Lecture 7

The Gauss - Markov Theorem

Theorem: Among all linear, unbiased estimators of β_1, say $\tilde{\beta}_1 = \sum_1^N c_i y_i$, the ordinary least squares estimator, $\hat{\beta}_1 = \sum_1^N w_i y_i$, has the smallest variance. ($\hat{\beta}_1$ is BLUE)

Proof: For $\tilde{\beta}_1$ to be unbiased we require that the weights c_i satisfy the two properties:

i) $\sum_1^N c_i = 0$

ii) $\sum_1^N c_i X_i = 0$.

If these two conditions hold, $E(\tilde{\beta}_1) = \beta_1$.

Now

$$\operatorname{var}(\tilde{\beta}_1) = E(\tilde{\beta}_1 - \beta_1)^2 = E\left(\sum_1^N c_i u_i\right)^2$$

$$= \sigma^2 \sum_1^N c_i^2 .$$

Consider

$$\operatorname{var}(\tilde{\beta}_1) = \sigma^2 \sum_1^N (c_i - w_i + w_i)^2$$

$$= \sigma^2 \sum_1^N (c_i - w_i)^2 + \sigma^2 \sum_1^N w_i^2$$

$$+ \sigma^2 \sum_1^N (c_i - w_i) w_i$$

$$= \sigma^2 \sum_1^N (c_i - w_i)^2 + \sigma^2 \sum_1^N w_i^2$$

(It can be shown that $\sum_1^N (c_i - w_i) w_i = 0$.)

Therefore

$$Var(\tilde{\beta}_1) = \sigma^2 \sum_1^N (c_i - w_i)^2 + Var(\hat{\beta}_1)$$

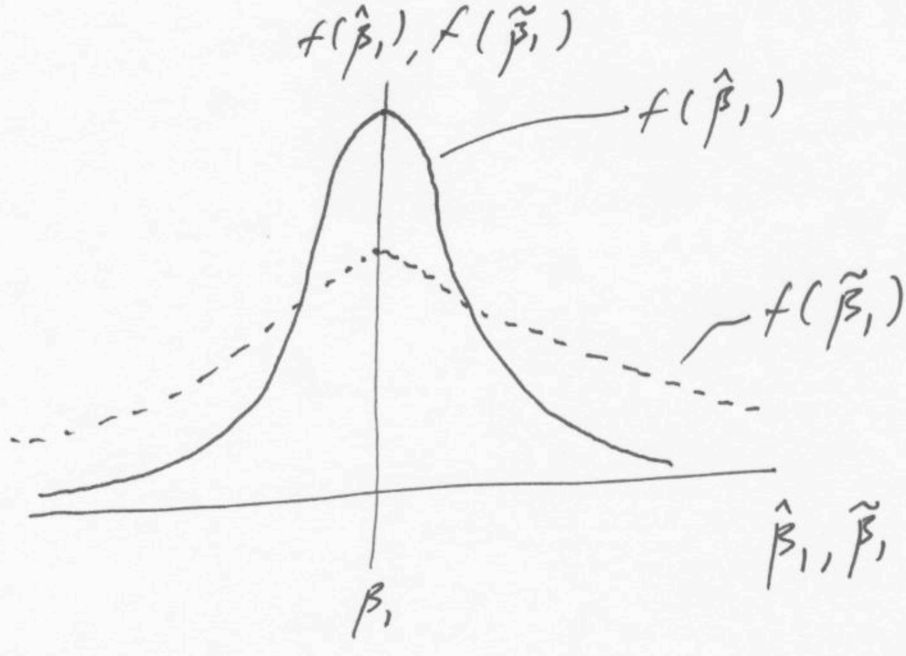
and $Var(\tilde{\beta}_1) \geq Var(\hat{\beta}_1)$.

Moreover, in courses in Mathematical statistics it can be shown that, if SLR.6 (normality of u_i) also holds, $\hat{\beta}_1$ is the minimum variance unbiased (MVU) estimator of β_1. That is, under the assumption of normality, $\hat{\beta}_1$ is the most efficient (i.e. has smallest sampling variance) among <u>all</u> unbiased estimators of β_1 irregardless of whether they are linear estimators or not.

Similarly, it can be shown that $\hat{\beta}_0$ is the best linear unbiased estimator (BLUE) of β_0.

With SLR.6, $\hat{\beta}_0$ is also the MVU estimator of β_0. Thus, if one is considering unbiased estimators of β_0 and β_1, the ordinary least squares estimators, $\hat{\beta}_0, \hat{\beta}_1$, are the best estimators you can choose.

We can represent this efficiency in the following way.



The variance of the sampling distribution of the OLS estimator, $\hat{\beta}_1$, is smaller than the variance of the sampling distribution of the competitor, $\tilde{\beta}_1$.