THE TRANSFER FUNCTION MODEL

Notation

Assume, for now, that the target variable y_t is stationary (i.e. has a constant mean, constant variance, and constant covariance) and that the proposed leading indicator x_t is stationary as well. The Transfer Function model is described by the following three equations.

$$y_t = \mu + \frac{\omega(B)}{\delta(B)} x_{t-b} + \varepsilon_t \tag{1}$$

$$\varepsilon_t = \frac{\theta(B)}{\phi(B)} a_t \tag{2}$$

$$(x_t - \mu_x) = \frac{\theta^*(B)}{\phi^*(B)} u_t \tag{3}$$

where μ is the intercept in equation (1), μ_x is the mean of x and a_t and u_t are white noise error terms that are uncorrelated with each other at all forward and backward lags, and the "backshift" polynomials are defined as follows:

$$\omega(B) = \omega_0 - \omega_1 B - \omega_2 B^2 - \cdots - \omega_r B^r$$

$$\delta(B) = 1 - \delta_1 B - \delta_2 B^2 - \cdots - \delta_s B^s$$

$$\theta(B) = 1 - \theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q$$

$$\phi(B) = 1 - \phi_1 B - \phi_2 B - \cdots - \phi_p B^p$$

$$\theta^*(B) = 1 - \theta_1^* B - \theta_2^* B^2 - \cdots - \theta_{q*}^* B^{q*}$$

$$\phi^*(B) = 1 - \phi_1^* B - \phi_2^* B^2 - \cdots - \phi_{p*}^* B^{p*} \quad .$$

It is assumed for stationarity and invertibility purposes that the roots of the polynomials $\delta(B)$, $\theta(B)$, $\phi(B)$, $\theta^*(B)$, and $\phi^*(B)$ are outside the unit circle (i.e. the roots are greater than one in magnitude if they are real and have a modulus greater than one if they are complex). Thus, corresponding to this assumption, it is quite important that the variables we are analyzing, y_t and x_t, be in stationary form. If they are not and, say, Δy_t and Δx_t are instead stationary, then Δy_t and Δx_t should replace y_t and x_t in equations (1) and (3) above.

Equation (1) is called the **systematic dynamics equation** of the Transfer Function model because it describes the dynamic relationship between the leading indicator, x_t, and the target variable y_t. What is the nature of this relationship? Obviously, when x_t changes it does not cause a change in y_t until b periods later. b is called the **delay parameter** in the Transfer Function model. The nature of the numerator and denominator polynomials, $\omega(B)$ and $\delta(B)$ determine whether the change in x_t has a **finitely-lived effect** on y_t (after a delay of b periods) or whether the change in x_t has an **infinitely-lived but diminishing effect** on y_t. (Recall that it is assumed that the roots of the autoregressive (denominator) polynomial $\delta(B)$ are all outside of the unit circle and this guarantees that if $\delta(B)$ is anything other than 1 (i.e. when $s > 0$) then the effect must be diminishing. We will develop more of the intuition on the relationship between x_t and y_t below.

Equation (2) is called the **error dynamics equation**. ε_t is the unobserved error in the systematic dynamics equation. Thus, on average, the dynamic relationship that exists between y_t and x_t is described by

$$y_t = \mu + \frac{\omega(B)}{\delta(B)} x_{t-b} \ .$$

The error ε_t represents the approximate nature of the relationship (1). Then equation (2) says that, to the extent that the deterministic part of the systematic dynamics between y_t and x_t (the right-hand-side of the above equation) is approximate, what is left over to be explained can be modeled by a Box-Jenkins ARMA(p,q) model. Notice that if the order of the numerator polynomial, r, is equal to zero, $\omega(B)$ reduces to ω_0. Likewise, if the order of the denominator polynomial, s, is equal to zero, $\delta(B)$ reduces to 1. Also notice that if r = 0, s = 0, and $\omega_0 = 0$, equations (1) and (2) imply that y_t follows a Box-Jenkins ARMA(p,q) model,

$$y_t = \mu + \frac{\theta(B)}{\phi(B)} a_t = \phi_0 + \frac{\theta(B)}{\phi(B)} a_t$$

and $\mu = \phi_0$. Therefore, the Box-Jenkins model for y_t is just **a special case** of the Transfer Function model where the leading indicator x_t has **no** effect on the target variable y_t. We, of course, will be very interested in distinguishing between the cases where x_t has no effect on y_t ($\omega(B)/\delta(B)=0$) and when x_t has a systematic effect on y_t ($\omega(B)/\delta(B)\neq 0$). In the former case, x_t turns out **not** to be a leading indicator of y_t while in the latter case x_t is a viable leading indicator of y_t. Thus, one of the roles of the **econometrician** is to determine whether or not the rational polynomial $\omega(B)/\delta(B)$ is or is not equal to zero. If $\omega(B)/\delta(B)$ is equal to zero, the econometrician (as compared to the **statistician** who doesn't know of or use x_t) should discard x_t as an aid in forecasting the target variable y_t and try to come up with another potential leading indicator, say z_t, that is useful in forecasting y_t.

One way the econometrician can determine whether or not x_t is a useful leading indicator in forecasting y_t (over and above the special case Box-Jenkins model for y_t) is to conduct an **out-of-sample forecasting experiment** and see if the forecasting accuracy of the Transfer Function model using x_t better than the forecasting accuracy of a simple Box-Jenkins model for y_t. (Recall, if $\omega(B)/\delta(B)=0$, equations (1) and (2) specialize to the Box-Jenkins model for, namely,

$$y_t = \mu + \frac{\theta(B)}{\phi(B)}a_t = \phi_0 + \frac{\theta(B)}{\phi(B)}a_t$$

where $\mu = \phi_0$, of course. We will discuss the nature of the out-of-sample forecasting experiments in more detail later.

