An Sir Epidemic Model with Partial Temporary Immunity

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Outline

Abstract

The SIR-epidemic model considers that recovered individuals are permanently immune, while the SIS model considers recovered individuals to be immediately re-susceptible. We study the case of temporary immunity in an SIR-based model with delayed coupling between the susceptible and removed classes, which results in a coupled set of delay-differential equations. We find conditions for which the non-zero endemic steady-state becomes unstable to periodic outbreaks. We then use analytical and numerical bifurcation analysis to describe how the severity and period of the outbreaks depends on the model parameters.

Delays in disease

"It is well understood in population biology that time delays of significant magnitudes relative to the generation time of an organism can induce oscillatory fluctuations in population abundance." Anderson & May, *Infectious Diseases of Humans*

- Physical origins
  - Latency time from infected to infectious.
  - Infectious time.
  - Maturation time of infants.
  - Temporary Immunity.
- Modeling
  - Constant coefficient ODEs: exponential distribution of immune times.
    "Easy" to analyze.
  - Integro-differential Es: arbitrary distributions.
    "Hard" to analyze.
  - Delay DEs: step distributions.
    All immune for the same fixed time τ.

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Overview

Our Goals:

- We study oscillatory epidemics with an SIR model.
  - Multiple-scales analysis of harmonic oscillations.
  - Matched-asymptotics analysis of pulsating oscillations.
  - Compare with numerical simulation and continuation methods.

Delay induced oscillations

- ODE: $x(t)' = rx(t)$
  - Let $x(t) = \exp(\lambda t)u$.
  - Characteristic equation: $\lambda = r$.
  - There exists a single real value λ, implying exponential growth or decay.
- DDE: $x(t)' = rx(t-\tau)$
  - Let $x(t) = \exp(\lambda t)u$.
  - Characteristic equation: $\lambda = re^{-\lambda\tau}$.
  - Let $\lambda = \sigma + i\omega$

$$\sigma = re^{-\sigma\tau} cos(\omega\tau), \quad \omega = -re^{-\sigma\tau}\sin(\omega\tau)$$

  - Transcendental equations with multiple solutions
  - Allows for oscillatory solutions to a first-order DDE.

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Outline

SIR Model

$$
\begin{aligned}
\frac{dS}{dt} &= \mu[1 - S(t)] - \beta I(t)S(t) + r_\gamma \gamma e^{-\mu\tau} I(t-\tau), \\
\frac{dI}{dt} &= \beta I(t)S(t) - (\mu + \gamma)I(t), \\
\frac{dR}{dt} &= \gamma I(t) - \mu R(t) - r_\gamma \gamma e^{-\mu\tau} I(t-\tau)
\end{aligned}
$$

- Equal birth and death rates.
  - Population size normalize to 1
  - $R(t) = 1 - [S(t) + I(t)]$
- Partial temporary immunity
  - $\gamma I(t-\tau)$: Temporary immunity.
  - $0 \leq r_\gamma \leq 1$: Fraction who are re-suscetible.
  - $0 < e^{-\mu\tau} \leq 1$: Fraction who survive to time τ.

$$\frac{dN(t)}{dt} = -\mu N(t) \rightarrow N(t) = N(0)e^{-\mu t}$$

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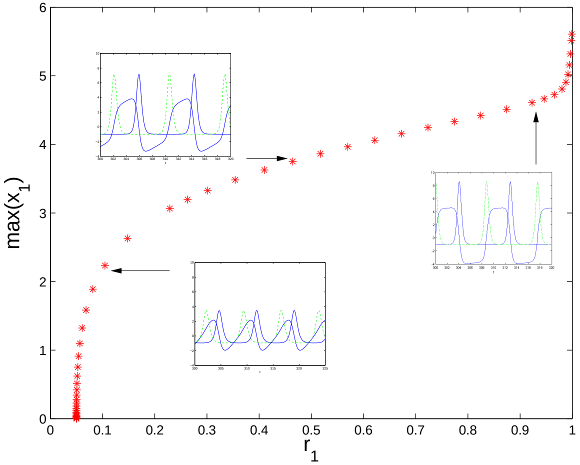
Delay induced oscillations in SIR model



