Lecture 25

Understanding the Dynamics of Pure (Finite) Distributed Lag Models and Autoregressive Distributed Lag Models

For beginners, see Section 10.2 in Wooldridge.

We are interested in coming to understand the effect that a change in an explanatory variable, say x_t, has on the dependent variable, y_t, in a time series regression model both in the short-run and the long-run.

Consider the pure distributed lag model

$$y_t = \alpha + \beta_0 x_t + \beta_1 x_{t-1} + \beta_2 x_{t-2} + u_t \qquad (1)$$

For convenience let $\alpha = 0$. Suppose that we apply OLS to model (1) and get the fitted regression

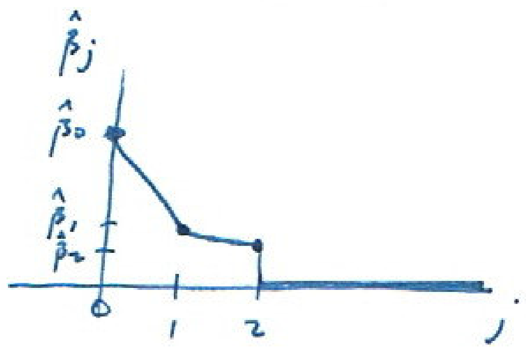
$$\hat{y}_t = \hat{\beta}_0 X_t + \hat{\beta}_1 X_{t-1} + \hat{\beta}_2 X_{t-2} \qquad (2)$$

How do we describe the effect of a change in X_t in this model? Consider the following scenario. Let $X_t = 0$ for $t = \ldots, -3, -2, -1$, $X_t = 1$ for $t = 0$, and $X_t = 0$ thereafter. This is called a "pulse" change in X_t. Now lets see what the effect on $\hat{y}_t$ is.

$$\hat{y}_t = 0 \qquad \text{for } t = \ldots, -3, -2, -1$$
$$\hat{y}_t = \hat{\beta}_0 \qquad \text{for } t = 0$$
$$\hat{y}_t = \hat{\beta}_1 \qquad \text{for } t = 1$$
$$y_t = \hat{\beta}_2 \qquad \text{for } t = 2$$
$$y_t = 0 \qquad \text{for } t = 3$$
$$\vdots$$

The Distributed Lag Function is then (assuming $\hat{\beta}_0 > \hat{\beta}_1 > \hat{\beta}_2 > 0$)

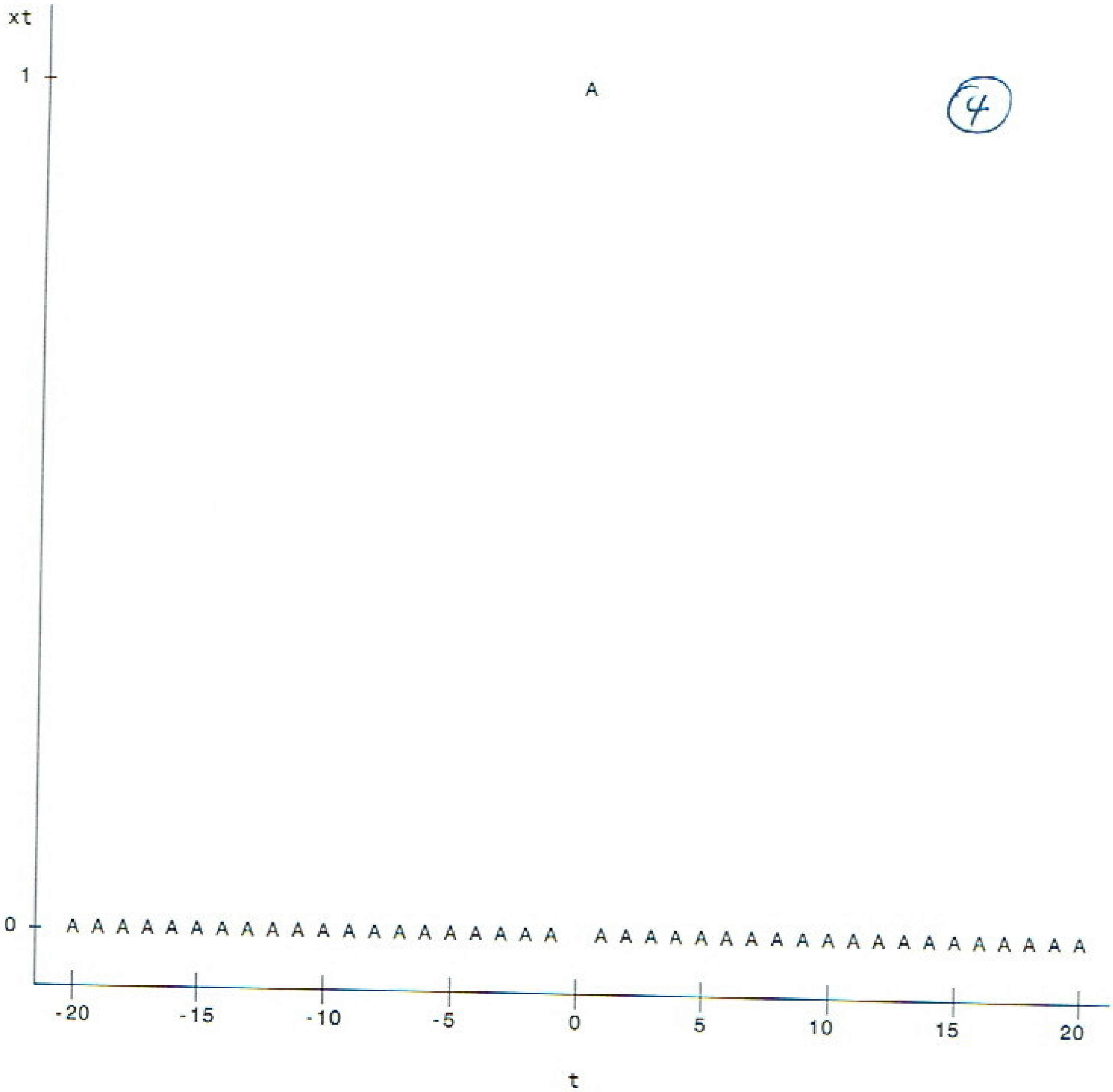
Distributed Lag Function for eq. (2)



