DETERMINISTIC TREND / DETERMINISTIC SEASON MODEL

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I. Introduction

The Deterministic Trend / Deterministic Season (DTDS) model is one of the first time series models proposed to handle trends and seasonality in economic and business data. It popularity depended on the relative ease of estimating such a model by the method of ordinary least squares and the interpretability of the model. Interestingly, **this model is a special case of the Unobservable Components model (UCM)** with fixed level ($\sigma_\eta^2 = 0$) and fixed slope ($\sigma_\xi^2 = 0$) resulting in the deterministic trend specification $\mu_t = \mu_0 + \beta_0 t$, fixed dummy seasonals ($\sigma_\omega^2 = 0$), assuming an autoregressive factor, $r_t = \rho r_{t-1} + \upsilon_t$, and no irregular component ($\sigma_\varepsilon^2 = 0$). The fact that the UCM encompasses the current simple model we are going to entertain just goes to highlight how far time series modeling has come over the last several decades. Actually, as you will see below, we generalize the autoregressive term to be an AR(r) process as compared to the AR(1) term assumed in the UCM. But the point is that, essentially, the current model is a special case of the UCM.

II. Notation for the DTDS Model

Assume the times series y_t is observed monthly. At first blush, one might think naively of writing the DTDS model as

$$y_t = \alpha + \beta t + \gamma_1 D_{t1} + \gamma_2 D_{t2} + \gamma_3 D_{t3} + \cdots + \gamma_{12} D_{t,12} + \varepsilon_t \qquad (1)$$

$$\varepsilon_t = \rho_1 \varepsilon_{t-1} + \rho_2 \varepsilon_{t-2} + \cdots + \rho_r \varepsilon_{t-r} + a_t \qquad\qquad (2)$$

where y_t is the target variable, $t = 1, 2, ..., T$, D_{tj} is a seasonal dummy variable that takes the value of one if the t-th observation is observed in the j-th month and 0 otherwise, and the errors ε_t follow an AR(r) process and the errors a_t are white noise.

However, a closer look indicates a redundancy in the sample design implied by the parametrization (1) and (2). In fact, the full set of seasonal dummy variables $D_{t1}, D_{t2}, \cdots, D_{t,12}$ is pefectly collinear with the intercept α. To avoid the so-called **Dummy Variable trap**, we may take one of three tacts:

- We can drop one of seasonal dummies, say the January seasonal D_{t1}.

This implies that the intercept, α, is the January intercept while the coefficients of the other months are "increments" to the January intercept. Let us call this the **"relative to January" parametrization**. That is, the February intercept is $(\alpha + \gamma_2)$, the March intercept is $(\alpha + \gamma_3)$, etc. Of course, the dropping of the January dummy is arbitrary and one could just as easily drop any other month. For now we will stick with dropping the January dummy and stick with the label chosen for the parametrization.

- We could drop the intercept in our model and then the seasonal dummy coefficients would represent the respective intercepts of the months. This, of course, requires that the regression be run "through the origin" and in this case some of the classical measures of goodness-of-fit like the coefficient of determination and the overall F-statistic are no longer applicable although traditional hypothesis testing can still be done in this context. One might call this the **"each season has its own intercept" parametrization**.

- We can keep all of the coefficients in (1) above but impose a restriction on the seasonal dummy coefficients that avoids the Dummy Variable trap. One restriction that is often imposed is setting the sum of the seasonal coefficients equal to zero. The advantage of this parametrization is that the signs of the coefficients reveal the "stronger" months (those with positive coefficients) versus the "weaker" months (those with negative coefficients). One might call this the **"zero sum constraint" parametrization**.

As it turns out, all of these parametrizations are **equivalent to each other** in the sense that the coefficients estimates one might get from using one parametrization can easily be translated into the coefficients estimates produced by either of the other two parametrizations. Since the relative to January parametrization is easier to implement in SAS, especially when using Proc Autoreg to estimate the autocorrelation structure (2), we will pursue this parametrization exclusively in the following discussion and write it as

$$y_t = \alpha + \beta t + \gamma_2 D_{t2} + \gamma_3 D_{t3} + \cdots + \gamma_{12} D_{t,12} + \varepsilon_t \tag{3}$$

$$\varepsilon_t = \rho_1 \varepsilon_{t-1} + \rho_2 \varepsilon_{t-2} + \cdots + \rho_r \varepsilon_{t-r} + a_t . \tag{4}$$

In terms of conventional additive time series decomposition, the **trend** is represented by the $\alpha + \beta t$ part of the model (α is the y_t intercept and β is the slope of the deterministic trend line, $\alpha + \beta t$), the part $\gamma_2 D_{t2} + \gamma_3 D_{t3} + \cdots + \gamma_{12} D_{t,12}$ is **seasonal part** of the model, and ε_t contains the **irregular part**, a_t, plus the **cyclical part**, $\rho_1 \varepsilon_{t-1} + \rho_2 \varepsilon_{t-2} + \cdots + \rho_r \varepsilon_{t-r}$, of the model.

III. Examining Some of the Details of the DTDS model

Let us look more closely at the DTDS model (3) and (4). In this form the model is a **linear** trend model as compared to a **quadratic** trend model. If $\alpha > 0$ and $\beta > 0$, then, generally speaking, the y_t data is positive at time $t = 0$, and has a positive slope to it. Of course we could have $\beta < 0$ which would imply that the data is declining. In actuality, if the data has "curvature" in it we could, instead, model the trend as $\alpha + \beta t + \theta t^2$. Of course, we can make the choice between the linear and quadratic trend forms of the data by closely inspecting the data and, as we will later see, using tests of hypotheses concerning θ. For now, we will assure that the trend in the data we are analyzing is linear, hence we assume trend $\alpha + \beta t$ for now.

What are the meanings of the seasonal dummy variables? Given the specification (1) we see that January's trend line is given by $\alpha + \beta t$, February's trend line is given by $(\alpha + \gamma_2) + \beta t$, ..., and December's trend line is given by $(\alpha + \gamma_{12}) + \beta t$. Thus, the "base"' trend line is January's trend line, while the γ_j, $j = 2, 3, ..., 12$ denote the "incremental" intercept coefficients that distinguishes the other months' trend lines (in particular the intercepts) from the trend line for the January months of each year. One can now see why this parametrization of the model is called the ``Relative to January'' parametrization. Therefore, in identifying the seasonal effects (both relative and absolute) we can to compare the magnitudes of α (the January intercept), $\alpha + \gamma_2$ (the February intercept), etc.. If the γ_j coefficient is positive, then the $j - th$ month is ``stronger'' than January, otherwise it is "weaker" than January. Strength of seasonal effect then is relative to January. Obviously, if all of the $\gamma_j's$ is positive then, by default, January's seasonal effect is the weakest.

