The Stark Difference Between the Prediction Intervals

of

Deterministic Trend Models and Stochastic Trend Models

Prediction Intervals Associated with a Simple Deterministic Trend Model

Consider the simple deterministic trend model

$$y_t = \beta_0 + \beta_1 t + a_t \tag{1}$$

where the error term a_t has the white noise error properties, $E(a_t) = 0$ for all t; $E(a_t^2) = \sigma_a^2$ for all t; and $E(a_s a_t) = 0$ for all $s \neq t$. In the derivation that follows we are going to assume, for simplicity, the sample size available to estimate the coefficients β_0 and β_1 is very large and thus the estimation risk in estimating these coefficients does not affect the accuracy of the forecasts produced by the Deterministic Trend Model. In essence, then, we are going to assume that the coefficients β_0 and β_1 are known with certainty when computing prediction confidence intervals.

As we know, the forecasting equation for the Deterministic Trend Model when the errors of the model are uncorrelated is

$$y_{T+h} = \beta_0 + \beta_1(T + h) \tag{2}$$

where we have assumed that the end of available data occurs at time T, h is the forecast horizon of interest, and again that the sample is so large that $\widehat{\beta_0} = \beta_0$ and $\widehat{\beta_1} = \beta_1$. Then the h-step-ahead forecast error is given by

$$e_{T+h} = y_{T+h} - \hat{y}_{T+h}$$

$$= \beta_0 + \beta_1(T + h) + a_{T+h} - \left(\beta_0 + \beta_1(T + h)\right) = a_{T+h} . \tag{3}$$

Therefore, the variance of the forecast error produced by the Deterministic Trend Model is

$$Var(e_{T+h}) = Var(a_{T+h}) = \sigma_a^2 . \tag{4}$$

Thus, the forecast error variance implied by the Deterministic Trend Model is **constant** and equal to σ_a^2 **no matter what the forecast horizon, h, is**. It follows that the 95% h-step-ahead prediction confidence interval implied by the Deterministic Trend Model is

$$\Pr(\hat{y}_{T+h} - 1.96\sigma_a < y_{T+h} < \hat{y}_{T+h} + 1.96\sigma_a) = 0.95 . \tag{5}$$

Most economists believe that the above predictive confidence interval is **too optimistic** given their experience "in the field." That is, future predictions $\hat{y}_{T+h}$ are likely to fall outside of such prediction confidence intervals much more than 5% of the time, especially so with the longer forecast

horizons. This is why most economists believe that business and economic data are subject to "stochastic" trends as compared to "deterministic" trends.

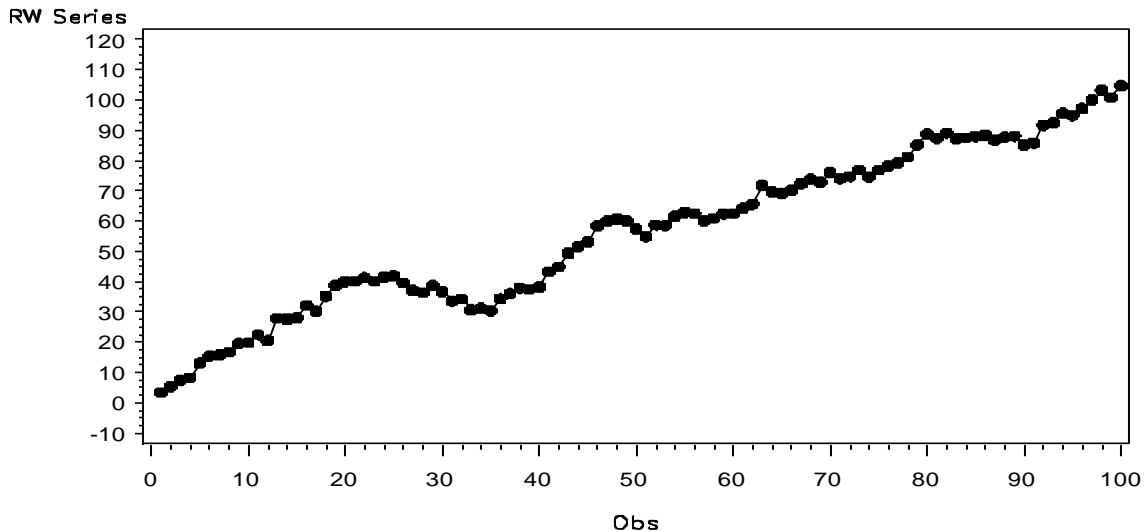A Very Simple Stochastic Trend Model and Its Prediction Intervals