Now go to my SAS program Dynamic.SAS. It generates the following graphs assuming that $\hat{\beta}_0 = 0.7$, $\hat{\beta}_1 = 0.5$, and $\hat{\beta}_2 = 0.3$ in eq. (2). See the pulse change in X_t on page 4 and the response of y_t to it on page 5. You can see that the effect on y_t is temporary because the change in X_t is temporary and the number of periods of effect is 3 corresponding to the number of distributed lag terms in eq. (2).

Pulse change in x(t) at time t = 0

Plot of xt*t. Legend: A = 1 obs, B = 2 obs, etc.



Effect of Pulse Change in x(t) on y(t) for Model 1:
$y(t) = 0.7x(t-1) + 0.5x(t-2) + 0.3x(t-3)$

Plot of yt*t. Legend: A = 1 obs, B = 2 obs, etc.

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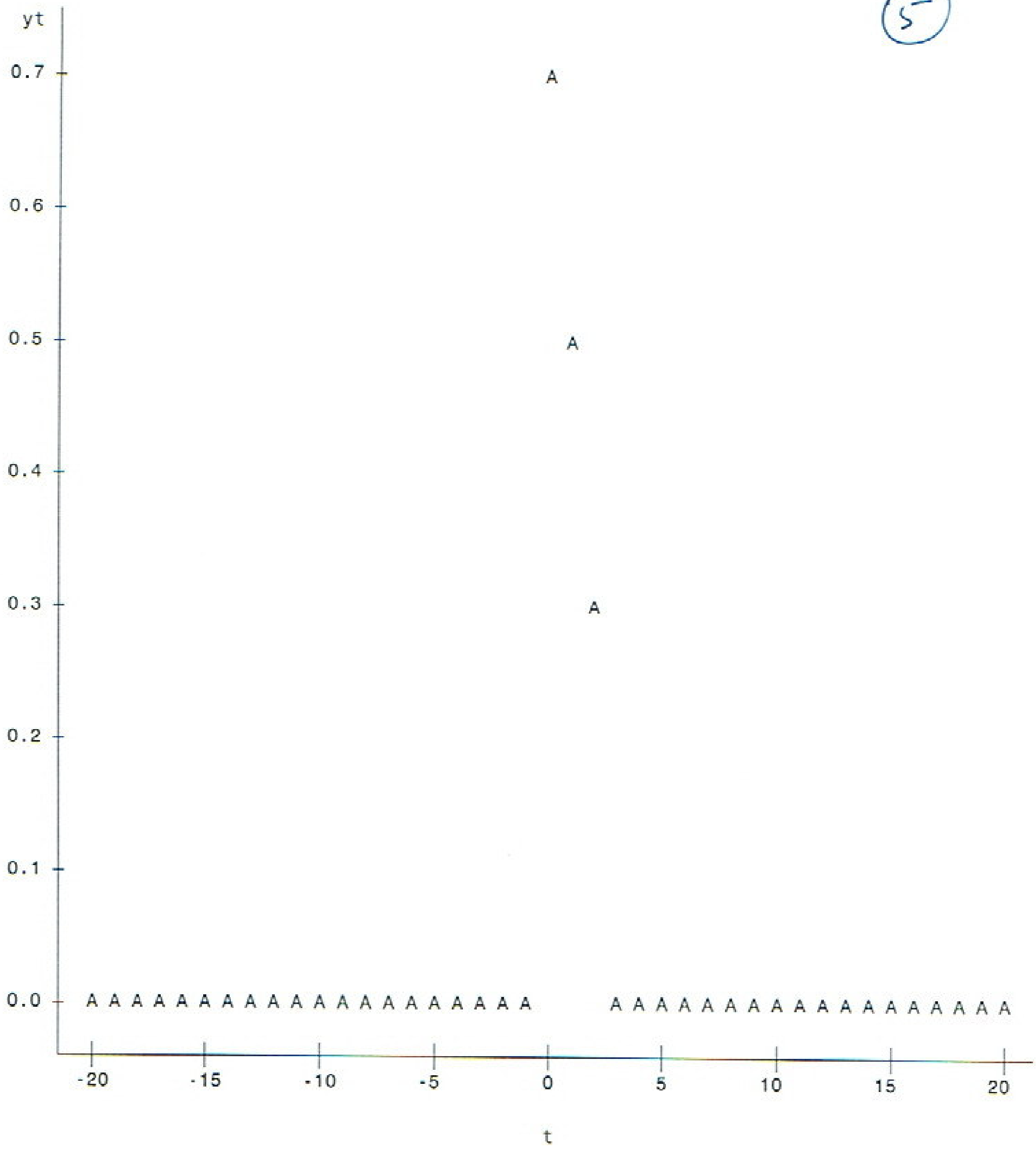


Now consider a second scenario where

$X_t = 0$ for $t = \cdots -3, -2, -1$ and $X_t = 1$ for

$t = 0, 1, 2, \cdots$. X_t takes a "step" change.