Equation (3) is called the leading indicator Box-Jenkins equation. In the Transfer Function model the leading indicator x_t is assumed to be **purely exogenous** in that x_t affects y_t but current and past values of y_t **do not** affect x_t. (This is sometimes called **one-way Granger Causality**.) In other words, x_t follows a stochastic process of its own, namely, an independent Box-Jenkins process ARMA(p*,q*). We here use p* and q* as AR and MA orders to distinguish them from the Box-Jenkins orders p and q of the error dynamics equation (2). The assumption that x_t is purely exogenous is a very important assumption when adopting the Transfer Function model to characterize the relationship between the leading indicator x_t and the target variable y_t. If this exogeneity assumption for x_t is **not** true, we need to use some other time series model to characterize the relationship between x_t and y_t. (One such model is called **the Vector Autoregressive model**, VAR for short, and given time in this class will be discussed later.)

One advantage of making a commitment to equation (3) is that, when forecasting y_t more than b periods ahead, for example, y_{T+b+1}, y_{T+b+2}, $\cdots$, etc., we need **future values** of x_t, namely x_{T+1}, x_{T+2}, $\cdots$, etc. When we are forecasting , say, y_{T+b+1} the Box-Jenkins model for x_t (equation(3)) can be estimated and used to produce the forecast $\hat{x}_{T+1}$, which in turn can be used in the estimated equation (1) to produce a b+1 forecast of y_t, namely, $\hat{y}_{T+b+1}$. Without an estimated version of equation (3) we can't use an estimated version of equation (1) to produce forecasts beyond b periods ahead.

Thus, the Transfer Function model of equations (1), (2), and (3) are dependent on the selection of the backshift order b, the polynomial orders r, s, p, q, p*, and q* and implicitly on the orders of differencing, say d and d*, that are required to make y_t and x_t stationary, respectively. If a d-order difference is needed to make y_t stationary and a d*-order difference is required to make x_t stationary, $\Delta^d y_t$ should replace y_t in equation (1) and $\Delta^{d*} x_t$ should replace x_t in equation (3) above and μ_x should be changed to be $\mu_{\Delta^{d*}x}$, the mean of the d*-differenced x_t series. From a notational perspective, we can represent equations (1) - (3) as TF(b, r, s, p, q, p*, q*, d, d*).

Before we go on, let's make some concrete choices of the Transfer Function orders d, d*, b, r, s, p, q, p*, and q* so that we can more fully appreciate the nature of the Transfer Function model represented by equations (1) – (3). Let d = d* = 0 (thus y_t and x_t are already stationary), b=1, r=1, s=0, p=0, q=0, p*=1 and q*=0. Also, for simplicity let's assume that the y-intercept in equation (1) is zero (μ = 0) and that the mean of x_t is zero ($\mu_x = 0$). Then, the Transfer Function model for this specific case can be written as

$$y_t = \frac{(\omega_0 - \omega_1 B)}{1} x_{t-1} + \varepsilon_t \qquad (1')$$

$$\varepsilon_t = a_t \qquad (2')$$

$$x_t = \phi_1^* x_{t-1} + u_t \qquad . \qquad (3')$$

In this case the systematic dynamics equation is a two-period distributed lag in x_t with a one-period delay, the error of the systematic dynamics equation is white noise (a_t) and the purely exogenous leading indicator follows an AR(1) Box-Jenkins process.

As another illustration, let d=d*=1, b=2, r=1, s=1, p=0, q=0, p*=0, q*=1, $\mu = \mu_{\Delta x} = 0$. Then the Transfer Function model takes the specific form

$$\Delta y_t = \frac{(\omega_0 - \omega_1 B)}{(1 - \delta_1 B)} \Delta x_{t-2} + \varepsilon_t$$

$$(1 - \delta_1 B)\Delta y_t = \omega_0 \Delta x_{t-2} - \omega_1 \Delta x_{t-3} + \delta_1 \Delta y_{t-1} + \varepsilon_t - \delta_1 \varepsilon_{t-1}$$

$$\Delta y_t = \omega_0 \Delta x_{t-2} - \omega_1 \Delta x_{t-3} + \delta_1 \Delta y_{t-1} + \varepsilon_t - \delta_1 \varepsilon_{t-1} \qquad (1'')$$

$$\varepsilon_t = a_t \qquad (2'')$$

$$\Delta x_t = u_t - \theta_1^* u_{t-1} \quad . \qquad (3'')$$

In this case, the systematic dynamics equation consists of Δy_t being explained by a two-period distributed lag in Δx_t with a two-period delay **and** a one-period lag of the endogenous variable Δy_t. In this model, not only does Δx_t have a two-period delay effect on y_t but **last period's** change in y_t (Δy_{t-1}) also has an effect on this period's change in y_t. Also the error term in the systematic dynamics equation (1'') follows an MA(1) process. From equation (2'') we can see that the error term ε_t is white noise, and from equation (3'') we see that the change in the leading indicator (Δx_t) follows an MA(1) process with MA(1) parameter θ_1^*.

Thus with the various choices of d, d*, b, r, s, p, q, p*, and q* we can have a very sophisticated description of the relationship that exists between the target variable y_t and the proposed leading indicator x_t.

Impulse Response Function

For the moment let us consider the deterministic form (i.e. without the error term ε_t) of the systematic dynamics equation

$$y_t = \frac{\omega(B)}{\delta(B)} x_{t-b}$$

$$= \frac{(\omega_0 - \omega_1 B - \cdots - \omega_r B^r)}{(1 - \delta_1 B - \cdots - \delta_s B^s)} x_{t-b} \qquad (4)$$

where, for simplicity, we let $\mu = 0$. Assuming that the roots of the polynomial $\delta(B)$ are all outside of the unit circle, we can write (4) in the impulse response form

$$y_t = \upsilon_0 x_{t-b} + \upsilon_1 x_{t-b-1} + \upsilon_2 x_{t-b-2} + \cdots , \qquad (5)$$

an infinite distributed lag in $x_{t-b}, x_{t-b-1,}, x_{t-b-2}, \cdots$. The coefficients $\upsilon_0, \upsilon_1, \upsilon_2, \cdots$ are called the **impulse response coefficients** associated with the (deterministic) systematic dynamics equation (4). The interpretation of these coefficients is as follows: Consider increasing x one unit at time t=0 and in the next period (t=1) returning it to its original value. υ_0 is called the **impact coefficient** and represents the initial impact that the one-period, one-unit increase in x has on y after a delay of b periods. υ_1 is the **delay-1 coefficient** that represents the effect that a one-period, one-period change in x has on y after a delay of b+1 periods. $\upsilon_2, \upsilon_3, \cdots$, have similar interpretations and are called the **delay-2, delay-3**, etc. impulse response coefficients.

