SIAM REV.
Vol. 40, No. 2, pp. 324–326, June 1998

009

ELIMINATING GIBB'S EFFECT FROM SEPARATION OF VARIABLES SOLUTIONS*

T. E. PETERSON†

Abstract. To solve the Laplace equation on a rectangular domain with given nonhomogeneous Dirichlet boundary conditions, separation of variables is frequently used. However the convergence of the resulting series solution will often be slow, due to the presence of Gibb's effects. A simple trick can avoid this and greatly increase the speed of convergence. This can illustrate to students how use of mathematical insight can give a practical benefit.

Key words. Gibb's effect, separation of variables

AMS subject classifications. 35C10, 35J05

PII. S0036144596316224

Let $\Omega = [0, X] \times [0, Y]$ and consider the boundary value problem

$$\Delta u = 0 \text{ in } \Omega, \quad u = g \text{ on } \partial\Omega,$$

where g is a continuous function defined on the boundary $\partial\Omega$ of Ω. Such a problem is commonly used to illustrate the technique of separation of variables; see for example [BC], [G], [H], [W]. Begin by letting $u = u_1 + u_2 + u_3 + u_4$, where each u_i is nonzero on only one edge of Ω. More precisely, let

$$g_1(x) = g(x, 0), \quad g_2(y) = g(X, y), \quad g_3(x) = g(x, Y), \quad g_4(y) = g(0, y).$$

The function u_1 is defined by $\Delta u_1 = 0$ in Ω and the boundary conditions

$$\begin{aligned} &u_1(x, 0) = g_1(x),\ u_1(x, Y) = 0,\ 0 < x < X, \\ &u_1(0, y) = u_1(X, y) = 0,\ 0 < y < Y, \end{aligned}$$

with analogous definitions for u_2, u_3, and u_4. Next, the separation of variables technique is applied to each u_i, resulting in

$$\begin{aligned} u_1(x, y) &= \sum_{n=1}^{\infty} a_n \sin(n\pi x/X) \sinh(n\pi(Y - y)/X), \\ &\qquad a_n = \frac{2}{X \sinh(n\pi Y/X)} \int_0^X g_1(x) \sin(n\pi x/X)\, dx, \\ u_2(x, y) &= \sum_{n=1}^{\infty} b_n \sinh(n\pi x/Y) \sin(n\pi y/Y), \\ &\qquad b_n = \frac{2}{Y \sinh(n\pi X/Y)} \int_0^Y g_2(y) \sin(n\pi y/Y)\, dy, \\ u_3(x, y) &= \sum_{n=1}^{\infty} c_n \sin(n\pi x/X) \sinh(n\pi y/X), \end{aligned}$$

*Received by the editors November 1, 1996; accepted for publication January 24, 1997.
http://www.siam.org/journals/sirev/40-2/31622.html

†Department of Mathematical Sciences, George Mason University, Fairfax, VA 22030 (tpeters1@gmu.edu).

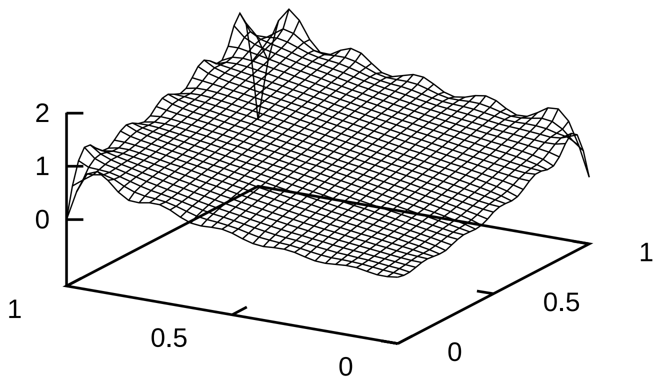

FIG. 1. *Solution obtained using ten terms.*

$$c_n = \frac{2}{X \sinh(n\pi Y/X)} \int_0^X g_3(x) \sin(n\pi x/X)\, dx,$$

$$u_4(x, y) = \sum_{n=1}^{\infty} d_n \sinh(n\pi(X - x)/Y) \sin(n\pi y/Y),$$

$$d_n = \frac{2}{Y \sinh(n\pi X/Y)} \int_0^Y g_4(y) \sin(n\pi y/Y)\, dy.$$

The solution u to the boundary value problem may now be expressed as an infinite series.