The "average" seasonal effect is, of course,

$$\bar{\alpha} = [\alpha + (\alpha + \gamma_2) + \cdots + (\alpha + \gamma_{12})/12 = [12\alpha + \sum_{j=2}^{12}\gamma_j]/12 = \alpha + \sum_{j=2}^{12}\gamma_j/12. \quad (5)$$

Then the "strong months" in terms of seasonal effect are those whose trend intercept α (for January) and $\alpha + \gamma_j$ (for the other months) are greater than $\bar{\alpha}$ while the "weak months" have α or $\alpha + \gamma_j$ intercepts that are less than $\bar{\alpha}$. If one wants to "standardize" the seasonal effects, one could do so by forming the coefficients

$$\gamma_1^* = \frac{\alpha}{\bar{\alpha}}, \ \gamma_2^* = \frac{(\alpha_2 + \gamma_2 - \bar{\alpha})}{\bar{\alpha}}, \ ...., \ \gamma_{12}^* = \frac{(\alpha_{12} + \gamma_{12} - \bar{\alpha})}{\bar{\alpha}} \ . \quad (6)$$

These coefficients are such that their signs helps us distinguish between the **positive** (seasonally strong) months and the **negative** (seasonally weak) months. Also their absolute maginitudes can be used for comparing the seasonal effects across months.

For example, consider the standardized seasonal coefficients (6) for the Plano Tax Revenue data as produced by the program Plano_Test_Seasonality.sas:

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Table 1

Standardized seasonal effects by month. They sum to zero.
Strong months are positive and weak months are negative.
Their magnitudes can be compared.

sum	d1a	d2a	d3a	d4a	d5a	d6a
-2.7756E-16	-0.41046	1.43772	-0.44900	-0.56675	0.60094	-0.30948

d7a	d8a	d9a	d10a	d11a	d12a
-0.39709	0.60483	-0.33027	-0.37695	0.65101	-0.45451

The variable "sum" represents the sum of the standardized coefficients which is equal to zero up to minute computing rounding error. Obviously, from a seasonal effect standpoint, the stronger months are 2, 5, 8, and 11 (February, May, August, and November) while the other months are weaker. The strongest month is February while the weakest month is April.

But what about the cycle part of this model? To understand how the equation (2) is capturing the cyclical part of the data. Consider the AR(1) version of (4),

$$\varepsilon_t = \rho \varepsilon_{t-1} + a_t \qquad (4')$$

where we require, $|\rho| < 1$ in order for the ε_t process to be stationary (i.e. have constant mean, variance and covariance with respect to time). (We simply use ρ rather than ρ_1 to simplify the notation somewhat.) In the Appendix we show that

(i) $E(\varepsilon_t) = 0$

(ii) $Var(\varepsilon_t) = E(\varepsilon_t^2) = \dfrac{\sigma_a^2}{1-\rho^2}$

(iii) $Cov(\varepsilon_t, \varepsilon_{t-j}) = E(\varepsilon_t \varepsilon_{t-1}) = \rho^j \dfrac{\sigma_a^2}{1-\rho^2}$ \qquad (7)

(iv) $Corr(\varepsilon_t, \varepsilon_{t-j}) = \rho^j$

For example, in the special AR(1) model $\varepsilon_t = 0.8\,\varepsilon_{t-1} + a_t$ the autocorrelation function of

the errors is represented graphically by the following graph.

ρ_j
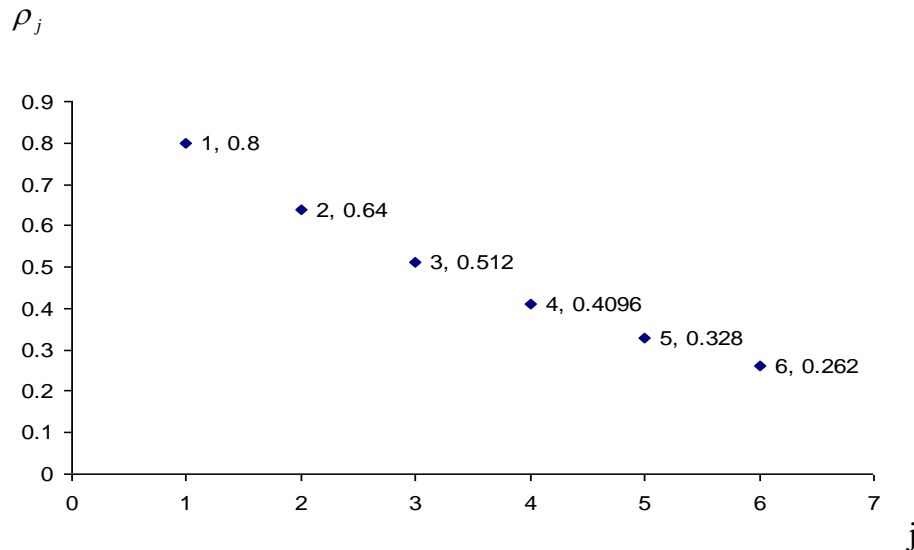
Given this autocorrelation function, we can see that adjacent errors ε_t, ε_{t-1} have an autocorrelation coefficient of 0.8 but that errors ε_t and ε_{t-j} that are j-periods apart have a smaller correlation of 0.8^j. The further the errors ε_t are apart, the less the correlation is between them.

But how does such an error process model cyclicity in economic and business data? Well, apart form the trend and seasonal effect, if the data (y_t) is **above** the trend plus seasonal effect (i.e. $\varepsilon_t > 0$) then the subsequent y_t is likely to be **above** the trend plus seasonal effect in the next period (i.e. $\varepsilon_{t+1} > 0$). Similarly if $\varepsilon_t < 0$ then there is a substantial possibility of $\varepsilon_{t+1} < 0$. That is, given above average behavior of y_t in time t, there is likely to be above average behavior of y_{t+1} in time $t+1$ and also quite likely so for several subsequent periods. Similarly, given below average behavior of y_t in time t, there likely to be below average behavior of y_{t+1} in time $t+1$ and for several periods thereafter. But this is typical of the cycles we finds in macroeconomic and business data associated with the overall business cycle of an economy.