Using DDE-Biftool, Engelborghs et al. and DDE-Solver, Thompson & Shampine

Steady-states and nondimensionalization

- Steady states ($\mathcal{R}_0$ = Basic reproductive number)

$$S_c = \frac{1}{\mathcal{R}_0}, \quad I_c = \frac{\frac{\mu}{\beta}(\mathcal{R}_0 - 1)}{1 - \frac{r_\gamma \gamma}{\mu+\gamma}}, \quad \text{where } \mathcal{R}_0 = \frac{\beta}{\mu + \gamma}$$

- Deviations from non-zero endemic state

$$I = I_c(1+y), \quad S = S_c(1 + \sqrt{\frac{I_c}{S_c}}x)$$

$$s = \beta\sqrt{S_c I_c}\, t \quad \text{then let } s \to t$$

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Nondimensionalized model

$$\text{Susceptible:} \quad \frac{dx}{dt} = -y - \epsilon x(a + by) + ry(t - \tau),$$

$$\text{Infectious:} \quad \frac{dy}{dt} = x(1 + y),$$

where

$$\epsilon = \sqrt{\mu} \ll 1$$

$$r = \frac{r_\gamma \gamma}{\mu + \gamma} e^{-\mu\tau}, \quad 0 \leq r < 1$$

$$\epsilon a = \frac{\mu + \beta I_c}{\beta\sqrt{S_c I_c}}, \quad \epsilon b = S_c \sqrt{I_c}$$

$$\gamma = O(\frac{1}{\mu}), \quad \beta = O(\frac{1}{\mu^2}), \quad r = O(1)$$

Outline

Hopf bifurcation to periodic outbreaks

- $(x, y) = (0, 0) \Rightarrow (S(t), I(t)) = (S_c, I_c)$.
- Full immunity, $r = 0$
  Weakly ($\epsilon \ll 1$) stable focus.
- Partial immunity, $r \neq 0$
  Examine linear stability

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ue^{\lambda t} \\ ve^{\lambda t} \end{pmatrix},$$

Characteristic equation for λ:

$$\lambda^2 + \epsilon a \lambda + 1 - re^{-\lambda \tau} = 0$$

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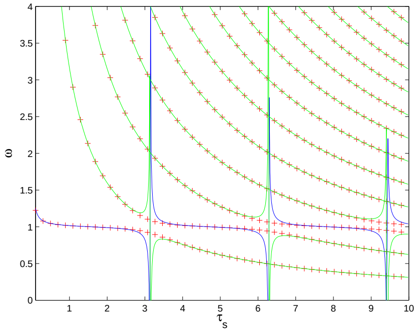
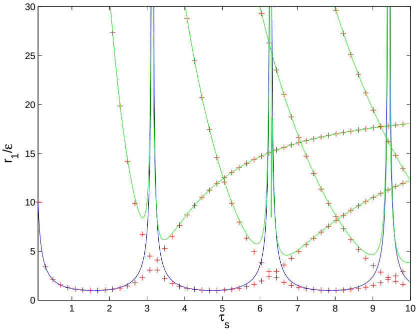
Linear-stability results

Frequency ω



Immunity parameter *r*



Outline

Introduction to delays

SIR Model with delay

Hopf bifurcation to oscillations

Small amplitude and multiple scales

Large amplitude and patched asymptotics

Summary

Small-amplitude outbreaks for low resusceptibility

- Weak immunity $\Rightarrow r = \epsilon r_1$. Local to Hopf.
- Multiple-scales analysis for DDEs, $T = \epsilon t$.

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \epsilon \frac{\partial}{\partial T}$$

$$y(t) = \epsilon^{1/2} y_1(t, T) + \epsilon y_2(t, T) + \ldots$$

$$\begin{aligned} y(t-\tau) \quad &\rightarrow \quad y(t-\tau, T-\epsilon\tau) \\ & \quad\quad y(t-\tau, T) - \epsilon\tau \frac{\partial}{\partial T} y(t-\tau, T) + \ldots . \end{aligned}$$

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Warning! Taylor series with delay can be misleading

From R.D. Driver, “Ordinary and Delay Differential Equations”

$$x' = -2x(t) + x(t-\tau)$$

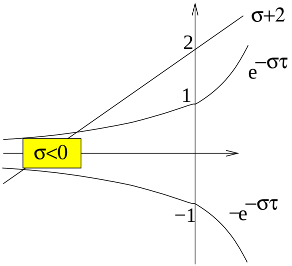
Let $x = e^{\lambda t}$

$$\lambda = -2 + e^{-\lambda\tau}$$

$$\sigma + 2 = e^{-\sigma\tau}\cos(\omega\tau), \quad \omega = -e^{-\sigma\tau}\sin(\omega\tau)$$