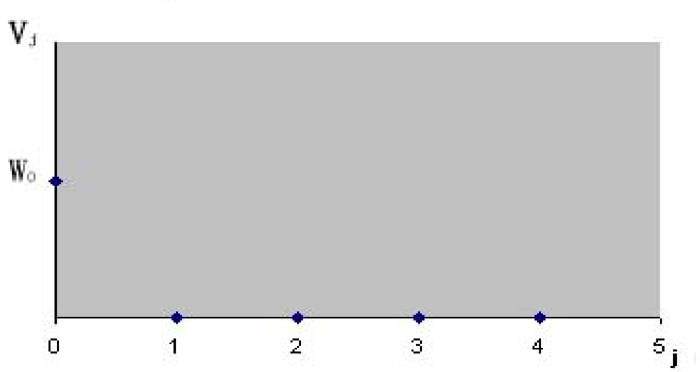
For example, let b=0, r=0, and s=0. Then the (deterministic) systematic dynamics equation (4) becomes

$$y_t = \omega_0 x_t \quad . \tag{4'}$$

Furthermore, let x_t be 0 for all time periods prior to and following t = 0, but equal to 1 at time period t=0. Now what impact does this type of change on x have on y_t? Well, y_t is equal to zero except at time t=0 and then it is equal to ω_0. Therefore, the impulse response function for equation (4') is

$$\upsilon_j = \begin{cases} w_0 & \text{for } j = 0 \\ 0 & \text{for } j = 1,2,.. \end{cases}$$

This can be plotted as



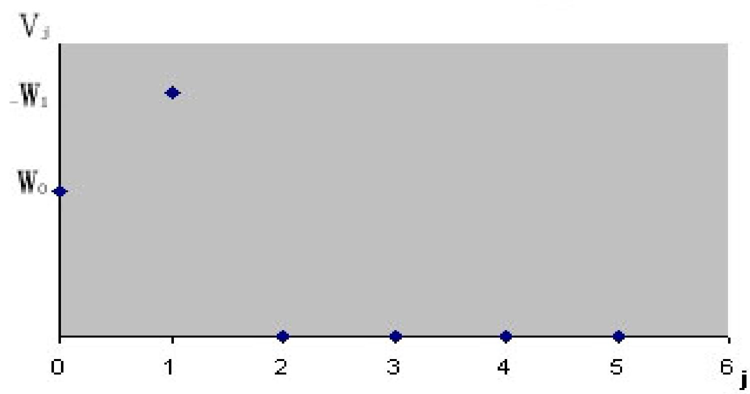
Now consider the case where $b = 0, r = 1, s = 0$. Therefore, the (deterministic) systematic dynamics equation becomes

$$y_t = w_0 x_t - w_1 x_{t-1} \tag{4''}$$

Let x_t have the one-period change of equation (6). What is the impact of this change of y_t? Well, y_t is zero except at time t=0 and then $y_0 = w_0$. In the following period $y_1 = -w_1$, Therefore, the impulse response function for (4'') is

$$v_j = \begin{cases} w_0 & \text{for j} = 0 \\ -w_1 & \text{for j} = 1 \\ 0 & \text{otherwise} \end{cases}$$

That is $v_0 = w_0$ and $v_1 = -w_1$, letting $w_1 < 0$ and $w_0 < |w_1|$, we have the plot of the impulse responses:



Then a one-period, one –unit change on x_t gives rise to an immediate change in y_t of w_0 units and then one more change in y_t of $(-w_1 - w_0)$ units $(y_0 = w_0, y_1 = -w_1)$ in period one and thereafter y_t resumes the value of 0.

Of course, if we had contemplated the functions $y_t = w_0 x_{t-2}$ and $y_t = w_0 x_{t-2} - w_1 x_{t-3}$. The corresponding impulse functions would have been just like the ones above but moved to the right by two periods, the amount of the new delay of $b = 2$ instead of $b = 0$. Now consider one more deterministic, systematic dynamics equation $b = 0, r = 1, s = 1$:

$$y_t = w_0 x_t - w_1 x_{t-1} + \delta_1 y_{t-1} \qquad (4''')$$

where we assume $0 < \delta_1 < 1$, and in particular, $0 < \delta_1 < 1$. Again let x_t evolve as in equation (6) and let's see what happens to y_t over time

$y_t = 0$ for t=………,-3,-2,-1

$y_0 = w_0$

$$y_1 = -w_1 + \delta_1 y_0 = -w_1 + \delta_1 w_0 = c \text{, say}$$

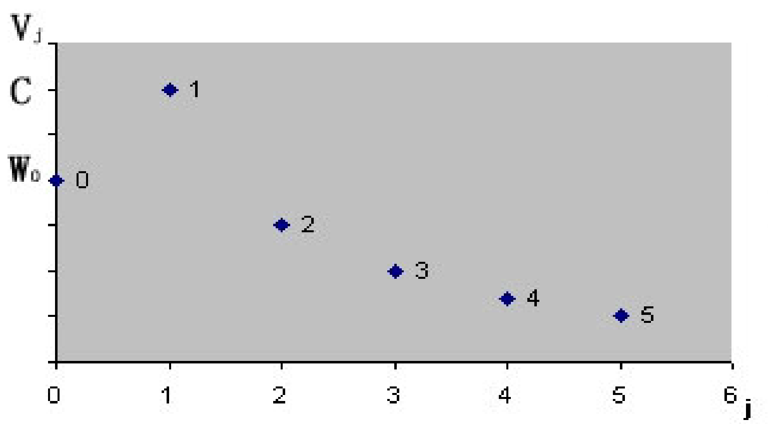
$$y_2 = \delta_1 y_1 = \delta_0(-w_1 + \delta_1 w_0) = \delta_1 c$$

$$y_3 = \delta_1 y_2 = \delta_1^2 y_1 = \delta_1^2 c \text{ etc.}$$

The impulse response function then becomes

$$v_j = \begin{cases} w_0 \quad \text{for j} = 0 \\ -w_1 + \delta_1 \mathrm{w}_0 = \mathrm{c}, \text{for j} = 1 \\ \delta_1^{\text{j-1}}\mathrm{c}, \ \text{otherwise} \end{cases}$$

