* Note: $x \geq 0$ so $u = \ldots \begin{cases} 1/2 & 0<x<ct \\ 1 & ct<x \end{cases}$

b) DofD: For a given observation pt., the source region $(x_0 t_0)$ that transmits info to (x,t)

DofI: For a given source pt $(x_0 t_0)$, the set of obs. pts (x,t) that receive info from $(x_0 t_0)$

Alternate calculation of boundary integral.

$$\frac{1}{2}\int_{-\infty}^{\infty} f(x_0)\,\delta(ct-|x-x_0|)\,dx_0$$

$$= \frac{1}{2}\int_{0}^{\infty} \delta(ct-|x-x_0|)\,dx_0$$

$$= \frac{1}{2}\int_{0}^{x} (x_0<x) \;+\; \frac{1}{2}\int_{x}^{\infty} (x_0>x)$$

$$\therefore\ x-x_0>0 \qquad\qquad \therefore\ x-x_0<0$$

$$|x-x_0| = x-x_0 \qquad\qquad |x-x_0| = -x+x_0$$

$$= \frac{1}{2}\int_0^x \underbrace{\delta(ct-x+x_0)}_{1 \text{ if}}\,dx_0 \;+\; \frac{1}{2}\int_x^{\infty}\underbrace{\delta(ct+x-x_0)}_{1\text{ if}}\,dx_0$$

$$ct-x+x_0=0 \qquad\qquad ct+x-x_0=0$$

$$x_0 = x-ct \qquad\qquad x_0 = x+ct$$

Require $0<x-ct<x$ $\qquad$ Require $x<x+ct<\infty$

$0<t<x/c$ $\qquad$ Always true for $t\geq 0$

$$= \begin{cases} 0+\frac{1}{2} = \frac{1}{2}, & 0<x<ct \\ \frac{1}{2}+\frac{1}{2} = 1, & ct<x \end{cases}$$

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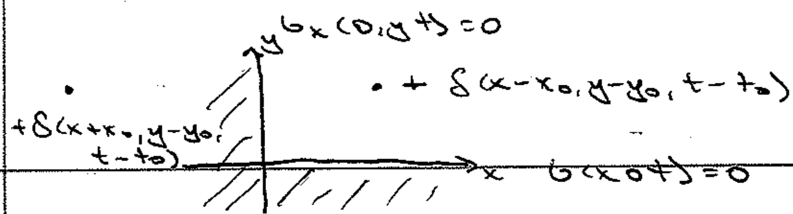

$-\delta(x+x_0, y+y_0, t-t_0) \quad -\delta(x-x_0, y+y_0, t-t_0)$

$$G_\infty(\underline{x}, t, \underline{x}_0, t_0) = \frac{1}{2\pi c} \frac{H(c(t-t_0)-r)}{\sqrt{c^2(t-t_0)^2 - r^2}} \qquad r = \left((x-x_0)^2 + (y-y_0)^2\right)^{1/2}$$

$$G = +G_\infty(x,y,t,x_0,y_0,t_0) + G_\infty(x,y,t,-x_0,y_0,t_0)$$
$$- G_\infty(x,y,t,x_0,-y_0,t_0) - G_\infty(x,y,t,-x_0,-y_0,t_0)$$

WAVE PROP.

1D: Step function (front) moving w/ speed c that propagates unchanged in time. Observer "sees" the source forever.

2D: Propagating step function as in 1D but the amplitude decreases like $\frac{1}{t}$ after the front passes.

3D: Propagating δ function traveling w/ speed c. Once the front passes the amplitude is 0 again.

2D & 3D: Energy is spread on circle or sphere in 2D & 3D, respectively, which leads to amplitude decay.

HEAT

Gaussian in space w/ width/standard deviation that increases like $\sqrt{t}$. The amplitude decreases as $1/t^{n/2}$ as energy is spread in space.

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$$\int_{t_i}^{t_f}\int_a^b v\,Lu\,dx\,dt = \int_{t_i}^{t_f}\int_a^b v\,(u_{tt}+\alpha u_x-\beta u_{xx})\,dx\,dt$$

$$= \underbrace{\int_a^b\int_{t_i}^{t_f} v\,u_{tt}\,dt\,dx}_{\text{IBP twice}\ \dots} + \alpha \underbrace{\int_{t_i}^{t_f}\int_a^b v\,u_x\,dx\,dt}_{\text{once}\ \dots} - \beta \underbrace{\int_{t_i}^{t_f}\int_a^b v\,u_{xx}\,dx\,dt}_{\text{twice}}$$

$$= \int_a^b\left[(v u_t - u v_t)\Big|_{t_i}^{t_f} + \int_{t_i}^{t_f} u\,v_{tt}\,dt\right]dx$$
$$+ \alpha\int_{t_i}^{t_f}\left[u v\Big|_a^b - \int_a^b u\,v_x\,dx\right]dt$$
$$- \beta\int_{t_i}^{t_f}\left[(v u_x - u v_x)\Big|_a^b + \int_a^b v\,u_{xx}\,dx\right]dt$$

$$= \int_{t_i}^{t_f}\int_a^b u\left[v_{tt} - \alpha v_x - \beta v_{xx}\right]dx\,dt$$
$$+ \int_a^b (v u_t - u v_t)\Big|_{t_i}^{t_f}\,dx$$
$$+ \int_{t_i}^{t_f} (\alpha u v - \beta v u_x + \beta u v_x)\Big|_a^b\,dt$$

$$= \int_{t_i}^{t_f}\int_a^b u\,L^* v\,dx\,dt + R$$

$$L^* = \frac{\partial^2}{\partial t^2} - \alpha\frac{\partial}{\partial x} - \beta\frac{\partial^2}{\partial x^2}$$

$$R = \int_a^b\left[(v u_t - u v_t)\Big|_{t_f} - (v\cancelto{g(x)}{u_t} - \cancelto{0}{u}\,v_t)\Big|_{t_i=0}\right]dx$$
$$+ \int_{t_i=0}^{t_f}\Big[(\alpha \cancelto{h}{u}\,v - \beta v u_x + \beta \cancelto{h}{u}\,v_x)\Big|_b$$
$$- (\alpha \cancelto{0}{u}\,v - \beta v u_x + \beta \cancelto{0}{u}\,v_x)\Big|_a$$

ADJOINT: $L^* G^* = \delta(x-x_0)\,\delta(t-t_0)$ $\qquad L \neq L^*$

$G^*\big|_{x=b} = 0 \qquad G^*\big|_{x=a} = 0 \qquad BC = BC^*$

$G^*\big|_{t>t_0} = 0 \qquad G^*_t\big|_{t>t_0} = 0 \qquad IC \neq IC^*$