9.4.2 a & b 6333-9p4p2-2.gif

- Given $Lu = f \quad B_1(u)|_a = \alpha \quad B_2(u)|_b = \beta$
- let $u = \bar{u} + r$ where $B_1(r)|_a = \alpha \quad B_2(r)|_b = \beta$

Then
$$L\bar{u} = \bar{f} \quad B_1(\bar{u})|_a = 0 \quad B_2(\bar{u})|_b = 0$$
where $\bar{f} = f - Lr$

- u & $\bar{u}$ satisfy the same homog problem
$$Lu_H = 0 \quad B_1(u_H)|_a = 0 \quad B_2(u_H)|_b = 0$$

$\therefore$ If $u_H \neq 0 \Rightarrow \bar{u}_H \neq 0$ and $\bar{f}$ must satisfy F.A.

- Given $L^* v = 0 \quad B_1^*(v)|_\alpha = 0 \quad B_2^*(v)|_\alpha = 0$
$$\int_a^b v \bar{f}\, dx = 0 \Rightarrow \int_a^b v f\, dx = \int_a^b v Lr$$
IBP to show this is equivalent to answer in text.

- $\bar{u}_H \neq 0 \Rightarrow \lambda = 0$ is an eigenvalue.

ie. Given $L\phi = -\lambda \sigma \phi \quad B_1(\phi)|_a = 0 \quad B_2(\phi)|_b = 0$
let $\lambda_1 = 0$ w/ $\phi_1 \neq 0$

- Assume L is sturm-Liouville ($L = L^*$) so S.L. Theory applies
Then F.A. becomes
$$\int_a^b \phi_1 \bar{f}\, dx = 0 \text{ because } \bar{u}_H = v = \phi_1 \text{ is the Homog solution.}$$

- Solve by E-function expansion.
let $\bar{u} = \sum_n \bar{U}_n \phi_n \quad \bar{f} = \sigma \tilde{f}$ w/ $\tilde{f} = \sum_n \tilde{F}_n \phi_n$

- F.A. says $\int_a^b \phi_1 \sigma \tilde{f}\, dx = \int_a^b \phi_1 \sigma \sum_n \tilde{F}_n \phi_n dx = \sum_n \tilde{F}_n \int_a^b \sigma \phi_1 \phi_n dx = 0$
For $n \neq 1$ Orthog of ϕ suffices
$n = 1 \quad \tilde{F}_1 \|\phi_n\|^2 = 0 \Rightarrow \tilde{F}_1 = 0$ If not, no sol. If so, ∞ sol.

- Sub into PDE.
$$L\left(\sum_n \bar{U}_n \phi_n\right) = \sigma \sum_n \tilde{F}_n \phi_n$$
$$\sum_n \bar{U}_n \underbrace{L\phi_n}_{-\sigma\phi_n\lambda_n} = \quad ''$$
$$-\sum_n \bar{U}_n \lambda_n \int_a^b \sigma \phi_n \phi_m\, dx = \sum_n \tilde{F}_n \int_a^b \sigma \phi_n \phi_m dx$$
$$-\bar{U}_n \lambda_n = \tilde{F}_n$$

If $n \neq 1 \quad \bar{U}_n = -\tilde{F}_n / \lambda_n$
$n = 1 \quad \bar{U}_1 \cdot 0 = 0 \quad \bar{U}_1$ undefined.

If F.A. satisfied $\bar{U}_1$ is undefined ∞ # of solutions.

- let $u = \sum_n U_n \phi_n \quad r = \sum_n R_n \phi_n$

Then $U_1 = \bar{U}_1 + \underbrace{R_1}_{\text{fixed \& known}}$

$\therefore U_1$ also unspecified. ∞ # of solutions.