Dynamic.sas generates the cumulative

multiplier effect of this step change

in X_t. See the step change in X_t on page 7

and the cumulative multiplier effect on y_t on page 8

when $\hat{\beta}_0 = 0.7$, $\hat{\beta}_1 = 0.5$, and $\hat{\beta}_2 = 0.3$ in eq. (2).

The total cumulative effect on y_t is

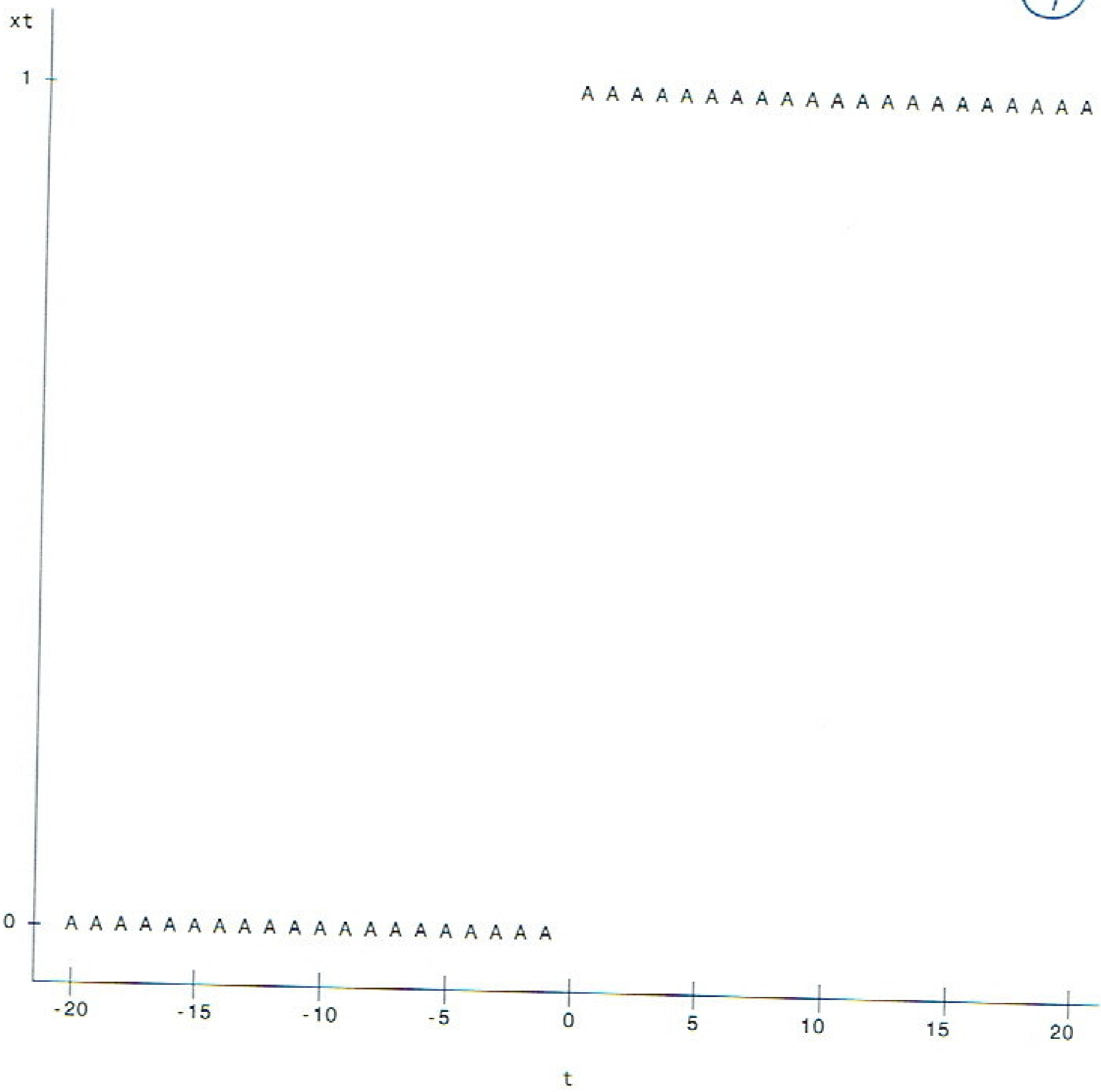
$\hat{\beta}_0 + \hat{\beta}_1 + \hat{\beta}_2 = 0.7 + 0.5 + 0.3 = 1.5$.

(This is sometimes called the <u>Long-Run Propensity</u> of a Pure Finite Distributed Lag Model.)

Step Change in x(t) at t = 0

Plot of xt*t. Legend: A = 1 obs, B = 2 obs, etc.