Which, plotted is



Where, for plotting purposes, we have assumed that $c > w_0, 0 < \delta_1 < 1$. Therefore, given the (deterministic) systematic dynamics equation of (4''') we see that a one period, one-unit change on x_t at t=0 results in y_t being $w_0, c, \delta_1 c, \delta_1^2 c, ...$, in time periods t=0,1,2,3,….. , respectively. That is, from its original "equilibrium", $y_0 = 0$, the successive deviations of y from this original equilibrium beginning with time t=0 are units and then one more change in y_t of $(-w_1 - w_0)$ units $(y_0 = w_0, y_1 = -w_1)$ in period one and thereafter y_t resumes the value of 0 $w_0, c, \delta_1 c, \delta_1^2 c, ...$, etc. until long enough into the future of settles back down to its original equilibrium of $y = 0$. Of course, if the equation (4'') we had let $b = 2$ instead of $b = 0$, we would have the same impulse responses as before but they would be delayed two periods and the impulse response graph immediately above would be shifted to the right by two periods.

We can, of course, generalize from this set of algebraic exercises. Again consider the general (deterministic) systematic dynamics equation

$$y_t = \frac{w(B)}{f(B)} x_{t-b} = \frac{(w_0 - w_1 B - .... - w_r B_r)}{(1-\delta_1 B - \delta_2 B^2 - ....\delta_s B^s)} x_{t-b} \qquad (4)$$

Is there anything in general that we can say about the impulse responses associated with such an equation. The answer is yes! Given b=0,1,2,….., or some integer, we know that the impulse responses are zero for lags j=0,1,2,…..,b-1, until j=b and then v_0 will equal to w_0. If s=0 and the denominator polynomial is the scalar $\delta(B) = 1$, then there will be r impulse responses after j=b that will be non-zero. In summary there will be r+1 non-zero impulse responses beginning with the lag j=b when the systematic dynamics follows the equation

$$y_t = w_0 x_{t-b} - w_1 x_{t-b-1} - w_2 x_{t-b-2} - ...... - w_r x_{t-b-1} \qquad (4'''')$$

But now consider when $s \neq 0$ and is same positive integer. Then we can say that beginning with lag j=b there will be a total of r+1 irregular impulse responses before the impulse responses begin a systematic decay to zero, either exponentially or sinusoidal (we don't know exactly until we know the signs and magnitudes of the $\delta_1, \delta_2, ..., \delta_s$ coefficients), The decay of the impulse responses is guaranteed by the assumption that the roots of $\delta(B)$, lies outside the unit circle.

In the case of the model (4''') the impulse response function will be given by

$$v_j = \{ w_j, \qquad \text{j} = 0,1,2,.........$$

In the case when $s \neq 0$ we have

$$y_t = w_0 x_{t-b} - w_1 x_{t-b-1} - w_2 x_{t-b-2} - ...... - w_r x_{t-b-r} + \delta_1 y_{t-1} + \delta_2 y_{t-2} + ... + \delta_s y_{t-s} \qquad (4''''')$$

Here the impulse response function will be of the form

$$v_j = \begin{cases} v_0, v_1, ..., v_r, & \text{"irregular" responses(r + 1 of them)} \\ v_{r+1}, v_{r+2}, ..., \text{exponentially or sinusoidal declining responses beginning } v_{r+1} \text{ with and continuing.} \end{cases}$$

In summary then, we can look at the number of periods delay before the first nonzero impulse response occurs and we will be able to determine the value of b. Thereafter, r will be determined by the number of "irregular" (not part of the decay) impulse responses before the impulse responses either because all zero. Thereafter or decay away to zero Now whether the impulse responses decay to zero or cut off to zero determines whether or not s=0 or $s \neq 0$ and there is an autoregressive part to the (deterministic) systematic dynamics equation. If the impulse responses cut off to zero after r+1 irregular responses the s must equal zero. Otherwise s=1,2,…… or same positive integer.

As you can see, the impulse response function can help us identify the b, r, and s orders in the systematic dynamics equation. When we add back in the error term of the systematic dynamics equation

$$y_t = u + \frac{w(B)}{f(B)} x_{t-b} + \varepsilon_t \qquad (1)$$

Then the impulse response coefficients need to be interpreted as the expected level of y_t at various subsequent periods give a one-time, one-unit change on x_t. Of course, when the polynomials $w(B)$ and $\delta(B)$ are estimated from the data resulting in $\hat{w}(B) = \hat{w}_0 - \hat{w}_1 B - ....\hat{w}_r B^r$ and $1 - \hat{\delta}_1 B - \hat{\delta}_2 B^2 - .....\hat{\delta}_s B^s$ we get the estimated impulse response polynomial

$$\hat{v}(B) = \frac{\hat{w}(B)}{\hat{\delta}(B)} = (\hat{v}_0 + \hat{v}_1 B + \hat{v}_2 B^2 + ...)$$

and the estimated impulse response function $(\hat{v}_j, j = 0,1,2,......)$ is not always as informative as theoretical impulse response function ($v_j, j = 0,1,2,......$)

<u>The Cross-Correlation Function</u>

One drawback of using the theoretical impulse response function and its empirical counterpart, the estimated (sample) impulse response function is that the choice of the scales of measurement of y_t and x_t (or alternatively the scales of measurement of $\Delta^d y_t$ and $\Delta^d x_t$) affects the magnitude (but not the pattern) of the impulse response coefficients. Alternatively, we can construct a function called the cross correlation function that mimics the delay, irregular spike, and cutting off or declining behavior of the impulse response function yet the correlations are, by design, between –1 and +1 and the invariant to the choice of the scales of measurement of y_t and x_t.

