# Stochastic Extinction in the Presence of Delayed Feedback

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# Abstract

Extinction processes are stochastic events that occur in many applications of finite populations such as reaction kinetics, population dynamics, and bio-chemical reactions. We consider the problem of stochastic extinction as a rare event occurring in systems with delayed feedback. We derive a general formulation of the probability of extinction, and show analytically and numerically, how delay modulates the exponent of the mean time to extinction in systems with both Gaussian and non-Gaussian noise.

# Stochastic Delay Differential Equations

- SDDE:

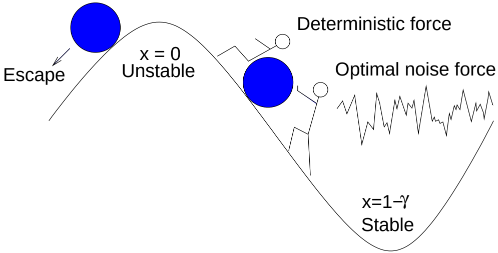
$$\dot{x}(t) = F(x(t), x(t-\tau)) + g(x(t))\xi(t)$$

- Deterministic part with delay:

$$F(x(t), x(t-\tau)) = x(1-x) - \gamma x_\tau, \qquad x_\tau = x(t-\tau)$$

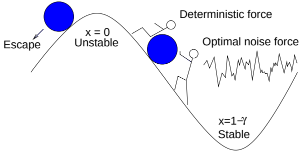
- Noise intensity $D$:

$$\xi(t) = \sqrt{2D}[\delta \text{ correlated Gaussian}]$$



# Stochastic systems with delay

- Networks with finite signal transit times.
- Lasers with external reflections.
- Machine tool cutting.
- Epidemiology (e.g. temportary immunity)
- ...



# Variational formulation

- Probability of a large (rare) fluctuation:

$$\mathcal{P}_x[x] = \exp(-R/D), \quad R = \min \mathcal{R}[x,\xi,\lambda]$$

where

$$\mathcal{R}[x,\xi,\lambda] = \frac{1}{2}\int \xi^2(t)dt + \int \lambda(t)[\dot{x}(t) - F(x,x_\tau) - g(x(t))\xi(t)]dt$$

- $\lambda$: Lagrange multiplier.
- Minimize the exponent such that $\nabla\mathcal{R} = 0$.

*Find the noise force $\xi$ that maximizes the probability of escape $\mathcal{P}$, under the constraint $\dot{x} = F + g\xi$.*

# Main results

- General theory for both Guassian and non-Guassian noise that can handle delay.
- Accurate predictions for how delay $\tau$, delay amplitude $\gamma$ and noise intensity $D$, affect the switching/escape time out of the basin of attraction of the stable steady state.

# Maximize the probablity of basin escape

- Minimize the exponent such that $\nabla \mathcal{R} = 0$.

$$\begin{aligned} \dot{x} &= \lambda g^2(x) + F(x, x_\tau) \\ \dot{\lambda} &= -\lambda^2 g(x) \frac{\partial g}{\partial x}(x) - \lambda \frac{\partial F}{\partial x}(x, x_\tau) - \lambda_{-\tau} \frac{\partial F}{\partial x_\tau}(x_{-\tau}, x) \end{aligned}$$

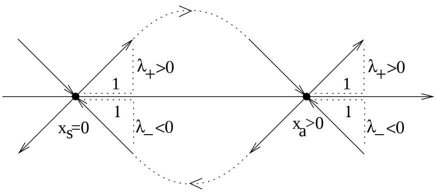
Advanced term: $x_{-\tau} = x(t + \tau)$

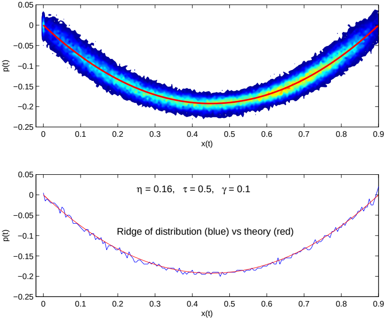
- Hamiltonian form:

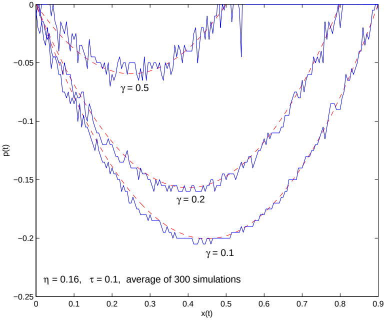
$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial \lambda}(x, \lambda, x_\tau) \\ \dot{\lambda} &= -\frac{\partial H}{\partial x}(x, \lambda, x_\tau) - \frac{\partial H}{\partial x_\tau}(x(t+\tau), \lambda(t+\tau), x(t)) \end{aligned}$$

$$H(x, x_\tau, \lambda) = \frac{\lambda^2 g^2(x)}{2} + \lambda F(x, x_\tau).$$

# Optimal noise path



# Melnikov approach to computing the action $\mathcal{R}$

- Solution exists for the problem when $\tau = 0$.
- Solutions for $\tau \neq 0$ (not small) remain close.