7

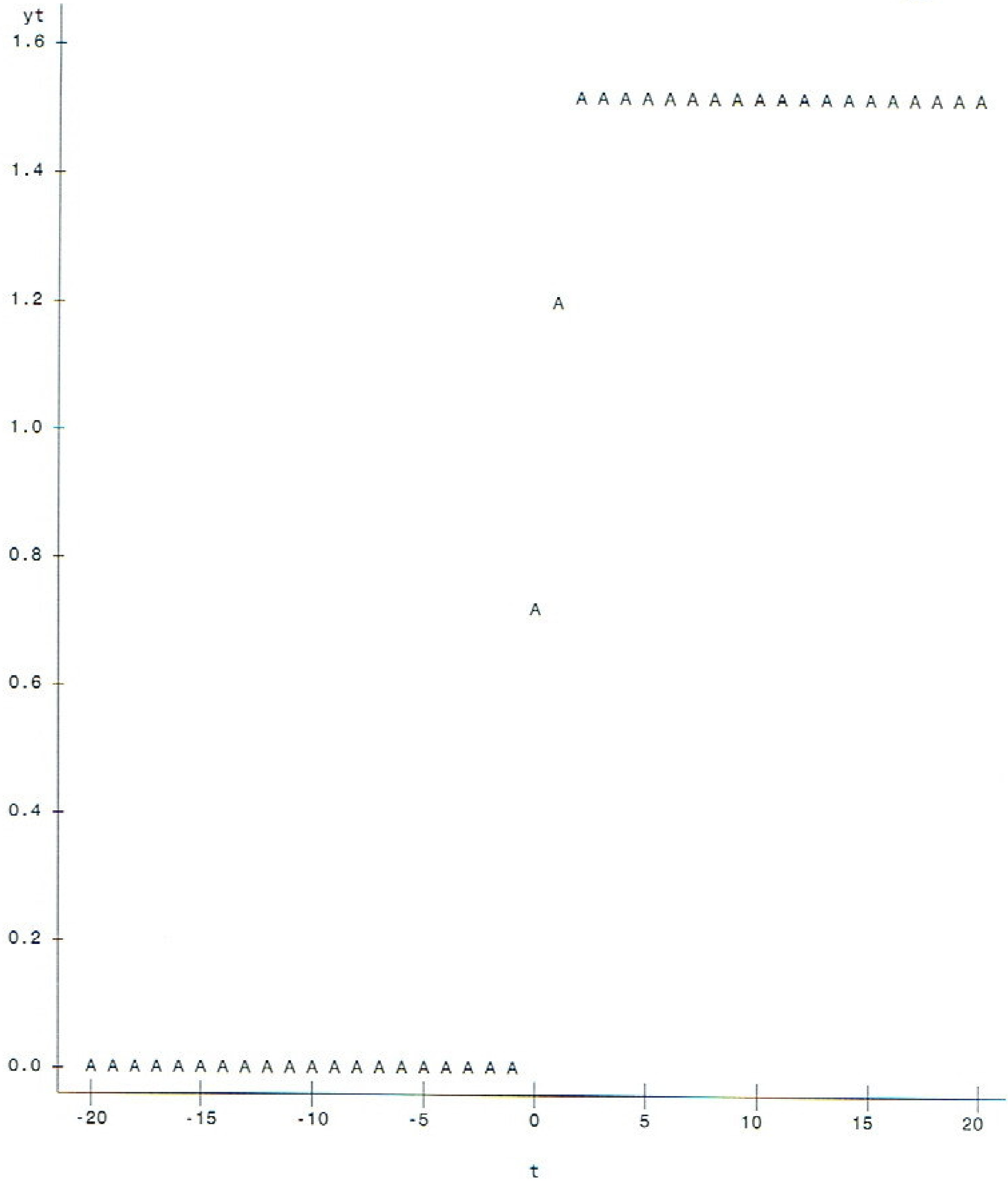


Cumulative Multipler Effect on y(t) Arising from Step Change in x(t) for Model 1:

y(t) = 0.7x(t-1) + 0.5x(t-2)+ 0.3x(t-3)

Plot of yt*t. Legend: A = 1 obs, B = 2 obs, etc.

8



Now consider the autoregressive, distributed lag model

$$\hat{y}_t = \hat{\beta}_0 x_t + \hat{\beta}_1 x_{t-1} + \hat{\beta}_2 x_{t-1} + \hat{\delta}_1 y_{t-1} . \quad (3)$$

Notice the addition of the new ~~extra~~ autoregressive term $\hat{\delta}_1 y_{t-1}$. The presence of this auto-regressive term changes the dynamic response of $\hat{y}_t$ to a change in x_t, whether it be a pulse change or a step change.

First, let us consider the pulse change $x_t = 0$ for $t = \cdots -3, -2, -1$, $x_t = 1$ for $t = 0$, and thereafter $x_t = 0$ for $t = 1, 2, \cdots$. The effect on $\hat{y}_t$ is

$$\hat{y}_t = 0 \quad \text{for } t = \ldots, -3, -2, -1$$

$$\hat{y}_0 = \hat{\beta}_0$$

$$\hat{y}_1 = \hat{\beta}_1 + \hat{\delta}_1 \hat{\beta}_0$$

$$\hat{y}_2 = \hat{\beta}_2 + \hat{\delta}_1 (\hat{\beta}_1 + \hat{\delta}_1 \hat{\beta}_0)$$

$$\hat{y}_3 = \hat{\delta}_1 \hat{y}_2$$

$$\hat{y}_4 = \hat{\delta}_1^2 \hat{y}_2$$

$$\vdots$$

$$\hat{y}_s = \hat{\delta}_1^{s-2} \hat{y}_2$$

$$\vdots$$

$$\hat{y}_\infty = 0 \qquad \text{as long as } |\hat{\delta}_1| < 1.$$

Now go to the Dynamic.sas program again and let $\hat{\beta}_0 = 0.7$, $\hat{\beta}_1 = 0.5$, $\hat{\beta}_2 = 0.3$ and $\hat{\delta}_1 = 0.8$ in eq. (3). See the pulse change in X_t on page 11 and the response of y_t to it on page 12. As you can see the effect of the pulse change in X_t is infinite but

Pulse change in x(t) at time t = 0

Plot of xt*t. Legend: A = 1 obs, B = 2 obs, etc.