Let's turn to the definition of the cross correlation function. Let w_t and z_t be two stationary time series that are potentially related to each other. Consider the following notation: Let

$$\gamma_{wz}(j) = E(w_t - \mu_w)(z_{t+j} - u_z) \tag{7}$$

j=-3,-2,-1,0,1,2,3

denote the cross-correlation between w_t and z_t at lag j. Notice that the lags j can be either positive or negative. For example, if $\gamma_{wz}(j) > 0$, then if w_t is above (below) its mean μ_w now then, more likely than not, z_t will be above its mean two periods from now. Of course $r_{wz}(j)$ is not invariant to the scales of measurement one might choose for w_t and z_t (100's, 1000's, 10000's etc.) but the cross-correlation at lag j between w_t and z_t is:

$$\rho_{wz}(j) = \frac{\gamma_{ww}(j)}{\sqrt{\text{var}(w_t)\,\text{var}(z_t)}} = \frac{\gamma_{wz}(j)}{\gamma_{ww}(0)\gamma_{zz}(0)} \tag{8}$$

where $\gamma_{zz}(j) = E(z_t - \mu_z)(z_{t+j} - u_z)$

and $\gamma_{ww}(j) = E(w_t - \mu_w)(w_{t+j} - u_w)$

are the autocovariance functions of w_t and z_t, respectively, and thus $\gamma_w(0)$ and $\gamma_z(0)$ are the variance w_t and z_t, respectively. By construction $-1 < \rho_{wz}(j) < 1$ and this is the case regardless of the choice of the scale of measurement of w_t and z_t. For example, if $\rho_{wz}(j) = 0.8$ then the correlation of w_t now with z_t two periods from now is 0.8 and if, say w_t is above its mean μ_w, now then, were likely than not, z_t will be above its mean, μ_z, two periods from now.

Of course, if w_t is purely exogenous with respect to z_t the $\rho_{wz}(j) = 0.8$ for j=…. , -3,-2,-1. That is , previous deviations of z_t from its mean do not affect current and future deviations of w_t from its mean. However, if w_t does affect z_t either concurrently or in the future(as would be expected of w_t is a leading indicator of z_t) then measure of the $\rho_{wz}(j)$ for j=0,1,2,……. Will be one-zero.

Let us then derive the cross-correlation functions for some simple transfer function models: consider the case of $\mu_x = 0, \mu = 0, b = b, r = 0, s = 0, p = 0, q = 0, p^* = 0, q^* = 0, d = 0, d^* = 0$. We have

$$y_t = w_0 x_{t-b} + \varepsilon_t \tag{9}$$

where $\varepsilon_t = a_t$ (10)

and $x_t = u_t$ (11)

is the white noise process.

Then

$$\gamma_{xy}(j) = E(x_t y_{t+j}) = E\left[x_t(w_0 x_{t-b+j} + a_{t+j})\right] = w_0 E(x_t x_{t-b+j}) + E(x_t a_{t+j}) = w_0 E(x_t x_{t-b+j}) =$$
$$= w_0 \gamma_{xx}(j-b)$$

Since, by assumption, x_t and a_{t+j} are uncorrelated at all leads and lags, following from

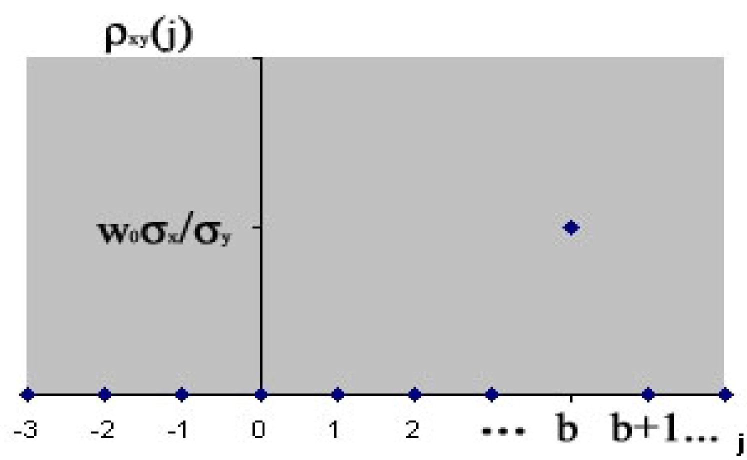
$E(x_t a_{t+j}) = E(\mu_t a_{t+j}) = 0$ for all j. Therefore

$$\gamma_{xy}(j) = \begin{cases} w_0 \sigma^2{}_x & \text{for } j = b \\ 0 & \text{otherwise} \end{cases}$$

is the autocovariance function for x and y given model (9)-(11). The cross-correction function for x and y given this model is

$$\rho_{xy}(j) = \frac{\gamma_{xy}(j)}{\sqrt{\gamma_{xx}(0)\sigma_{yy}(0)}} = \begin{cases} \dfrac{w_0 \sigma_x^2}{\sqrt{\sigma_x^2 \sigma_y^2}} = w_0 \dfrac{\sigma_x}{\sigma_y} & for \;\; j = b \\ 0 & otherwise \end{cases}$$

In summary, the cross-correlation function for the model (9)-(10) is plotted as



where here we have assumed that $w_0 > 0$. There is one spike, after a delay of b periods, and then also, reflecting the exogeneity of x_t vis-a-vis y_t the negative it cuts off logs of the cross-correlation function are all zero $\rho_{xy}(j) = 0$ for $j = ...-3,-2,-1$. That is the "signature" (vis-à-vis the cross-correlation function) of the model (9)-(11) where r=0,s=0,and b=1.