IV. Estimation and Prediction in the DTDS Model

Suppose that we want to use model (3)-(4) for forecasting purposes. First, we have to estimate the coefficients of the model. A first approximation can be obtained by the so-called method of **Ordinary Least Squares** to get estimates of the parameters α, β, γ_2, ..., γ_{12} which we label $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}_2$, ..., $\hat{\gamma}_{12}$. The Ordinary Least Squares estimates are then derived by minimizing the sum of squared errors of the model, namely,

$$S = \sum_{t=1}^{T}(y_t - \hat{\alpha} - \hat{\beta}t - \hat{\gamma}_2 D_{t2} - \cdots - \hat{\gamma}_{12}D_{t,12})^2, \tag{8}$$

The Ordinary Least Squares errors of the model are

$$\hat{\varepsilon}_t = (y_t - \hat{\alpha} - \hat{\beta}t - \hat{\gamma}_2 D_{t2} - \cdots - \hat{\gamma}_{12}D_{t,12}). \tag{9}$$

In a similar manner we can consider the "fitted" AR(r) error model

$$\hat{\varepsilon}_t = \rho_1 \hat{\varepsilon}_{t-1} + \rho_2 \hat{\varepsilon}_{t-2} + \cdots + \rho_r \hat{\varepsilon}_{t-r} + a_t, \tag{10}$$

and use Ordinary Least Squares squares to obtain estimates of the autoregressive parameters ρ_1, ρ_2, ..., ρ_r in (10), say, $\hat{\rho}_1$, $\hat{\rho}_2$, ..., $\hat{\rho}_r$. Finally, with the Ordinary Least Squares estimates of $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}_2$, ..., $\hat{\gamma}_{12}$, $\hat{\rho}_1$, ..., and $\hat{\rho}_r$, we can forecast y_{t+1} (a one-step-ahead forecast) using the minimum mean square error formula

$$\hat{y}_{t+1} = \hat{\alpha} + \hat{\beta}(t+1) + \hat{\gamma}_2 D_{t+1,2} + \cdots + \hat{\gamma}_{12}D_{t+1,12} + \hat{\varepsilon}_{t+1} \tag{11}$$

$$\text{where} \quad \hat{\varepsilon}_{t+1} = \rho_1 \hat{\varepsilon}_t + \rho_2 \hat{\varepsilon}_{t-1} + \cdots + \rho_r \hat{\varepsilon}_{t-r+1} \tag{12}$$

is the forecasted error for time period $t+1$. The other forecasts $\hat{y}_{t+2}, \hat{y}_{t+3}, \cdots$ can be obtained by using (11) and (12) recursively.

In terms of software implementation of this model, in SAS we can use PROC FORECAST to produce the forecasts in this way. However, a more efficient way to estimate the present model is to use **Generalized Least Squares** to estimate the regressive parameters α, β, γ_2, ..., γ_{12} along with the error parameters ρ_1, ρ_2, ..., ρ_r. PROC AUTOREG in SAS can be used to accomplish this task. The term Generalized Least Squares derives its name from the fact that the variables of original model (3) are

transformed in such a way so that the resulting errors of the transformed equation statisfy the assumptions of Ordinary Least Squares, namely that the errors of the transformed model are independently and identically distributed with zero mean and constant variance. For example, if r = 1 and the errors ε_t of the DTDS model follow an AR(1) process, a popular Generalized Least Squares transformation of the data is the so-called **Cochrane-Orcutt transformation**. This transformation calls for transforming the original model (3) into the Generalized Least Squares equation

$$y_t^* = \alpha^* + \beta t^* + \gamma_2 D_{t2}^* + \cdots + \gamma_{12} D_{t,12}^* + a_t \quad \text{for } t = 2,3,\cdots,T. \quad (13)$$

where $\alpha^* = \alpha - \alpha\rho + \rho\beta$, $t^* = (1-\rho)t$, $D_{t2}^* = D_{t2} - \rho D_{t-1,2}, \cdots, \ D_{t,12}^* - \rho D_{t-1,12}^*$. Note that the derived error term a_t is now white noise as required by Ordinary Least Squares. See the appendix for this derivation. To implement (Feasible) Generalized Least Squares one first estimates ρ by Ordinary Least Squares producing

$$\hat{\rho} = \frac{\displaystyle\sum_{t=2}^{T} \hat{\varepsilon}_t \hat{\varepsilon}_{t-1}}{\displaystyle\sum_{t=1}^{T} \hat{\varepsilon}_t^2} \quad . \quad (14)$$

and then applying Ordinary Least Squares to the transformed equation (13) with $\hat{\rho}$ substituted in for ρ. Feasible Generalized Least Squares can be extended in a similar fashion to the higher order AR(r) error process (4) and SAS Proc Autoreg provides the means for doing so.

V. Prediction in the DTDS Model

In general the predictions produced by the the deterministic trend/ deterministic seasonal model (3)-(4) revert to the estimated trend line plus seasonal effects as the forecast horizon, h, approaches infinity ($h \rightarrow \infty$). This result is most easily shown for the case where the model has AR(1) errors. In this case, the h-ahead forecast of y is

$$\hat{y}_{t+h} = \hat{\alpha} + \hat{\beta}(t+h) + \hat{\gamma}_2 D_{t+h,2} + \cdots + \hat{\gamma}_{12} D_{t+h,12} + \hat{\varepsilon}_{t+h} \quad (14)$$

where $\hat{\varepsilon}_{t+h} = \hat{\rho}^h \hat{\varepsilon}_t$. Since $|\hat{\rho}| < 1$, we have $\hat{\varepsilon}_{t+h} \to 0$ as $h \to \infty$, and the h-step ahead forecast of the model approaches the trend plus seasonal part

$\hat{\alpha} + \hat{\beta}(t + h) + \hat{\gamma}_2 D_{t+h,2} + \cdots + \hat{\gamma}_{12} D_{t+h,12}$. That is, although the deterministic trend and deterministic seasonal effects remain in the forecast profile, as the forecast horizon goes to infinity, the cyclical part eventually vanishes in its effect. That makes sense in the respect that the trend and seasonal effects are assumed to be deterministic (i.e. fixed through time) while the cycle is modeled as being stochastic and mean reverting.