Consider the following very simple stochastic trend model, called the **Random Walk with Drift** model:

$$y_t = \emptyset_0 + y_{t-1} + a_t \quad . \tag{6}$$

The below graph represents 100 observations data generated by the Monte Carlo method with the drift term, $\emptyset_0 = 1.0$, the initial value of y being $y_0 = 0$, and random normal errors a_t being drawn independently from a N(0,4) distribution.

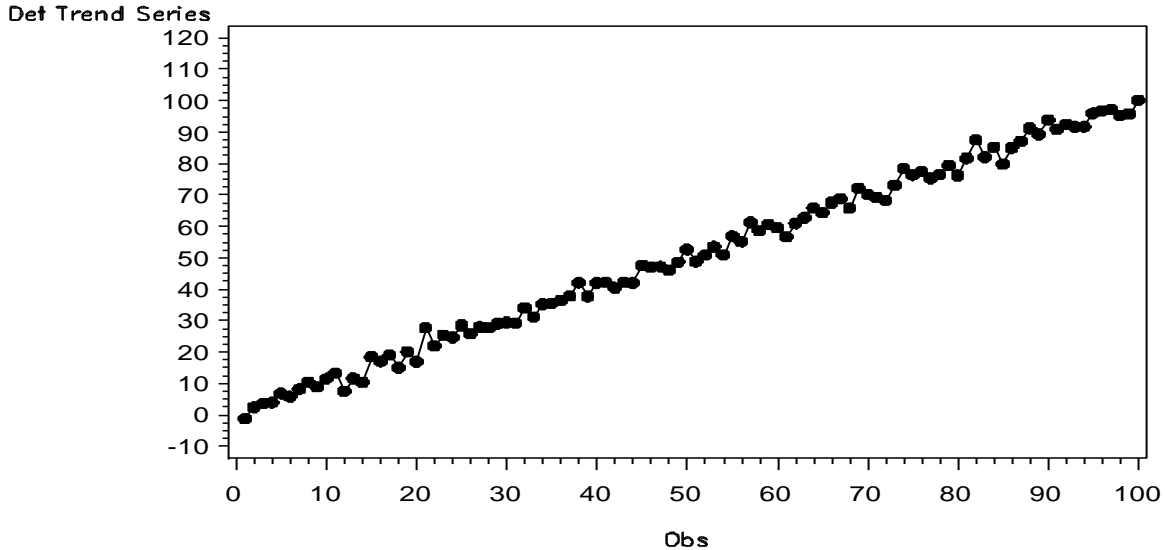# Monte Carlo Random Walk With Drift Data
X=Time Y=RW Series

Notice how slowly this data is turning around an imaginary straight line that we might draw through the center of this data (as in a least squares line). When you draw your line through the data you will notice how slowly the data turns around the line in that it doesn't cross the line very often out of 100 observations. We call such data "slow-turning" data and, as such, are indicative of a "stochastic" trend as implied by the Random Walk with Drift model.

In contrast, consider the following "deterministic" trend data:

Deterministic Trend Data
X=Time Y= DetTrend Series

These 100 observations have been generated by the Deterministic Trend Model $y_t = 0.0 + 1.0t + a_t$ with a_t being drawn independently from a N(0,4) distribution. The two times series depicted here have the same slope but the data are turning very differently around the two respective trends. In the stochastic trend case, the data are very slowly turning around the "stochastic" trend while in the deterministic trend case, the data are very quickly turning around the "deterministic" trend. It is also said that the data around the deterministic trend is "trend reverting" while the stochastic trend data is not.