Now consider a second model

$$y_t = w_0 x_{t-b} - w_1 x_{t-b-1} + \varepsilon_t \tag{12}$$

$$\varepsilon_t = a_t \tag{13}$$

$$x_t = \mu_t \tag{14}$$

$$b = b, r = 1, s = 0, p = 0, q = 0, p^* = 0, q^* = 0, d = 0, d^* = 0$$

The covariance function is defined to be

$$\begin{aligned}\gamma_{xy}(j) &= E(x_t y_{t+j}) = E\left[x_t(w_0 x_{t-b+j} - w_1 x_{t-b-1+j} + a_{t+j})\right] \\ &= E(w_0 x_t x_{t-b+j}) + E(-w_1 x_t x_{t-b-1+j}) + E(x_t a_{t+j}) = w_0 \gamma_{xx}(j-b) - w_1 \gamma_{xx}(j-b-1)\end{aligned}$$

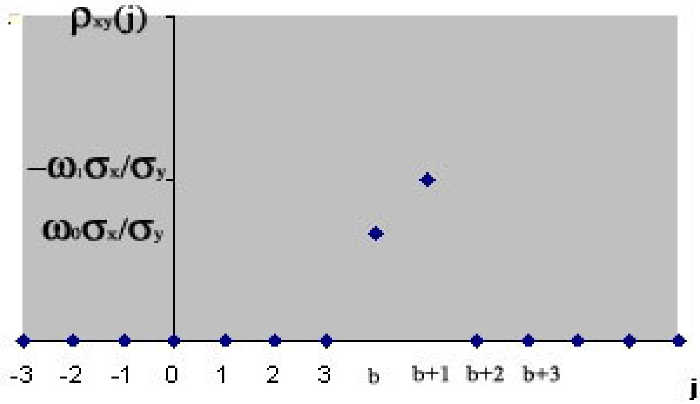
where we have $E(x_t a_{t+j}) = E(u_t a_{t+j}) = 0$ for all j. Then

$$\gamma_{xy}(j) = \begin{cases} w_0 \sigma^2{}_x & \text{for } j = b \\ -w_1 \sigma_x^2 & \text{for } j = b+1 \\ 0 & \text{otherwise} \end{cases}$$

This translates into the cross-correlation function of

$$\rho_{xy}(j) = \begin{cases} w_0 \dfrac{\sigma_x}{\sigma_y} & \textit{for } j = b \\ -w_1 \dfrac{\sigma_x}{\sigma_y} & \textit{for } j = b+1 \\ 0 & \textit{otherwise} \end{cases}$$

In graphical form of the cross-correlation function can be plotted as follows:



In this graph we have assumed that $0 < w_0 < -w_1$.

Finally, consider the model $b = b, r = 1, s = 1, p = 0, q = 0, p^* = 0, q^* = 0, d = 0, d^* = 0$

$$y_t = w_0 x_{t-b} - w_1 x_{t-b-1} + \delta_1 y_{t-1} + \varepsilon_t \tag{15}$$

$$\varepsilon_t = a_t \tag{16}$$

$$x_t = a_t \tag{17}$$

$$\begin{aligned}\gamma_{xy}(\mathrm{j}) &= \mathrm{E}(\mathrm{x}_t \mathrm{y}_{t+j}) = \mathrm{E}\left[\mathrm{x}_t(\mathrm{w}_0 \mathrm{x}_{t-b+j} - w_1 \mathrm{x}_{t\text{-b-1+j}} + \delta_1 y_{t-1+j} + \mathrm{a}_{t+j})\right] \\ &= \mathrm{E}(\mathrm{w}_0 \mathrm{x}_t \mathrm{x}_{t-b+j}) + \mathrm{E}(\text{-}\mathrm{w}_1 \mathrm{x}_t x_{t\text{-b-1+j}}) + \delta_1 E(x_t y_{t-1+j}) + \mathrm{E}(\mathrm{x}_t \mathrm{a}_{t+j}) \\ &= w_0 \gamma_{xx}(j-b) - w_1 \gamma_{xx}(j-b-1) + \delta_1 \gamma_{xx}(j-1)\end{aligned}$$

where we have $\mathrm{E}(\mathrm{x}_t \mathrm{a}_{t+j}) = \mathrm{E}(\mathrm{u}_t \mathrm{a}_{t+j}) = 0$ for all j. Then for j<1, $\gamma(j-1) = 0$ because the covariance between x_t and previous lags of y_t, acutely, $y_{t-1}, y_{t-2}, \ldots$, are all zero by the pure exogeneity of the leading indicator equation. In the model (15)-(17), $\gamma_{xy}(0) = \gamma_{xy}(1) = ... = \gamma_{xy}(b-1) = 0$ because $\gamma_{xy}(j) = E(x_t y_{t+j}) = E(\mu_t y_{t+j}) = 0$ for j<b as well. When j=b however we have

$$\gamma_{xy}(b+1) = w_0 \gamma_{xx}(0) = w_0 \sigma_x^2 .$$

Also for j=b+1 we have

$$\gamma_{xy}(b+1) = -w_1 \gamma_{xx}(0) + \delta_1 \gamma_{xy}(b) = -w_1 \sigma_x^2 + \delta_1 w_0 \sigma_x^2 = m,$$

For j=b+2 we have

$$\gamma_{xy}(b+2) = \delta_1 \gamma_{xy}(b+1) = \delta_1 m,$$

For j=b+3 we have

$$\gamma_{xy}(b+3) = \delta_1 \gamma_{xy}(b+2) = \delta_1^{\,2} m,$$

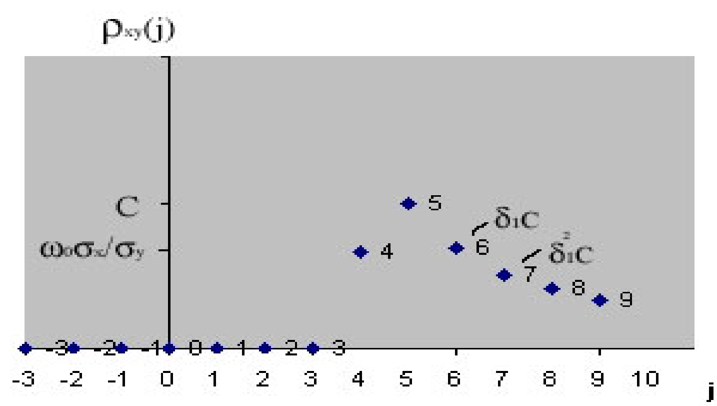
In general then our covariance function is

$$\gamma_{xy}(j) = \begin{cases} w_0 \sigma^2{}_x & \text{for j = b} \\ -w_1 \sigma_x^2 + \delta_1 w_0 \sigma_x^2 = \text{m} & \text{for j = b + 1} \\ \delta_1^{\,s} m & \text{for j = b + 1 + s and s = 1,2,3,4.......} \end{cases}$$