The standard error of the h-step-ahead forecast $\hat{y}_{t+h}$, denoted by $se(\hat{y}_{t+h})$, is the estimated of the square root of

$$E(\varepsilon_{t+h})^2 = E(y_{t+h} - \alpha - \beta(t+h) - \gamma_2 D_{t+h,2} - \cdots - \gamma_{12} D_{t+h,12} - \rho^h \varepsilon_t)^2. \quad (15)$$

In the Appendix we show that

$$E(\varepsilon_{t+h}^2) = \frac{\sigma_a^2}{1 - \rho^2}(1 - \rho^{2h}) \ . \quad (16)$$

This variance can then be consistently estimated by

$$\hat{Var}(\varepsilon_{t+h}) = \frac{\hat{\sigma}_a^2}{(1 - \hat{\rho}^2)}(1 - \hat{\rho}^{2h}) \quad (17)$$

where $\hat{\rho}$ can be obtained from (14) and

$$\hat{\sigma}_a^2 = \sum_{t=2} \hat{a}_t^2 /(T - 1) \quad (18)$$

with being the Ordinary Least Squares residuals obtained from (13).

It follows that the standard error of the h-step-ahead forecast of y_{t+h} is

$$se(\hat{y}_{t+h}) = \sqrt{\hat{Var}(\varepsilon_{t+h})} = \sqrt{\frac{\hat{\sigma}_a^2}{(1 - \hat{\rho}^2)}(1 - \hat{\rho}^{2h})} \ . \quad (19)$$

Notice that this standard error approaches the following limit as $h \to \infty$

$$se(\hat{y}_{t+\infty}) = \sqrt{\frac{\hat{\sigma}_a^2}{1 - \hat{\rho}^2}} \quad (20)$$

which, in turn, approaches the standard deviation of ε_t, namely

$$sd(\varepsilon_t) = \sqrt{\sigma_a^2/(1-\rho^2)} \tag{21}$$

as the sample size of the time series goes to infinity.

That the Deterministic Trend / Deterministic Seasonal model's representation of the uncertainty associated with future forecast errors approaches **a finite limit** $\sqrt{\sigma_a^2/(1-\rho^2)}$ as the horizon of the forecast goes to infinity is seen by many forecasters as a major limitation of the model. Intuitively, this result arises from the fact that the DTDS model is a deterministic model and mean reverting in nature. In contrast, we will see that in the Box-Jenkins models (as well as the UC and exponential smoothing models) the trend in the data is treated as being stochastic and thus the standard errors of the Box-Jenkins forecasts approach infinity (i.e. are unbounded) as the forecast horizon approaches infinity ($h \to \infty$). As a consequence, many forecasters believe that in practical forecasting problems in business and economics the confidence intervals produced by the DTDS model invariable **understate** the forecast uncertainty actually presented by the data. Of course, one can actually observe the out-of-sample prediction coverage of the competing forecast confidence intervals and can judge for oneself which of the competing forecast confidence intervals has the more accurate coverage rate.

Putting these disagreements aside, the $(1-\alpha)$% confidence interval for a h-step-ahead forecast produced by the Deterministic Trend/ Seasonal model is expressed as

$$Pr(\hat{y}_{t+h} - se(\hat{y}_{t+h})Z_{\alpha/2} < y_{t+h} < \hat{y}_{t+h} + se(\hat{y}_{t+h})Z_{\alpha/2}) = 1 - \alpha \tag{22}$$

where $1-\alpha$ is the chosen level of confidence (usually α =0.01, 0.05, or 0.10) and and $Z_{\alpha/2}$ is that value of the standard normal cumulative distribution such that $Pr(Z \geq Z_{\alpha/2}) = \alpha/2$. For example, when α =0.05 and we are interested in constructing a 95% prediction confidence interval for y_{t+h} we have $Z_{.025}$ =1.96 and the 95% prediction confidence interval for an h-step-ahead forecast of y_{t+h} is

$$[\hat{y}_{t+h} - se(\hat{y}_{t+h})1.96, \ \hat{y}_{t+h} + se(\hat{y}_{t+h})1.96] \qquad . \tag{23}$$

Thus, $\hat{y}_{t+h}$ is the **point** forecast and the (23) is the **prediction interval** forecast.

Using the SAS program Plano_Forecast.sas we generate 13 forecasts for December 2005 through December 2006 following the recursive use of (14) and calculating 95% prediction confidence intervals based on a formula similar to (19). (The forecasts were generated using an AR(1,3) model instead of the simple AR(1) model we used to explain the logic of standard errors of forecasts.)

Forecasts of Plano Sales Tax Revenues for December 2005 through December 2006 Produced by Deterministic Trend / Deterministic Season model with AR(1,3) errors p_ar3 = point forecasts, l_ar3 = lower 95% cl, u_ar3 = upper 95% cl

Obs	p_ar3	l_ar3	u_ar3	t
1	3984155.31	3455246.73	4513063.88	191
2	4044477.15	3509928.84	4579025.47	192
3	6279377.88	5743912.03	6814843.73	193
4	4100836.29	3496954.56	4704718.03	194
5	3968818.35	3357807.14	4579829.57	195
6	5383075.20	4771181.76	5994968.65	196
7	4343554.86	3706266.46	4980843.27	197
8	4245753.49	3602417.77	4889089.21	198
9	5437241.49	4792752.77	6081730.21	199
10	4394418.44	3739547.47	5049289.41	200
11	4346567.01	3687381.60	5005752.41	201
12	5544408.02	4883911.17	6204904.88	202
13	4322689.65	3655481.06	4989898.23	203

VI. The Choice of the Order r in the Autoregressive Part of the DTDS Model

In Proc Autoreg in SAS one has the option of specifying a maximum order of of the autoregression, say, r_{max}, and then letting the program sequentially eliminate all of the autoregressive terms that are not significant at a prespecified level. equenall

Consider the least square estimate of the r_{max}-order autoregression:

$$\hat{\varepsilon}_t = \rho_1 \hat{\varepsilon}_{t-1} + \cdots + \rho_r \hat{\varepsilon}_{t-r_{max}} + \hat{a}_t$$

Then the program sequentially test the following hypotheses, dropping those coefficients that are not statistically significant but retaining those that are.

$$H_{0,1} : \rho_{r_{max}} = 0 \quad vs \quad H_{1,1} : \rho_{r_{max}} \neq 0$$
$$H_{0,2} : \rho_{r_{max-1}} = 0 \quad vs \quad H_{1,2} : \rho_{r_{max-1}} \neq 0 \qquad \text{etc.} \tag{24}$$

If all autoregressive coefficients are found to be statistically insignificant, then Proc Autoreg simply reports the Ordinary Least Squares estimates and the associated test statistics.

VII. Tests for the Presence of Trend and Seasonality in the DTDS Model

The DTDS offers a very convenient framework of testing for the presence or absence of trend or seasonality or both in time series data that approximate mean reverting behavior. (Even if the data is not mean reverting in behavior but instead has a stochastic trend, the tests to be discussed could nevertheless be useful.)