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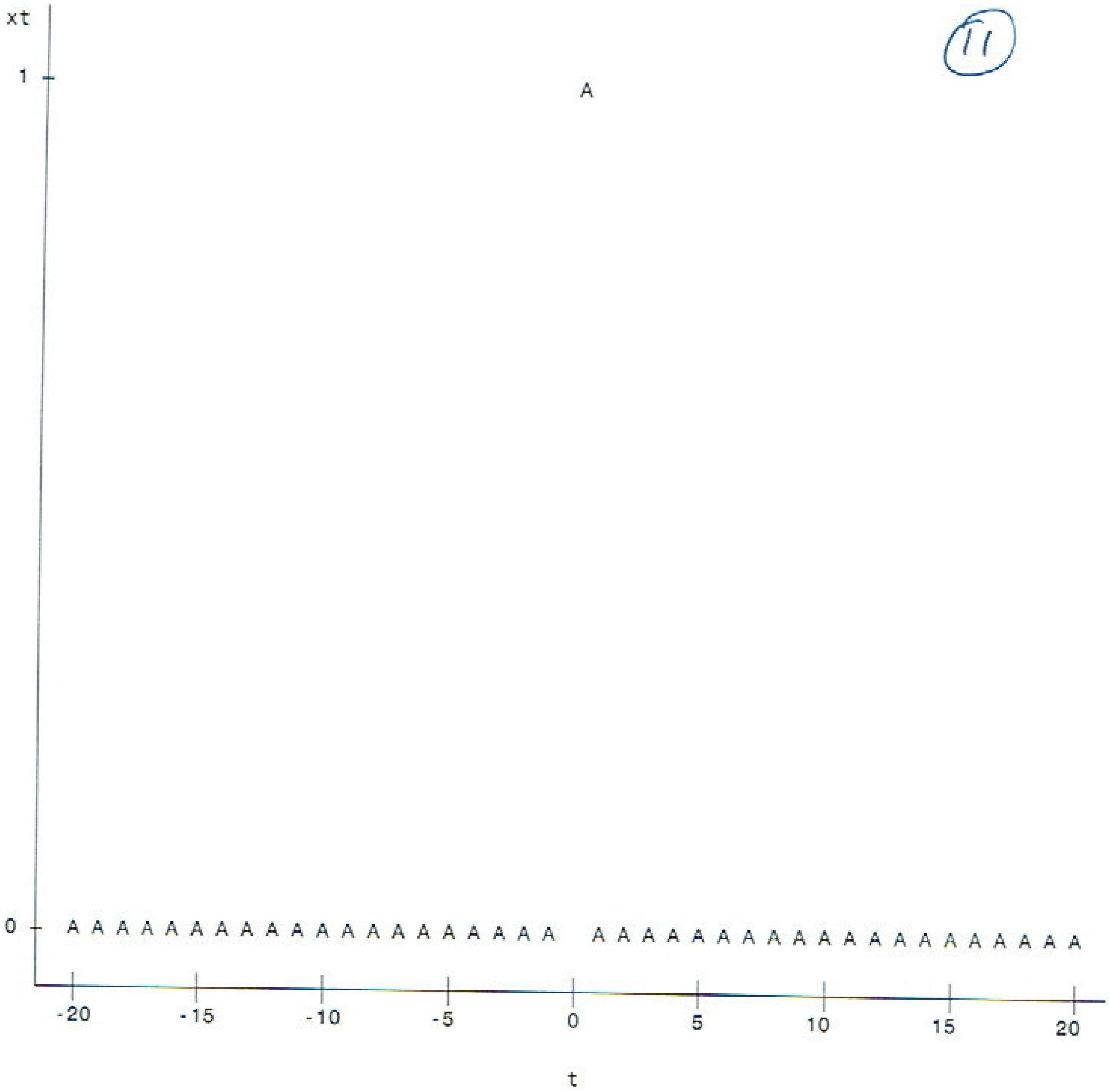
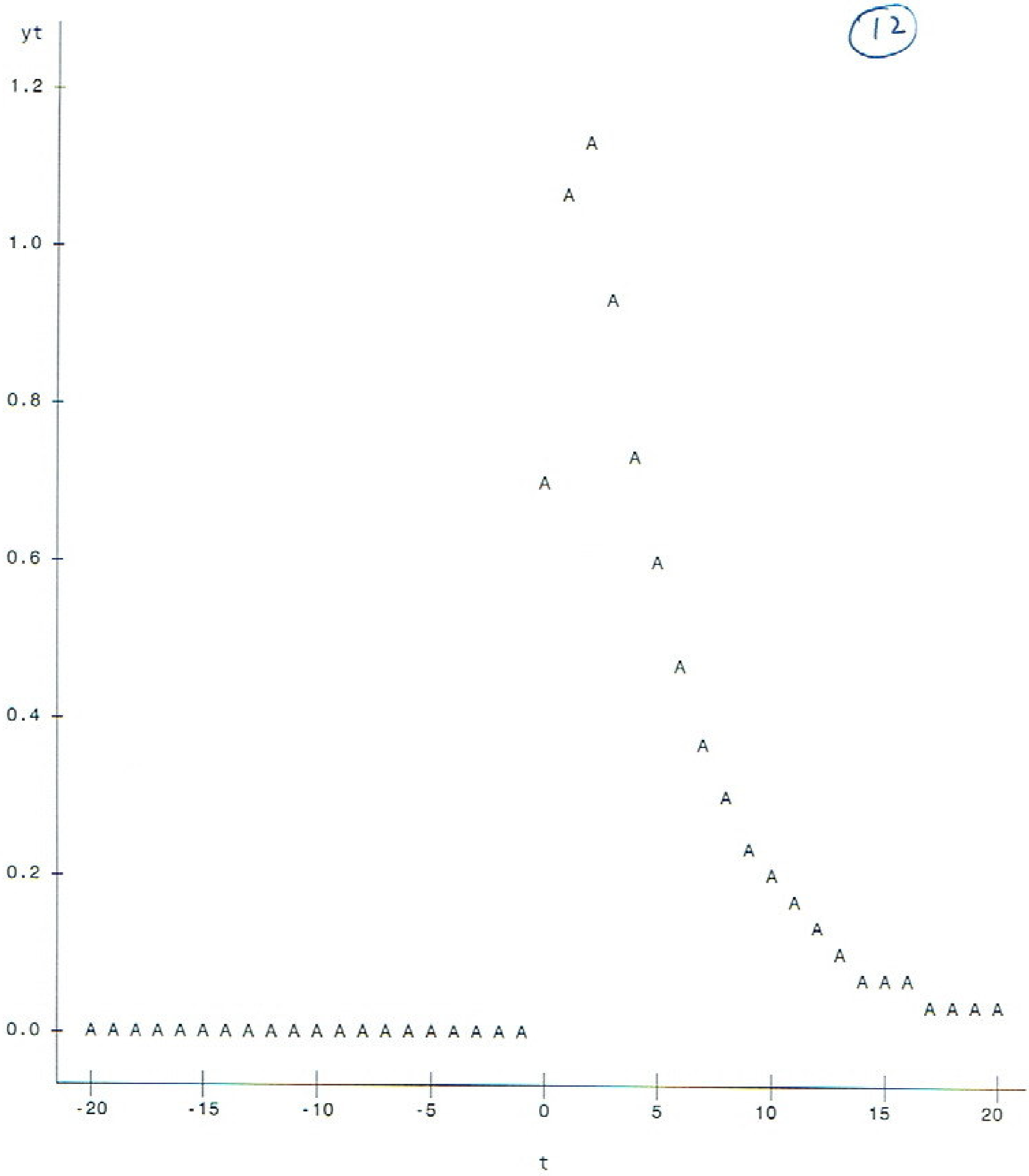