$$\delta_\tau x(t) \equiv x(t) - x(t-\tau) \ll 1$$

- The action be expressed as a perturbation problem:

$$\mathcal{R}[x,\xi,\lambda] = \mathcal{R}_0[x,\xi,\lambda] + \mathcal{R}_1[x,\xi,\lambda],$$

$$\begin{aligned}
\mathcal{R}_0[x,\xi,\lambda] &= \frac{1}{2}\int \xi^2(t)dt + \int \lambda(t)[\dot{x}(t) - F(x,x) - g(x(t))\xi(t)]dt \\
\mathcal{R}_1[x,\xi,\lambda] &= \int \lambda(t)[F(x,x) - F(x,x_\tau)]dt.
\end{aligned}$$

# Minimization of $\mathcal{R}_0$

- Optimal path equations:

$$\begin{aligned} \dot{x}_o &= \lambda_o g^2(x_o) + F(x_o, x_o) \\ \dot{\lambda}_o &= -\lambda_o^2 g(x_o) \frac{\partial g}{\partial x}(x_o) - \lambda_o \frac{\partial F}{\partial x}(x_o, x_o) - \lambda_o \frac{\partial F}{\partial x_\tau}(x_o, x_o). \end{aligned}$$

- Solutions:

$$\lambda_o(t) = -2\frac{F(x_o(t), x_o(t))}{g^2(x_o(t))}, \text{ and } \dot{x}_o(t) = -F(x_o(t), x_o(t)).$$

# Additive noise: $g(x) = 1$

- For the delayed logistic equation

$$\dot{x}(t) = x(1-x) - \gamma x(t-\tau) + \xi(t) \tag{1}$$

- After computing the correction $\mathcal{R}_1$, the action is:

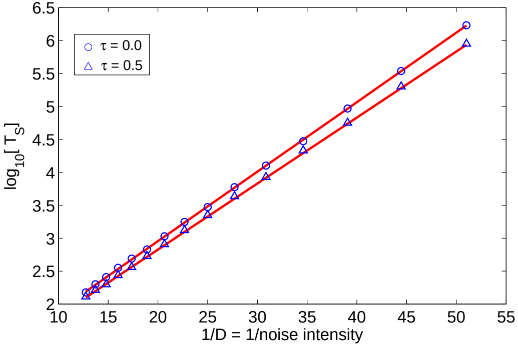
$$\mathcal{R}(\tau) \approx \frac{(1-\gamma)^3}{3}(1-\gamma\,\tau) + \mathcal{O}(\tau^2).$$

- Switching rate and switching time:

$$W_S = c\exp(-R/D) \qquad T_S = \frac{1}{W_S}$$

# Escape time vs. inverse noise intensity ($\gamma = 0.1$)

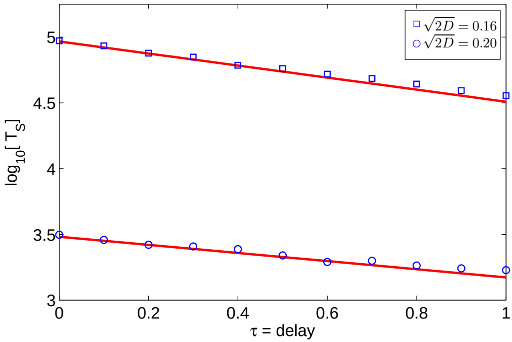
- Noise intensity ↓, escape time ↑
- Delay ↑, escape time ↓




Data points are the mean values taken over 1000 simulations.

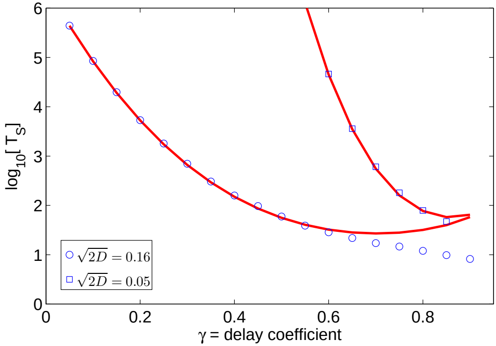
# Escape time vs. delay ($\gamma = 0.1$)

- Delay $\uparrow$, escape time $\downarrow$



# Escape time vs. dissipation ($\tau = 0.1$)

- Delay coefficient ↑, escape time ↓



# Multiplicative noise: $g(x) = \sqrt{x}$

- For the delayed logistic equation

$$\dot{x}(t) = x(1-x) - \gamma x(t-\tau) + \sqrt{x}\xi(t) \tag{2}$$

- After computing the correction $\mathcal{R}_1$, the action is:

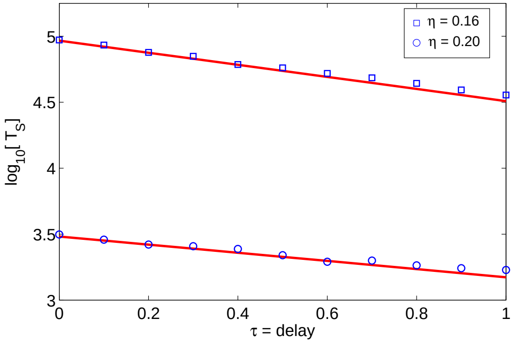
$$\mathcal{R}(\tau) \approx (1-\gamma)^3(1-\gamma\,\tau) + \mathcal{O}(\tau^2).$$

- Switching rate and switching time:

$$W_S = c\exp(-R/D) \qquad T_S = \frac{1}{W_S}$$

# Escape time vs. inverse noise intensity ($\gamma = 0.1$)

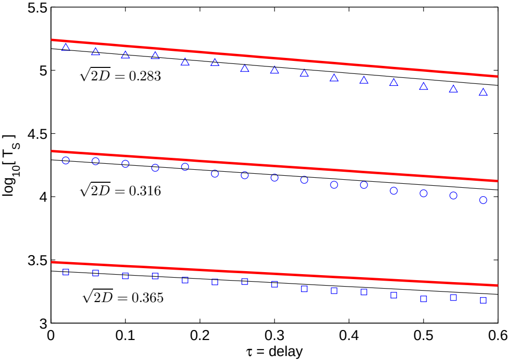
- Noise intensity ↓, escape time ↑
- Delay ↑, escape time ↓




Data points are the mean values taken over 2000 simulations.

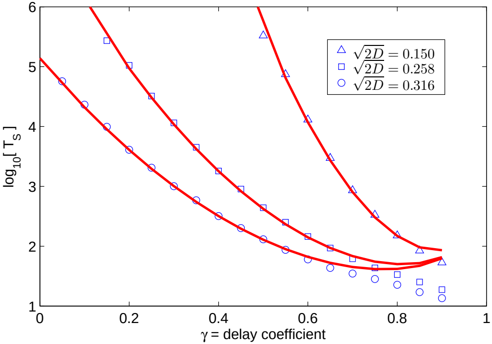
# Escape time vs. delay ($\gamma = 0.1$)

- Delay ↑, escape time ↓



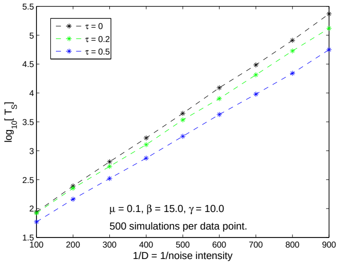
# Escape time vs. dissipation ($\tau = 0.1$)

- Delay coefficient ↑, escape time ↓

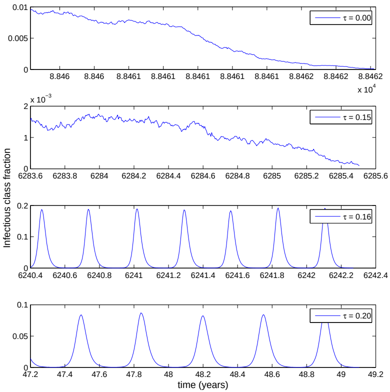


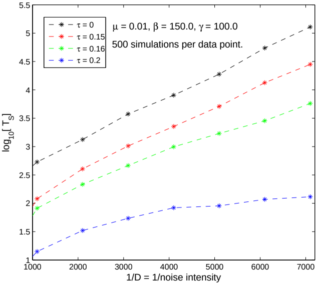
# SIRS Epidemic with temporary immunity

$$\begin{aligned}
\dot{S} &= \mu(1-S) - \beta S I + r_\gamma \gamma e^{-\mu\tau} I(t-\tau) - \xi(t) S + \eta_s(t)\sqrt{S} \\
\dot{I} &= \beta S I - (\mu+\gamma) I + \eta_i(t)\sqrt{I} \\
\dot{R} &= \gamma I - \mu R + r_\gamma \gamma e^{-\mu\tau} I(t-\tau) + \xi(t) S + \eta_r(t)\sqrt{R}
\end{aligned}$$



# Hopf to pulsations ⇒ early extinction



# Summary

- Variational approach based on optimal path that maximizes probability of escape.
  - Can consider both additive and multiplicative noise sources.
  - Generalizes to non-Guasssian noise sources.
  - Generalizes to consider the effect of delay.
- Melnikov approach based on small path deviations used to compute action.
  - Not necessarily small delay.
- Switching time: excellent fit between theory and simulations.
  - Larger noise intensity $\Rightarrow$ easier escape.
  - Larger delay coefficient $\Rightarrow$ saddles are closer $\Rightarrow$ easier escape.
  - Larger delay time
    $\Rightarrow$ farther back in history of state $x(t)$
    $\Rightarrow$ weaker repulsive force of $x = 0$ saddle

$$\dot{x} \sim x - \gamma x(t - \tau)$$

$\Rightarrow$ easier escape.