In practice, for example when seeking to produce a plot of the solution, one must take only a finite number of terms in the above series. Along each edge, one is essentially representing g by a Fourier sine series. Unless g is zero at both endpoints of the edge, a Gibb's effect will be present, and convergence of the series slow. Thus, to obtain an accurate approximation to the solution, a large number of terms will be needed.

The presence of Gibb's effects may be avoided by a simple trick. Let b be the bilinear interpolant of g over Ω;

$$b(x, y) = g(0, 0)\frac{(X - x)(Y - y)}{XY} + g(X, 0)\frac{x(Y - y)}{XY} + g(X, Y)\frac{xy}{XY} + g(0, Y)\frac{(X - x)y}{XY}.$$

Let $u = v + b$, and since any bilinear function is harmonic, v will satisfy

$$\Delta v = 0 \text{ in } \Omega, \quad v = g - b \text{ on } \partial\Omega.$$

The procedure outlined above can now be applied to v in place of u. The advantage is that the boundary values for v at the four corners are zero, so there will be no Gibb's effect.

The motivation for this procedure is the desire that the corner values be zero in order to avoid a Gibb's effect—subtracting the bilinear interpolant makes these values zero in the simplest way possible.

As an illustration consider $X = Y = 1$ and $g(x, y) = x^2 + y^2$. Figure 1 shows the solution obtained by the standard procedure, taking the first ten terms in the series solution. Figure 2 shows the solution obtained using the bilinear interpolant as described, also with ten terms. The superior quality of the latter solution is clearly evident.

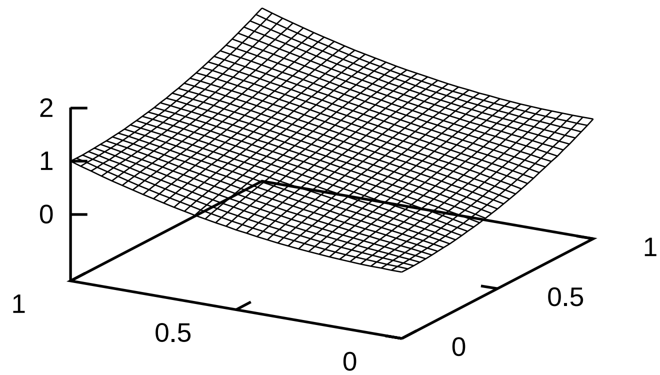

FIG. 2. *Solution obtained using bilinear interpolant and ten terms.*

This is undoubtedly not an original observation; however it does not seem to appear in any of the standard books on the topic.[1] It could provide a good example to students in an introductory course of using mathematical insight to obtain a practical benefit. A similar technique, using terms of the form $\arctan((y - y_0)/(x - x_0))$, can be used to handle jump discontinuities in the boundary data.

REFERENCES

[BC] BROWN AND CHURCHILL, *Fourier Series and Boundary Value Problems*, 5th ed., McGraw-Hill, New York, 1993.

[G] K. E. GUSTAFSON, *Introduction to Partial Differential Equations and Hilbert Space Methods*, 2nd ed., John Wiley & Sons, New York, 1987.

[H] R. HABERMAN, *Elementary Applied Partial Differential Equations*, 2nd ed., Prentice–Hall, Englewood Cliffs, NJ, 1987.

[W] H. F. WEINBERGER, *A First Course in Partial Differential Equations*, John Wiley & Sons, New York, 1965.

[1] A footnote in [G] does mention this technique, but as a means of extending existence theorems.