We have already derived the prediction intervals for the above deterministic trend data, now let's turn to deriving the prediction intervals for the stochastic trend data as typical of the Random Walk with Drift Model. First, it will be shown later that the minimum mean square, h-step-ahead prediction equation for the Random Walk with Drift model (6) is

$$\hat{y}_{T+h} = y_T + \emptyset_0 h. \tag{7}$$

Equation (7) says that when the data follows the Random Walk with Drift model (6) the best h-step-ahead forecasts fall on a straight line that starts with last available observation y_T and has a slope of $\emptyset_0$. Again we are assuming an infinite sample size and that $\widehat{\emptyset}_0 = \emptyset_0$. Thus we are abstracting from the effect of parameter estimation risk on our prediction interval calculations. The prediction error is then

$$e_{T+h} = y_{T+h} - \hat{y}_{T+h}. \tag{8}$$

But starting with $y_1 = \emptyset_0 + y_0 + a_1$ and solving forward to any time period t, we can see that

$$y_t = y_0 + \emptyset_0 t + a_1 + a_2 + \cdots + a_t. \tag{9}$$

Then for time period T + h we have

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$$y_{T+h} = y_0 + \emptyset_0(T + h) + a_1 + a_2 + \cdots + a_{T+h} . \tag{10}$$

Subtracting equation (7) from (10) we get,

$$e_{T+h} = a_{T+h} + a_{T+h-1} + \cdots + a_{T+1} . \tag{11}$$

Given the independence of the $a_t's$ we have the variance of the prediction error of the Random Walk with Drift Model to be

$$Var(e_{T+h}) = h\sigma_a^2 \tag{12}$$

instead of simply σ_a^2 as in the variance of the prediction error assumed for the Deterministic Trend Model. That is, the forecast error variance associated with the stochastic trend model (6) is increasing directly proportional to the forecast horizon of interest, h. It follows that the 95% h-step-ahead prediction confidence interval implied by the Stochastic Trend model of (6) is

$$\Pr\left(\hat{y}_{T+h} - 1.96\sqrt{h}\sigma_a < y_{T+h} < \hat{y}_{T+h} + 1.96\sqrt{h}\sigma_a\right) = 0.95 . \tag{13}$$

Therefore, even if the error variance σ_a^2 is the same for the Deterministic Trend Model and the Stochastic Trend Model, the prediction intervals for the Stochastic Trend Model are wider than the Deterministic Trend Model, except for the one-step-ahead horizon. Furthermore, the Stochastic Trend Model prediction intervals become wider with the increase in forecast horizon at the rate of $\sqrt{h}$.

Summary

In summary, the prediction intervals implied by the Deterministic Trend conception of time series and the Stochastic Trend conception are very different. Compare the prediction intervals (5) and (13). The variance of the prediction error of the Deterministic Trend Model is constant regardless of the forecast horizon, h. In contrast, the variance of the prediction error of the Stochastic Trend Model is increasing proportionately with the length of the time horizon, h. Given the experience that economists have had in predicting future events, needless to say, they have an orientation toward assuming a stochastic trend in data as compared to assuming a deterministic trend. Also most economic data have a tendency to be "slow turning" rather than "trend reverting" and, as such, are more indicative of having stochastic trends.

In fact there exists a statistical test called the **Augmented Dickey-Fuller test for a unit root (case 3)** that will allow us to test a time series to be one having stochastic trend (the null hypothesis) versus a deterministic trend (the alternative hypothesis). We will study this test later. Obviously, this distinction and being able to test for it is very important, especially as it relates to constructing prediction confidence intervals of our forecasts.

Two of the most frequently discussed classes of "stochastic" trend models are (1) the **Box-Jenkins model** and (2) the **Unobserved Components model**. We turn to studying these models next.