To allow for the greatest generality let us consider the following DTDS model where we have added a quadratic term to the time trend part of the model to allow for data that might have some **curvature** in its trend. Consider

$$y_t = \alpha + \beta t + \theta t^2 + \gamma_2 D_{t2} + \gamma_3 D_{t3} + \cdots + \gamma_{12} D_{t,12} + \varepsilon_t \tag{3A}$$

and

$$\varepsilon_t = \rho_1 \varepsilon_{t-1} + \rho_2 \varepsilon_{t-2} + \cdots + \rho_r \varepsilon_{t-r} + a_t \quad . \tag{4}$$

A. Testing for the Presence or Absence of Seasonality

If we want to test for the presence or absence of seasonality in our data then we are interested in the hypotheses

$$H_0 : \gamma_2 = \gamma_3 = \cdots = \gamma_{12} = 0$$

versus (25)

$$H_1 : \text{At least one of the } \gamma's \text{ is not equal to zero.}$$

The **null hypothesis** states that the slopes of the trends of the non-January months all equal the slope of the trend for January. As a result, all of the trends of the months exactly coincide and thus there is no seasonal variation in the data. The **alternative hypothesis** signifies the presence of seasonality in the data since one or more of the slopes of the trends of the non-January months do not coincide with the January trend.

The appropriate F-statistic to test this hypothesis is derived from the Generalized Least equation (13) or the AR(r) generalization of it. Let SSR_u be the sum of squared residuals that one obtains from the Generalized Least Squares equation and SSR_R be the sum of squared residuals that one obtains from the restricted Generalized Least Squares equation where the restrictions $\gamma_2 = \gamma_3 = \cdots = \gamma_{12} = 0$ have been imposed in the estimation process. The appropriate F-statistic is of the form

$$F = \frac{(SSR_R - SSR_U)/11}{SSR_u/(T - r - 14)} \tag{26}$$

which, under the assumed truth of the null hypothesis, H_0, has an F-distribution of 11 numerator degrees of freedom and (T-r-14) denominator degrees of freedom in repeated samples. The numerator degrees of freedom corresponds to the number of restrictions being imposed by the null hypothesis while the denominator degrees of freedom corresponds to the number of observations minus the number of parameters in the unrestricted (full) model. There are 14 coefficients to be estimated in (3A) and r

coefficients to be equation (4). Of course, if the autoregression (4) is simplified somewhat by backward section as detailed in Section VI above then r in (26) becomes **the maximum order** that is specified for the autoregressive coefficients in the model.

The intuition behind (26) is that, if seasonality is not present in the data (i.e. H_0 is true), the fit offered by the restricted model will be almost as good as the fit offered by the unrestricted (full) model and although SSR_R must be greater than equal to SSR_U by necessity, it will not be too much greater and, as a result, the F-statistic will be small implying a p-value ($\Pr(F > F_{observed})$) that is greater than the chosen size of the test (usually 0.01, 0.05, or 0.10). In contrast, if seasonality is present in the data then the restricted model will not provide a very good fit of the data and SSR_R will substantially exceed SSR_U. Then the F-statistic (26) will be large and will imply a p-value less than the chosen size of the test. In summary, a F-statistic derived from (26) that has a small value and a large p-value supports the supposition of no seasonality in the data while a large F-value and a small p-value suppoet the supposition of seasonality in the data.

As an example of this test, consider again the Plano Sales Tax Revenue data. The SAS program Plano_Test_Seasonality.sas provides the following results concerning the test statistic (26) :

Test 1				
Source	**DF**	**Mean Square**	**F Value**	**Pr > F**
Numerator	11	3.4355621E12	68.75	<.0001
Denominator	175	49972763534		

As expected, the numerator degrees of freedom equals 11. The denominator degrees of freedom is 175 = 190 observations – 11 seasonal dummy coefficients – the maximal order of the autoregression on the errors, 4. The numerator mean square error represents the calculated value for the numerator in (26) while the denominator mean square error represents the calculated value for the denominator in (26). In the Generalized Least

Squares estimation process it turned out that the quadratic term was not needed (t = -1.59 with two-side p-value = 0.1138) while the OLS residuals indicated the need to use Generalized Least squares for conducting statistical inference tests. See the below Durbin-Watson table where the DW statistics of orders 1 – 4 are all highly significant.

Durbin-Watson Statistics			
Order	DW	Pr < DW	Pr > DW
1	1.3805	<.0001	1.0000
2	1.5470	0.0013	0.9987
3	0.6783	<.0001	1.0000
4	1.4171	<.0001	0.9999
Note:	Pr<DW is the p-value for testing positive autocorrelation, and Pr>DW is the p-value for testing negative autocorrelation.		

As it turns out Proc Autoreg retained the first and third order autoregressive coefficients in (4) as revealed in the following Proc Autoreg tables:

Backward Elimination of Autoregressive Terms			
Lag	Estimate	t Value	Pr > |t|
2	0.023174	0.38	0.7075
4	0.030767	0.41	0.6852

Estimates of Autoregressive Parameters			
Lag	Coefficient	Standard Error	t Value
1	-0.168865	0.058024	-2.91
3	-0.604465	0.058024	-10.42

In addition, the fitted model appears to have at least near white noise residuals as indicated by the following Durbin-Watson table for the Generalized Least Squares residuals and the corresponding autocorrelation function (ACF) of the same residuals.

Durbin-Watson Statistics			
Order	DW	Pr < DW	Pr > DW
1	1.8819	0.2098	0.7902
2	2.2356	0.9491	0.0509
3	2.2310	0.9548	0.0452
4	1.8891	0.2944	0.7056
Note:	Pr<DW is the p-value for testing positive autocorrelation, and Pr>DW is the p-value for testing negative autocorrelation.		

B. Testing for Combinations of Trend and Seasonality

Other tests suggest themselves as well in the DTDS model. Consider the possibility of testing for **no trend in the data but in the presence of seasonality**. The null and alternative hypotheses for this test are

$$H_0 : \beta = \theta = 0$$

versus   (27)

$$H_1 : \text{Either } \beta \neq 0 \text{ or } \theta \neq 0 \text{ or both.}$$

In this case the appropriate F-statistic is

$$F = \frac{(SSR_R - SSR_U)/2}{SSR_u /(T - r - 14)}$$ (28)

which under the assumed truth of the null hypothesis is distributed in repeated samples as an F-distribution with two numerator degrees of freedom and the same denominator degrees of freedom implied by the unrestricted model (3A) and (4).

