MATH 6333

6333- 8p4q2

(8.4 2) $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$ $u(x,0) = f(x)$ $u(0,t) = A$ $u(L,t) = B$

- Consider $-\lambda \phi = \phi_{xx}$ $\phi(0) = \phi(L) = 0$

$$\phi_n = \sin \sqrt{\lambda_n}\, x \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2$$

$$\phi_m \frac{\partial u}{\partial t} = k \phi_m \frac{\partial^2 u}{\partial x^2}$$

$$= k \Big[u \underbrace{\frac{\partial^2 \phi_m}{\partial x^2}}_{-\lambda_m \phi_m} - \frac{\partial}{\partial x}\left(u \frac{\partial \phi_m}{\partial x} - \phi_m \frac{\partial u}{\partial x}\right)\Big]$$

$$\phi_m \frac{\partial u}{\partial t} = -k\lambda_m \phi_m u - k \frac{\partial}{\partial x}\left(u \frac{\partial \phi_m}{\partial x} - \phi_m \frac{\partial u}{\partial x}\right)$$

- let $u = \sum_{n=1}^{\infty} U_n(t) \phi_n(x)$ $\quad f(x) = \sum_{n=1}^{\infty} F_m \phi_n(x)$

- Apply $\int_0^L \cdot \, dx$

$$\frac{d}{dt} U_m(t) = -k\lambda_m U_m(t) - \frac{k}{\|\phi_m\|^2}\left(u \frac{\partial \phi_m}{\partial x} - \phi_m \frac{\partial u}{\partial x}\right)\Big|_0^L$$

$$- \frac{k2}{L}\Big(B \frac{m\pi}{L} \underbrace{\cos(m\pi)}_{(-1)^m} - A \frac{m\pi}{L}\Big)$$

$$U_m' = -k\lambda_m U_m - \frac{2k}{L}\left(\frac{m\pi}{L}\right)\left(B(-1)^m - A\right)$$

$m \to n$ $\quad U_n(t) = C_n e^{-k\lambda_n t} - \frac{2}{n\pi}\left(B(-1)^n - A\right)$

$$U_n(0) = C_n - \frac{2}{n\pi}\left(B(-1)^n - A\right) = F_n$$

$$C_n = F_n + \frac{2}{n\pi}\left(B(-1)^n - A\right)$$

$$u(x,t) = \sum_{n=1}^{\infty} \left[\left(F_n + \frac{2}{n\pi}\left(B(-1)^n - A\right)\right) e^{-k\lambda_n t} - \frac{2}{n\pi}\left(B(-1)^n - A\right)\right] \cdot \sin \frac{n\pi}{L} x$$