This implies that the correlation function for the model (15)-(17) is

$$\rho_{xy}(j) = \begin{cases} w_0 \dfrac{\sigma_x}{\sigma_y} & \textit{for}\ \ \text{j} = b \\ -w_1 \dfrac{\sigma_x}{\sigma_y} + \delta_1\ w_0 \dfrac{\sigma_x}{\sigma_y} = \text{c} & \textit{for}\ \ \text{j} = b+1 \\ \delta_1^s \text{c} & \textit{f}\text{or j = b + 1 + s, s = 1,2} \end{cases}$$

In graphical form the cross-correlation function can be plotted as



where in graphing we have assumed that $0 < w_0$, $c > w_0 \dfrac{\sigma_x}{\sigma_y} > 0$ and $0 < \delta_1 < 1$. This is a " signature" cross-correlation function for a transfer function model where b=b(there is a b period delay before the "spikes" begin), then there are(r-1) " irregular" spikes before an exponential or sinusoidal decay begins and then thereafter, instead of cutting off as when s=0, the cross-correlation function decays away when s=1>0.

The deviation of the cross-correlation function of a transfer function model is somewhat more complicated when x_t is not a white noise series. However, we can calculate the cross-correlation function of the "pre-whitened" y_t and x_t series and obtain analogous results to those we obtained before. Assuming that y_t and x_t are already stationary, for example, the pre-whitened series we need to cross-correlate are,

$$y_t^{\;+} = y_t \frac{\phi(B)}{\theta(B)}$$

and

$$x_t^{\;+} = x_t \frac{\phi(B)}{\theta(B)} = \frac{\theta(B)}{\phi(B)} \frac{\phi(B)}{\theta(B)} \mu_t = \mu_t$$

where the pre-filter is $\phi(B)/\theta(B) = (1 - \phi_1 B - \phi_2 B^2 - ......\phi_p B^p)/(1 - \theta_1 B - \theta_1 B^2 \cdots - \theta$. Cross-correlating the pre-filtered leading indicator (i.e the white noise errors of the leading indicator Box-Jenkins model), $x_t^{\;+} = \mu_t$, with the pre-filtered y series, $y_t^{\;+}$, produces a cross-correlation function which provides a pattern that allows us to identify the delay parameter, b and numerator and denominator polynomial orders, r and s, respectively, that correspond to the transfer function for the original data y_t and x_t. Applying the pre-filter $\dfrac{\phi(B)}{\theta(B)}$ to the systematic dynamics equation (1).

Summarizing, the cross-correlation function of a transfer function model with a systematic dynamics equation of

$$y_t = \mu + x_{t-b} \frac{w(B)}{\delta(B)} + \varepsilon_t$$

should have no spikes until j=b, then have r more "irregular" spikes (spikes not part of a systematic decaying pattern), followed by either a cutting off behavior if s=0, or a decaying pattern if s=1 (or some other positive integer) provides the pre-filter are transfer function-model

$$y_t \frac{\phi(B)}{\theta(B)} = x_{t-b} \frac{w(B)}{\delta(B)} \frac{\phi(B)}{\theta(B)} + \frac{\phi(B)}{\theta(B)} \varepsilon_t$$

$$y_t^+ = x_{t-b^+}^+ \frac{w(B)}{\delta(B)} + \varepsilon_t^+ = \frac{w(B)}{\delta(B)} \mu_{t-b} + \varepsilon_t^+ ,$$

where $\mu = 0$ has been conveniently(but without loss of result) imposed. Analyzing the cross-correlation function between y_t^+ and x_t^+ (ie. μ_t) clearly reveals the original rational (polynomial structure $\frac{w(B)}{\delta(B)}$) of the relationship between the original $y_t's$ and $x_t's$.

Sample Cross-correlation Function

When we discussed the theoretical ACF and PACF functions, it was noted that we had to construct sample estimates of them before proceeding to build a Box-Jenkins model. Similarly, we need to construct a sample cross-correlation function which hopefully closely resembles the theoretical cross-correlation function before we can build a transfer function linking a leading indicator, x_t, with a target variable y_t, say $c_{xx}(0)$ and $c_{yy}(0)$. What we need are the sample variances of the estimated pre-filtered series:

$$\hat{y}_t^+ = y_t \frac{\hat{\phi}(B)}{\hat{\theta}(B)}$$

and

$$\hat{x}_t^+ = x_t \frac{\hat{\phi}(B)}{\hat{\theta}(B)}$$

where the estimated pre-filter is

$$\frac{\hat{\phi}(B)}{\hat{\theta}(B)} = \frac{1 - \hat{\phi}_1 B - \hat{\phi}_2 B^2 - ......\hat{\phi}_p B^p}{1 - \hat{\theta}_1 B - \hat{\theta}_2 B^2 - ......\hat{\theta}_p B^p}$$

and the $\hat{\phi}_i$ and $\hat{\theta}_i$ have been obtained by estimating an appropriate Box-Jenkins model for x_t (we are implicitly assuming in this discussion that x_t and y_t are already stationary)

A consistent estimate of the variance of y_t^+ is

$$c_{y_t^+ y_t^+}(0) = \sum_{t=1}^{t} (\hat{y}_t^{\,+} - \bar{y}_t^{\,+}) \Big/ T \qquad (18)$$

where $\bar{y}_t^{\,+}$ = the sample mean of the $\hat{y}_t^{\,+}$, namely $\sum_{t=1}^{T} \frac{\hat{y}_t^{\,+}}{T}$

A consistent estimate of the variance of $x_t^{\,+}$ is

$$c_{x_t^+ x_t^+}(0) = \sum_{t=1}^{T} \hat{\mu}_t^2 \Big/ T \qquad (19)$$

where $\hat{\mu}_t$ are the Box-Jenkins white noise residuals for the leading indicator equation(3). A consistent cross-covariance estimate at lag j is given by

$$c_{x_t^+ y_t^+}(0) = \begin{cases} \frac{1}{T}\sum_{t=1}^{T-j} \hat{\mu}_t (\hat{y}_{t+j}^{\,+} - \bar{\hat{y}}_t^{\,+}) & \text{for j = 0,1,2,3,....} \\ \frac{1}{T}\sum_{t=1}^{T+j} \hat{\mu}_{t+j} (\hat{y}_t^{\,+} - \bar{\hat{y}}_t^{\,+}) & \text{for j = 0,-1,-2......} \end{cases} \qquad (20)$$