Consider the real-part equation

$$|\sigma + 2| \le e^{-\sigma\tau}$$



$\sigma < 0$: Exponentially decaying solutions

Small delay: $\tau \ll 1$

$$x' = -2x(t) + x(t-\tau)$$

$$x' = -2x(t) + [x(t) - \tau x'(t) + \frac{1}{2}\tau^2 x''(t) + \ldots]$$

Let $x = e^{\lambda t}$ and keep $O(\tau^2)$

$$\lambda = -2 + [1 - \tau\lambda + \frac{1}{2}\tau^2\tau^2]$$

$$\frac{1}{2}\tau^2\tau^2 - (\tau+1)\lambda + 1 = 0$$

$$\lambda = \frac{(\tau+1) \pm \sqrt{(\tau+1)^2 - 2\tau^2}}{\tau^2}$$

$\lambda_+ > 0$ for all τ: Exponentially growing solutions.
Must validate analytical results with numerical simulations.

Amplitude and frequency

- $y_1 = A(T)e^{it} + c.c.$

$$A_T = -\frac{a}{2}A - \frac{i}{6}|A|^2 A - \frac{i}{2}r_1 A(T - \tau_\epsilon)e^{-i\tau}$$

- Periodic solutions: $A(t) = Be^{i\omega t}$

$$\begin{aligned} 0 &= -aB - r_1 B \sin\tau, \\ \omega &= -\frac{1}{6}B^2 - \frac{r}{2}\cos\tau \end{aligned}$$

  Find $\omega = \omega(B)$ but not $B = B(r)$.
- Must go to higher order in analysis. $r = \epsilon^2 r_3$.

$$A_T = c_1 A + c_2|A|^2 A + c_3|A|^4 A - \frac{i}{2}r_3 A(T - \tau_\epsilon)e^{-i\tau}$$

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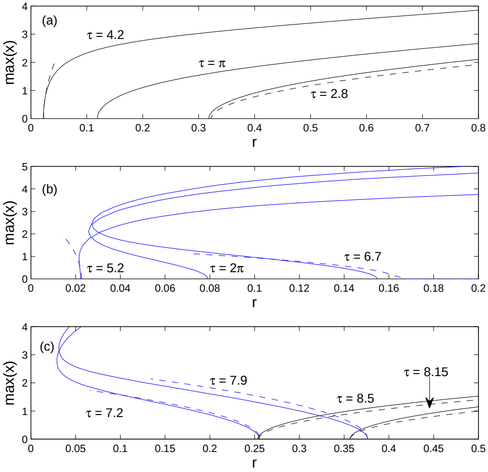
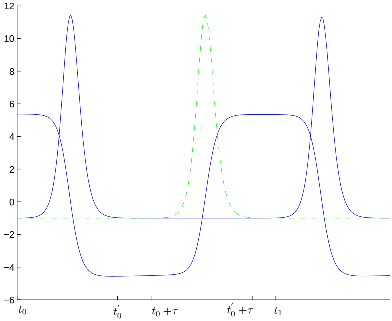
Modified multiple-scale analysis



Outline

Pulsating outbreaks for high re-susceptibility



- Inner I: $t \in [t_0, t_0^{'}]$: $y \gg 1$ and the delay term $y(t-\tau) \approx -1$
- Outer I: $t \in [t_0^{'}, t_0+\tau]$: y and $y(t-\tau) \approx -1$
- Inner II: $t \in [t_0+\tau, t_0^{'}+\tau]$: $y(t-\tau) \gg 1$ and $y \approx -1$
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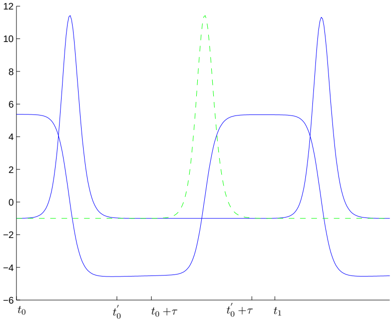

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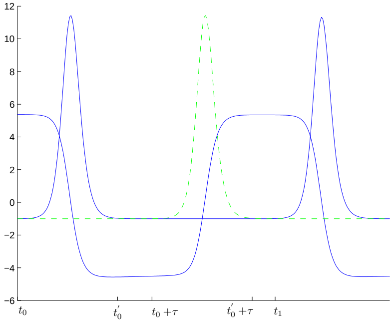