Interim Multipler Effects on y(t) of Pulse Change in x(t) for Model 2:
$y(t) = 0.7x(t-1) + 0.5x(t-2) + 0.3x(t-3) + 0.8y(t-1)$

Plot of yt*t. Legend: A = 1 obs, B = 2 obs, etc.

12



eventually diminishing down to zero. The autoregressive part of ~~the~~ model (3) is what causes the step dummy effect to be infinitely lived but diminishing to zero eventually (as long as $|\hat{\delta}_1| < 1$). The exponential decay begans at $t=2$.

The cumulative effect of a step change in X_t on y_t in model (3) is as follows:

$$\text{Cumulative effect} = \sum_{t=0}^{\infty} \hat{\gamma}_t$$

$$= \hat{\gamma}_0 + \hat{\gamma}_1 + \hat{\gamma}_2 (1 + \hat{\delta}_1 + \hat{\delta}_1^2 + \cdots)$$

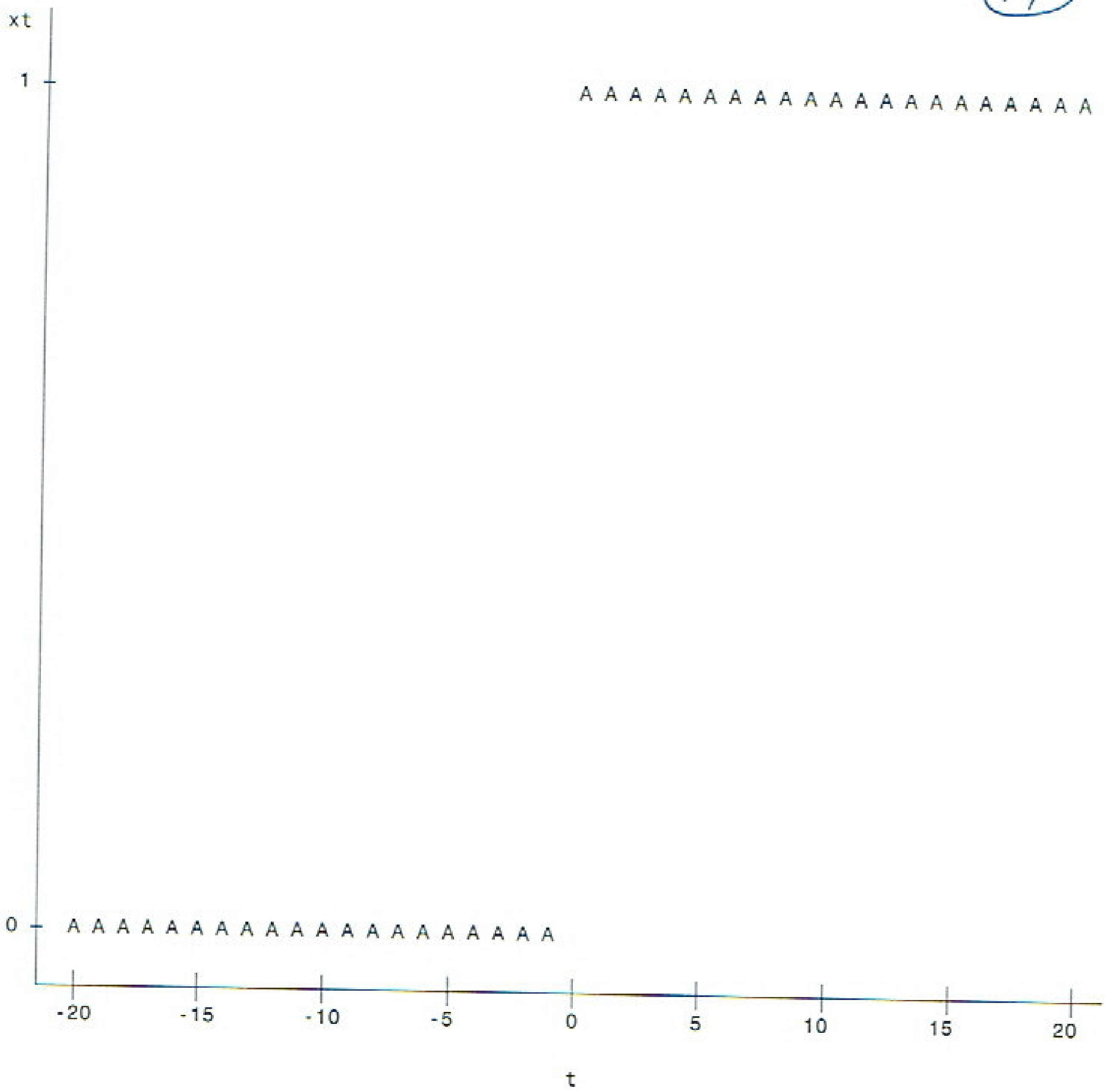
$$= \hat{\gamma}_0 + \hat{\gamma}_1 + \frac{\hat{\gamma}_2}{1-\hat{\delta}_1} . \quad (\text{as long as } |\hat{\delta}_1| < 1)$$