For testing that **the data is absent both trend <u>and</u> seasonality** we have the following null and alternative hypotheses:

$$H_0 : \beta = \theta = \gamma_2 = \cdots = \gamma_{12} = 0$$

versus   (29)

$$H_1 : \text{At least one of the above coefficients is not equal to zero.}$$

In this case the appropriate F-statistic is

$$F = \frac{(SSR_R - SSR_U)/13}{SSR_u/(T-r-14)} \quad . \tag{30}$$

This assumes of course that we have both the linear and quadratic terms in the unrestricted model. This test is equivalent to the **F-test for the overall significance** of the DTDS regression and is available in the standard ANOVA table that most statistical regression packages produce (including SAS).
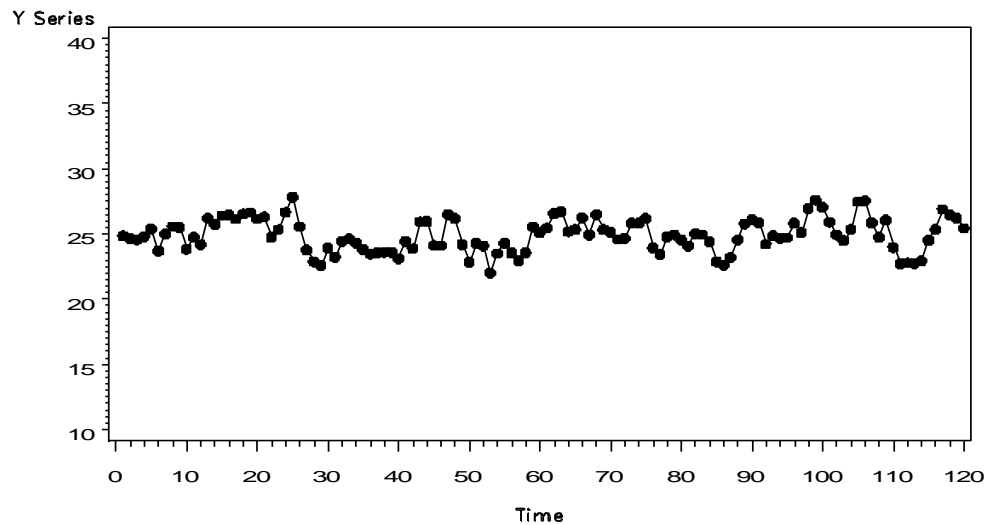
C. Some Data Sets Demonstrating the Various Types of Tests

Consider the following Monte Carlo data sets generated and tested by the SAS program Comprehensive Trend_Seasonal Test.sas all by the method of Generalized Least Squares.

Data with No Trend or Seasonal Effects

Figure 1

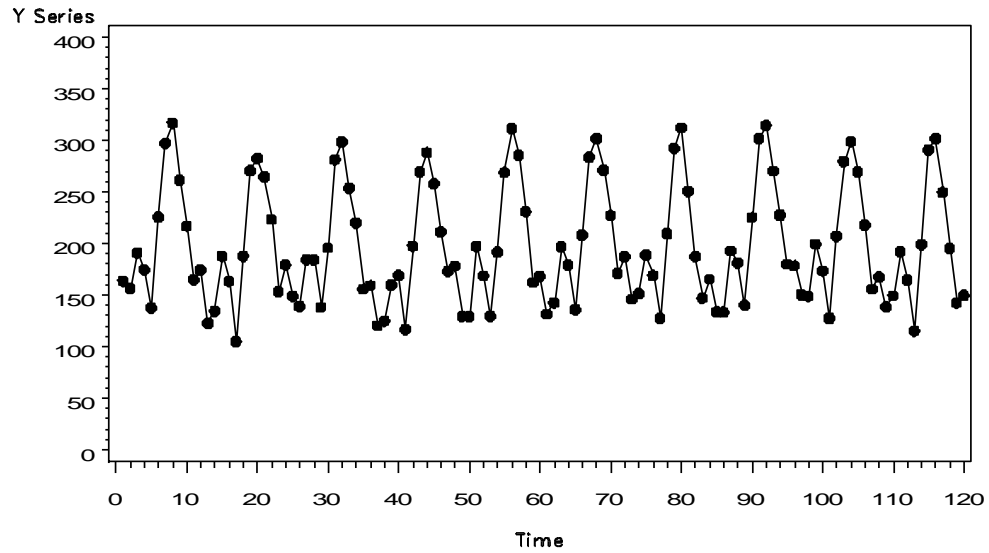AR(1) Series with Rho = 0.6 No Trend and No Seasonal Effects
X=Time Y=Y Series

Data with No Trend but Seasonal Effects

Figure 2

AR(1) Series with Rho = 0.6 with Fixed Seasonals and No Trend
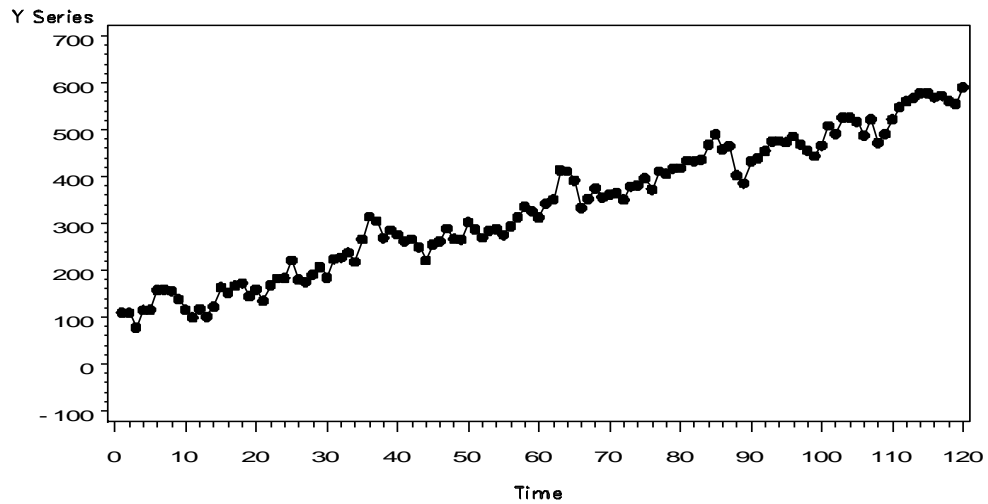X=Time Y=Y Series