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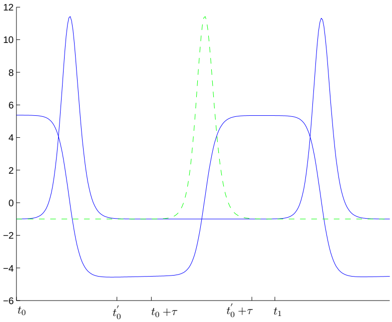

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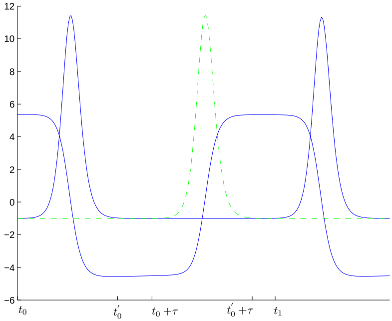

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Patching

- Find approximate solutions in each subinterval.
- Patch them together at the end points.
  - Inner I $(t_0^{'})$ = Outer 1 $(t_0^{'})$
  - Outer 1 $(t_0 + \tau)$ = Inner II $(t_0 + \tau)$
  - Inner II $(t_0{}' + \tau)$ = Outer II $(t_0^{'} + \tau)$
- End result is a map ...
  - Starting at Inner I (t_0)
  - End at Outer II (t_1)
- At time t_n, given the values (x_n, y_n) ...
  - We can predict the time of the next pulse t_{n+1} and hence the period P_n.
  - We can predict the values of the susceptible and infectious populations at the next pulse (x_{n+1}, y_{n+1}).
  - Thus, we have a map

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Map

$$\begin{aligned}
2x_{n+1} &= P_n + \epsilon[\frac{2}{3}bx_n^2 + x_n\gamma P_n - x_n\gamma\frac{r}{1-r}(2\tau - P_n) \\
&\quad -\frac{1}{2}(a-b)P_n^2] \\
0 &= P_n^2(1-r) + 2(2r-1)x_nP_n - 4\tau r x_n + \frac{4}{3}\epsilon r b x_n^2(\tau - P_n) \\
P_n &= t_{n+1} - t_n
\end{aligned}$$

Fixed points of the map $\Rightarrow$ periodic solutions of the flow

$$\begin{aligned}
x_f &= \tau - \frac{\epsilon\tau^2}{6r}[(2+r)\gamma + 2(1-r)b] \\
P_f &= 2\tau - \frac{\epsilon\tau^2}{3r}[(2+r)\gamma + 2b],
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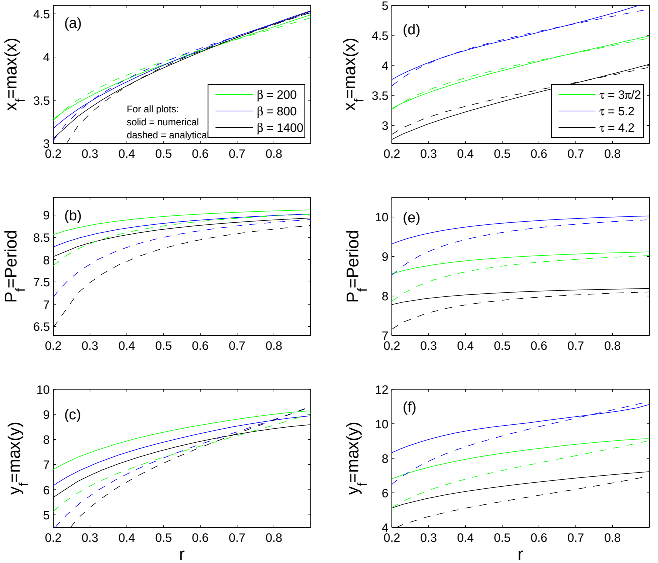
$$

Amplitude and period



Outline

Introduction to delays

SIR Model with delay

Hopf bifurcation to oscillations

Small amplitude and multiple scales

Large amplitude and patched asymptotics

Summary

Summary: as r increases...

- Hopf bifurcation
  - Severity and period of epidemics increase
- Low-resusceptibility
  - Multiple Scale analysis local to the bifurcation
  - Period related to natural relaxation period
- High-resusceptibility
  - Derive map based on pulsating epidemics
  - Period locked to delay time
  - Amplitude fixed to period by underlying Hamiltonian
  - As τ **increases**, period and intensity **increase**
  - As β **increases**, period and intensity **decrease**

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