see the step change in X_t on page 14 and the cumulative response of y_t to it on page 15. Here again $\hat{\beta}_0 = 0.7$, $\hat{\beta}_1 = 0.5$, $\hat{\beta}_2 = 0.3$, and $\hat{\delta}_1 = 0.8$ in eq. (3). Again the cumulative effect is sometimes called the <u>Long-Run Propensity</u> of the model.

Step Change in x(t) at t = 0

Plot of xt*t. Legend: A = 1 obs, B = 2 obs, etc.

T4



Cumulative Multipler Effects on y(t) from Step Change in x(t) for Model 2:
y(t) = 0.7x(t-1) + 0.5x(t-2)+ 0.3x(t-3)+ 0.8y(t-1)

Plot of yt*t. Legend: A = 1 obs, B = 2 obs, etc.

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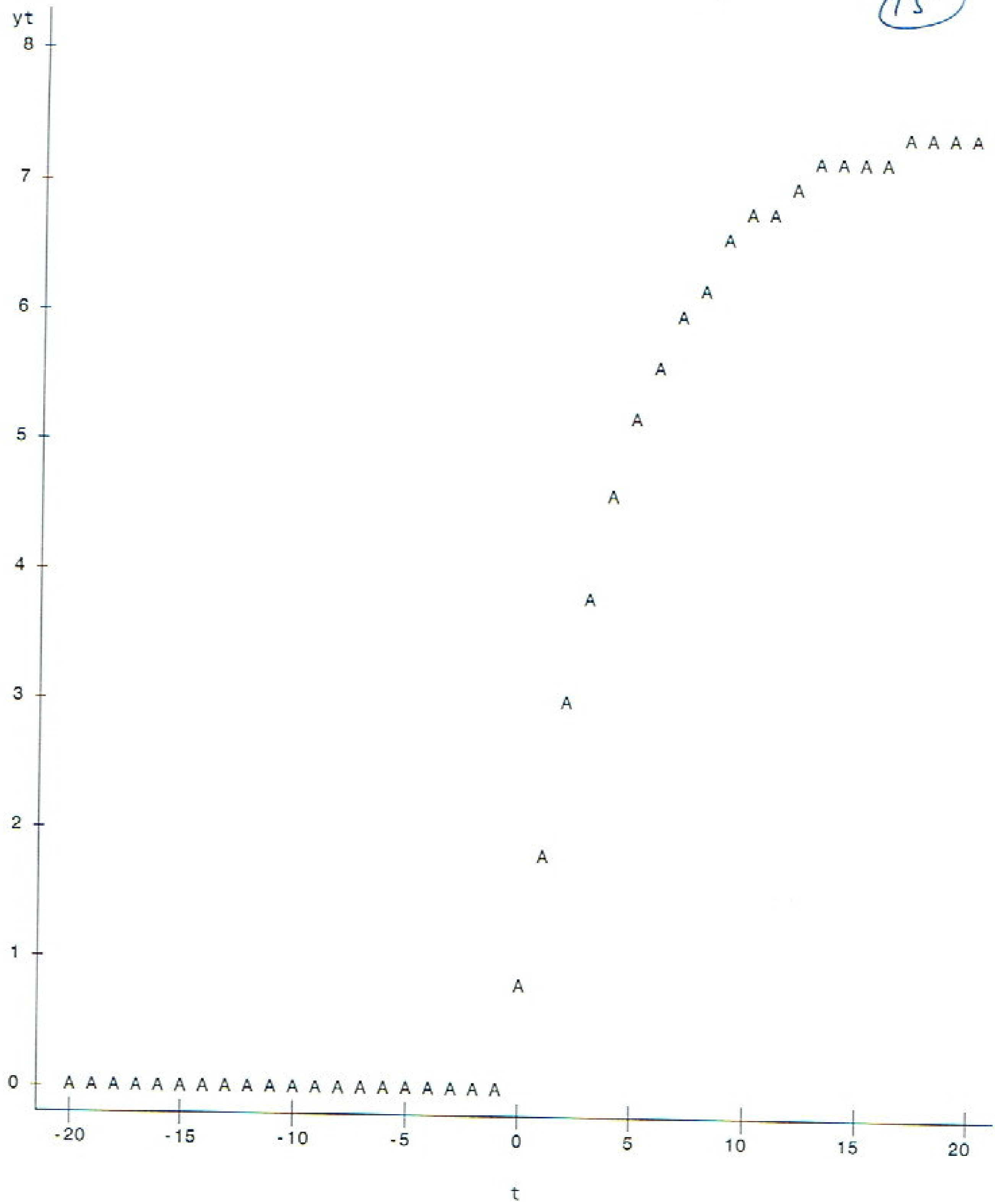