Data with Trend but No Seasonal Effects

Figure 3

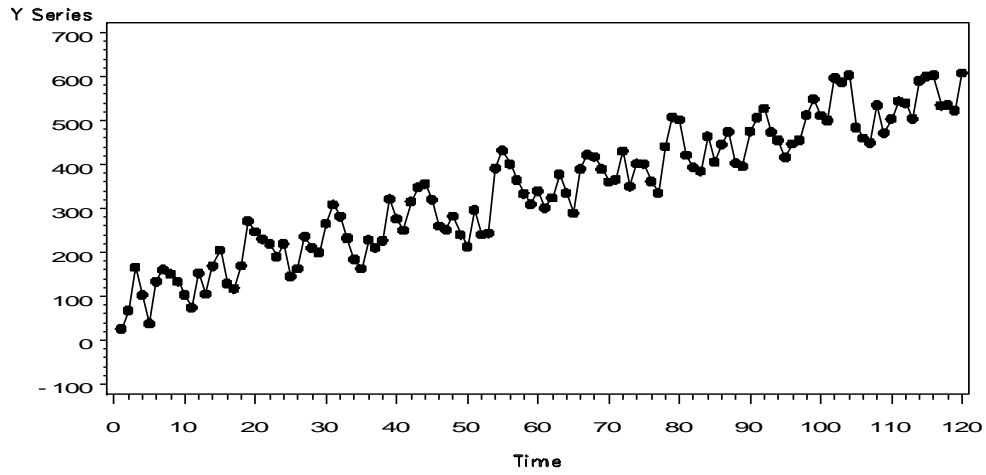Deterministic Trend Data with AR(1) Error but No Seasonal Effects
X=Time Y=Y Series

Data with Trend and Seasonal Effects

Figure 4

Deterministic Trend Data with AR(1) Errors and Seasonal Effects
X=Time Y=Y Series

For **Figure 1** the following test results are produced by the SAS program Comprehensive Trend_Seasonal Test.sas:

<table>
<tr><th colspan="5">Test Trend</th></tr>
<tr><th>Source</th><th>DF</th><th>Mean Square</th><th>F Value</th><th>Pr > F</th></tr>
<tr><td>Numerator</td><td>2</td><td>0.338063</td><td>0.35</td><td>0.7080</td></tr>
<tr><td>Denominator</td><td>105</td><td>0.975887</td><td></td><td></td></tr>
</table>

<table>
<tr><th colspan="5">Test Season</th></tr>
<tr><th>Source</th><th>DF</th><th>Mean Square</th><th>F Value</th><th>Pr > F</th></tr>
<tr><td>Numerator</td><td>11</td><td>0.541156</td><td>0.55</td><td>0.8611</td></tr>
<tr><td>Denominator</td><td>105</td><td>0.975887</td><td></td><td></td></tr>
</table>

Test Trend_Season				
Source	**DF**	**Mean Square**	**F Value**	**Pr > F**
Numerator	13	0.517788	0.53	0.9009
Denominator	105	0.975887		

You can see that none of these tests are statistically significant since all of the p-values of the tests are greater than, say, 0.05 thus implying the **absence of both trend and seasonal effects** as roughly indicated by a visual inspection of the data.

For **Figure 2** the test results are:

Test Trend				
Source	**DF**	**Mean Square**	**F Value**	**Pr > F**
Numerator	2	13.902751	0.17	0.8470
Denominator	105	83.577582		

Test Season				
Source	**DF**	**Mean Square**	**F Value**	**Pr > F**
Numerator	11	19227	230.05	<.0001
Denominator	105	83.577582		

Test Trend_Season				
Source	**DF**	**Mean Square**	**F Value**	**Pr > F**
Numerator	13	16284	194.84	<.0001
Denominator	105	83.577582		

Here, the seasonal test indicates **significant seasonal effects** (p<.0001) while the trend test indicates **no trend in the data**. The joint test of the absence of both trend and seasonal effects is naturally rejected.

For **Figure 3** the test results are:

Test Trend				
Source	**DF**	**Mean Square**	**F Value**	**Pr > F**
Numerator	2	214584	445.68	<.0001
Denominator	105	481.473152		

Test Season				
Source	**DF**	**Mean Square**	**F Value**	**Pr > F**
Numerator	11	167.325155	0.35	0.9724
Denominator	105	481.473152		

Test Trend_Season				
Source	**DF**	**Mean Square**	**F Value**	**Pr > F**
Numerator	13	33268	69.10	<.0001
Denominator	105	481.473152		

Here, the trend test indicates a significant trend while the seasonal effects are insignificant, as expected. The joint test of the absence of both trend and seasonal effects is naturally rejected.

For **Figure 4** the test results are:

Test Trend				
Source	**DF**	**Mean Square**	**F Value**	**Pr > F**
Numerator	2	447984	961.80	<.0001
Denominator	105	465.775770		

Test Season				
Source	**DF**	**Mean Square**	**F Value**	**Pr > F**
Numerator	11	16575	35.59	<.0001
Denominator	104	465.775770		

Test Trend_Season				
Source	**DF**	**Mean Square**	**F Value**	**Pr > F**
Numerator	13	85015	182.52	<.0001
Denominator	104	465.775770		

Here, the trend test indicates a significant trend while the seasonal test indicates significant seasonal effects as well. The joint test is also highly significant as expected.

VIII. Conclusion

The Deterministic Trend / Deterministic Seasonal model is one of the oldest models used for forecasting time series. Interestingly it is a special case of the UC model (apart from us assuming a higher order autoregressive error process). However, many years ago there did not exist the software or computing power to execute the UCM thus the applicability of the DTDS model historically. The DTDS does provide a useful testing framework for detecting trend, seasonal effects, and a combination thereof. The biggest drawback, however, may be the fact that the infinite horizon prediction confidence intervals are bounded which is not very realistic given that most economic and business data seem to have stochastic trends in them with little mean reversion. Despite this drawback, the DTDS model is quite interpretable, and in the presence of fixed seasonal effects is useful for identifying months which are seasonally strong as compared to those that are seasonally weak and for making comparisons of these strengths and weaknesses.