Now how do we interpret the dynamic response of the general fertility equation we estimated in fertil3.wf1?

$$\hat{y}_t = -0.77 + 0.28\, y_{t-1} + 0.11\, x_{t-2}$$

where $y_t = \Delta gfr_t$ and $x_t = \Delta pe_t$. In general let us consider the equation

$$y_t = \gamma + \delta_1 y_{t-1} + \beta_2 x_{t-2} .$$

Assume $x_t = 0$ for $t = \cdots, -3, -2, -1$ ~~0, 1, 2~~

Then the mean of y in this instance is

$$\bar{y} = \frac{\gamma}{1-\delta_1} .$$

Now consider a one period change in x_t at time $t = 0$ in the amount of Δx_t.

Then the effect on y_t is

$$y_0 = \bar{y}$$

$$y_1 = \bar{y}$$

$$y_2 = \bar{y} + \beta_2 \Delta X$$

$$y_3 = \gamma + \delta_1 (\bar{y} + \beta_2 \Delta X) = \gamma + \delta_1 y_2$$

$$y_4 = \gamma + \delta_1 y_3 = \gamma + \delta_1 (\gamma + \delta_1 y_2)$$

$$= \gamma (1 + \delta_1) + \delta_1^2 y_2$$

$\vdots$

$$y_s = \gamma (1 + \delta_1 + \delta_1^2 + \cdots + \delta_1^{s-3}) + \delta_1^{s-2} y_2$$

$\vdots$

$$y_\infty = \gamma (1 + \delta_1 + \delta_1^2 + \cdots) + \delta_1^\infty y_2$$

$$= \frac{\gamma}{1 - \delta_1} = \bar{y} \quad (\text{if } |\delta_1| < 1).$$

Now assume we have a pulse change in X_t from 0 to 10 at time = 0 (i.e. $\Delta X = 10$). What is the impact on our fertility equation? (In other words, we ~~[illegible]~~ go from having a zero change in the real personal exemption from one period to the next to a change in the real personal exemption of $\Delta X = \$10$ for period $t=0$ and return to no change in the personal exemption thereafter. We want to see how this pulse change in Δpe_t affects the change in the general fertility rate over time. Using the above estimated fertility equation we have

$$y_0 = \bar{y} = \frac{\gamma}{1-\rho_1} = \frac{-0.77}{1-0.28} = \frac{-0.77}{.72} = -1.07$$

With no change in the real personal exemption rate, we would expect there to be a <u>decline</u> in the general fertility rate of 1.07 births per 1000 women of child-bearing age each year. With this one period change of \$10 in the real personal exemption rate we can expect the following changes in the change in the general fertility rate:

$$\Delta gfr_0 = \bar{y} = -1.07$$

$$\Delta gfr_1 = -1.07$$

$$\Delta gfr_2 = -1.07 + 0.11(10) = 0.03$$

$$\Delta gfr_3 = -0.77 + 0.28(0.03) = -0.76$$

$$\Delta gfr_3 = -0.77 + 0.28(-0.76) = -0.98$$

$$\Delta gfr_4 = -0.77 + 0.28(-0.98) = -1.04$$

$$\vdots$$

$$\Delta gfr_\infty = \bar{y} = -1.07.$$

So the pulse change of \$10 in the real personal exemption rate only briefly gives rise to an increase in the fertility rate (period 2 after the change) and then thereafter we return to the long-run average decline in the general fertility rate of -1.07 births per thousand per year.

What happens when we have a permanent increase in the real personal exemption of \$10 per year? What effect does this policy have on the change rate of the general fertility rate?

Going back to the general notation of

$$y_t = \gamma + \delta_1 y_{t-1} + \beta_2 X_{t-2}$$

let us assume that $X_t = 0$ for $t = \cdots, -3, -2, -1$ and that X_t goes from zero to ΔX at time zero and remains at the new level thereafter.

The original "equilibrium" (long-run value) of y with $X_t = 0$ is

$$\bar{y} = \frac{\gamma}{1-\delta_1}.$$

It can be shown that the new equilibrium value of y when X_t goes from 0 to the ΔX level permanently is

$$\bar{\bar{y}} = \frac{\gamma + \beta_2 \Delta X}{1-\delta_2}.$$

Therefore the total impact of going from $X_t = 0$ to $X_t = \Delta X$ thereafter is

$$\bar{\bar{y}} - \bar{y} = \frac{\gamma + \beta_2 \Delta X}{1-\delta_2} - \frac{\gamma}{1-\delta_2}$$

$$= \frac{\beta_2 \Delta X}{1-\delta_1}.$$

(In essense this total impact multiplier effect is equal to the sum of the interim impacts we detailed on p. 17.)

In our general fertility example we have

$$\bar{\bar{y}} - \bar{y} = \frac{0.11(\$10)}{1-0.28} = 1.52$$

If we make the change in the real personal exemption go from zero change each year to a $10 increase each year we go from having an average decline in the general fertility rate of −1.07 children born per one thousand women of child bearing age to an average <u>increase</u> in the general fertility rate of $-1.07 + 1.52 = 0.45$ children per year. Of course, we cannot increase the real personal exemption by $10 per year forever nor can we increase the general fertility rate of women by 0.45 children per year because there is a limit to the ability of the U.S. Treasury to finance such a prolonged increase in the real exemption as well as there is a limit to the number of children that women can bear in their child-bearing years. Anyway, you have an idea how we can use an estimated Autoregressive Distributed Lag Model to look at the effects of various policy initiatives on a dependent variable of interest both in the short-run as well as in the long-run.