<div align="center">**Appendix**</div>

Properties of the AR(1) Error Term:

Now we prove the results stated in (7). By backward substitution in (4'), it is easy to show that

$$\varepsilon_t = a_t + \rho a_{t-1} + \rho^2 a_{t-2} + \cdots \qquad\qquad (4'')$$

It follows that the mean of the ε_t is

$$\begin{aligned} E(\varepsilon_t) &= E(a_t + \rho a_{t-1} + \rho^2 a_{t-2} \cdots) \\ &= E(a_t) + \rho E(a_{t-1}) + \rho^2 E(a_{t-2}) + \cdots \\ &= 0 + \rho \cdot 0 + \rho^2 \cdot 0 \\ &= 0. \end{aligned}$$

The variance of the ε_t is

$$\begin{aligned} E(\varepsilon_t - 0)^2 &= E(\varepsilon_t^2) \\ &= E(a_t^2 + \rho^2 a_{t-1}^2 + \rho^4 a_{t-2}^2 + \cdots \\ &\quad + \rho a_t a_{t-1} + \rho^2 a_t a_{t-2} + \cdots \\ &\quad + \text{additional terms involving } a_s a_{t-1} \text{ where } s \neq t) \\ &= E(a_t^2) + \rho^2 E(a_{t-1}^2) + \rho^4 E(a_{t-2}^2) + \cdots \\ &= \sigma_a^2 + \rho^2 \sigma_a^2 + \rho^4 \sigma_a^2 + \cdots \\ &= \sigma_a^2 (1 + \rho^2 + \rho^4 + \cdots) \\ &= \sigma_a^2 \cdot \frac{1}{1 - \rho^2} = \frac{\sigma_a^2}{1 - \rho^2} \end{aligned}$$

The covariance between ε_t and ε_{t-1} is

$$\begin{aligned}
E(\varepsilon_t\varepsilon_{t-1}) \;&=\; E(a_t + \rho a_{t-1} + \rho^2 a_{t-2} + \cdots)\\
&\quad\cdot (a_{t-1} + \rho a_{t-2} + \rho^2 a_{t-3} + \cdots)\\
&=\; E(\rho a_{t-1}^2 + \rho^3 a_{t-2}^2 + \rho^5 a_{t-3}^2 + \cdots)\\
&=\; \rho\sigma_a^2 + \rho^3\sigma_a^2 + \rho^5\sigma_a^2 + \cdots\\
&=\; \rho\sigma_a^2(1 + \rho^2 + \rho^4 + \cdots)\\
&=\; \rho\frac{\sigma_a^2}{1-\rho^2}\\
&=\; \rho\sigma_\varepsilon^2
\end{aligned}$$

In a similar manner it can be shown that

$$E(\varepsilon_t\varepsilon_{t-j}) = \rho^j\sigma_\varepsilon^2$$

Then the autocorrelation function for the AR(1) model (4') is

$$\begin{aligned}
Corr(\varepsilon_t\varepsilon_{t-j}) = \rho_j \;&=\; \frac{cov(\varepsilon_t\varepsilon_{t-j}))}{\sqrt{Var(\varepsilon_t)Var(\varepsilon_{t-j})}}\\
&=\; \frac{\rho^j\sigma_\varepsilon^2}{\sqrt{\sigma_\varepsilon^2\sigma_\varepsilon^2}}\\
&=\; \rho^j
\end{aligned}$$

$$j = 1,2,\ldots$$

Derivation of the Cochrane-Orcutt equation (13) for use in obtaining the estimates of the parameters in the DTDS model with AR(1) errors

$$y_t = \alpha + \beta t + \gamma_2 D_{t2} + \cdots + \gamma_{12}D_{t,12} + \varepsilon_t \tag{3}$$

Lagging (3) one period provides

$$y_{t-1} = \alpha + \beta(t-1) + \gamma_2 D_{t-1,2} + \cdots + \gamma_{12}D_{t-1,12} + \varepsilon_{t-1}\ . \tag{3'}$$

Multiplying (3') by ρ provides

$$\rho y_{t-1} = \rho\alpha + \rho\beta(t-1) + \rho\gamma_2 D_{t-1,2} + \cdots + \rho\gamma_{12}D_{t-1,12} + \rho\varepsilon_{t-1}\ . \tag{3''}$$

Subtracting (3'') from (3') provides

$$y_t - \rho y_{t-1} = (\alpha - \alpha\rho + \rho\beta) + \beta(1-\rho)t + \gamma_2(D_{t2} - \rho D_{t-1,2}) + \cdots + \gamma_{12}(D_{t,12} - \rho D_{t-1,12})$$

$$+ \varepsilon_t - \rho \varepsilon_{t-1}$$

$$y_t^* = \alpha^* + \beta t^* + \gamma_2 D_{t,2}^* + \cdots + \gamma_{12} D_{t,12}^* + a_t \tag{13}$$

as required.

Derivation of the h-step-ahead standard error of $\hat{y}_{t+h}$:

Given the AR(1) error process, by recursive substitution we can show that

$$
\begin{aligned}
\varepsilon_t &= \rho(\rho \varepsilon_{t-2} + a_{t-1}) + a_t \\
&= \rho^2 \varepsilon_{t-2} + \rho a_{t-1} + a_t \\
&= \rho^2 (\rho \varepsilon_{t-3} + a_{t-2)} + \rho a_{t-1} + a_t \\
&= \rho^3 \varepsilon_{t-3} + \rho^2 a_{t-2} + \rho a_{t-1} + a_t \\
&= \rho^h \varepsilon_{t-h} + \rho^{h-1} a_{t-h+1} + \cdots + \rho a_{t-1} + a_t
\end{aligned}
$$

It then follows that

$$\varepsilon_{t+h} = \rho^h \varepsilon_t + a_{t+h} + \rho a_{t+h-1} + \cdots + \rho^{h-1} a_{t+1} \ .$$

Therefore

$$E(\varepsilon_{t+h}^2) = E(\rho^h \varepsilon_t + a_{t+h} + \rho a_{t+h-1} + \cdots + \rho^{h-1} a_{t+1})^2$$

But at time t ε_t is no longer random and, as a result, we have

$$
\begin{aligned}
E(\varepsilon_{t+h}^2) &= E(a_{t+h} + \rho a_{t+h-1} + \cdots + \rho^{h-1} a_{t+1})^2 \\
&= E(a_{t+h}^2) + \rho^2 E(a_{t+h-1}^2) + \cdots + \rho^{2(h-1)} E(a_{t+1}^2) \\
&= \sigma_a^2 (1 + \rho^2 + \cdots + \rho^{2(h-1)}) \\
&= \sigma_a^2 [(1 + \rho^2 + \rho^4 + \cdots) - \rho^{2h}(1 + \rho^2 + \rho^4 + \cdots)] \\
&= \sigma_a^2 [\frac{1}{1-\rho^2} - \rho^{2h}(\frac{1}{1-\rho^2})] \\
&= \sigma_a^2 [\frac{1}{1-\rho^2}(1 - \rho^{2h})]
\end{aligned}
$$

as desired.