Finally the sample cross-correlation function is

$$\gamma_{x_t^+ y_t^+}(j) = \left\{ \frac{c_{x_t^+ y_t^+}(j)}{\sqrt{c_{x_t^+ x_t^+}(0) c_{y_t^+ y_t^+}(0)}} \right. \quad \text{j=.....-3,-2,-1,0,1,2,3.....} \qquad (21)$$

This is a consistent estimate of the theoretical cross correlation function

$$\rho_{x_t^+ y_t^+}(j) = \left\{ \frac{\gamma_{x_t^+ y_t^+}(j)}{\sqrt{\gamma_{x_t^+ x_t^+}(0) \gamma_{y_t^+ y_t^+}(0)}} \right.$$

Under the assumption that x_t and y_t are totally correlated with each other (and thus that the pre-filtered $x^+{}_t$ and $y^+{}_t$ are unrelated to each other), the standard error of the estimates, $\gamma_{x_t^+ y_t^+}(j)$, is approximately (in large samples) $\frac{1}{\sqrt{T}}$, that is, $SE(\gamma_{x_t^+ y_t^+}(j)) = \frac{1}{\sqrt{T}}$ when $\rho_{x_t^+ y_t^+}(j) = 0$. Then if an observed sample cross-correlation coefficient, say $\gamma_{x_t^+ y_t^+}(j)$, is outside of the 95% confidence interval $(-1.96/\sqrt{T}, 1.96/\sqrt{T})$, one could conclude that the theoretical cross-correlation between $x^+{}_t$ and $y^+{}_t$ at lag j, $\rho_{x_t^+ y_t^+}(j)$, is nonzero, using the above confidence interval, hopefully, we can distinguish between significant spikes in the sample cross-correlation function and the "zero" values at certain lags. What may be difficult to discuss in the sample cross-correlation function-is "cutting off" behavior and "tailing off" behavior which is the distinguishing characteristic between transfer function models with s=0 (cutting off) versus s=1(or some other positive integer) when tailing off.

Identification of Transfer Function Models

The steps for identify a TF $(b, r, s, p, q, p^*, q^*, d, d^*)$
Model are as follows:

(1) Visually inspect plots of the leading indicator x_t and target variable y_t and determine the order of differencing needed to transfer x_t (or possibly $\log x_t$) to stationarity difference (d^*)and the order of differencing needed to transfer y_t (or possibly $\log y_t$) to stationarity (d). The stationary form of x_t is then $\Delta^{d*} x_t$ (or possibly $\Delta^{d*} \log(x_t)$) while the stationary form of y_t is then $\Delta^d y_t$ (or possibly $\Delta^d \log(y_t)$).

(2) Fit a Box-Jenkins model for the stationary form of the leading indicator, x_t, namely $\Delta^{d*} x_t$. You will then determine the orders p^* and q^* for the ARIMA (p^*, d^*, q^*) model of the leading indicator x_t.

(3) Given estimated Box-Jenkins model for the leading indicator x_t form the estimated pre-filtered values $(\Delta^d y_t)^+ = \frac{\hat{\phi}(B)}{\hat{\theta}(B)} \Delta^d y_t$ and $(\Delta^{d*} x_t)^+ = \frac{\hat{\phi}(B)}{\hat{\theta}(B)} \Delta^{d*} x_t$.
Calculate the sample cross-correlation function between these estimated pre-filtered series. Use the 95% confidence interval $(-1.96/\sqrt{T}, 1.96/\sqrt{T})$ to determine which sample cross-correlations are significant and which ones are not. Choose b t be the lag at which the first significant cross-correlation occurs, then choose r based on the number of "irregular" spikes in the sample cross-correlation minus one and then choose s=0, if the sample cross-correlation function cuts off, and s=1(or some other positive integer) if, after the irregular spikes, the sample cross-correlation function systematically tails off.

(4) Estimate the suggested transfer model using the b, r, and s values you determined in step (3). Also for your chosen value of b, estimate additional models, if any, suggested by the sample cross-correlation function. Between the competing models choose the model that has the smallest goodness-of-fit measures, AIC and SRC, white noise residuals and statistically significant coefficient (apart from possibly the μ).

(5) In certain instances you may not be able to find values of r and s, for you given b value, which will produce white noise residuals. If so, you need to fit a Box-Jenkins model to the residuals $\hat{\varepsilon}_t$. Obtain reasonable values for p and q for the Box-Jenkins model of the residuals of your systematic dynamics equation (1). This is called "mapping up" the auto correlation in the residuals of the systematic dynamics equation. In so doing will have tentative values of b, r, s, and p and q, that produce the smallest goodness of fit measures AIC and SBC, white residuals, and statistically significant coefficients.

(6) Before making the tentative model of step(5) the final model of choice, you need to examine the t-statistics of four overfitting models. Given a b value (the delay parameter), there is one over-fitting model for each of the dimensions, r, s, p, and q incrementing one order while holding the rest of the orders fixed at the tentative choice. If each the t-statistics of the overfitting parameters of the four overfitting models are each statistic less than 1.96 or the absolute value of the t statistic is less than 1.96 or the p-value of the t-statistic is greater than 0.05), then "fall back" to the tentative model of step (5) and make final choice.

(7) Use your transfer-function-model model in an out of sample forecasting experiment and compare the forecasting accuracy of the model (using either MAE, MSE, or the boss's loss function of choice) with the forecasting accuracy of a properly chosen Box-Jenkins model produces more accurate forecasts (using the leading indicator) then the Box-Jenkins model (which ignores the leading indicator) then the leading indicator would appear to be useful and the transfer function model should be used for future forecasting tasks. On the other hand, if the Box-Jenkins model should prove to be more accurate than the Transfer Function model, we should drop consideration of the leading indicator x_t and consider building a different transfer function using another proposed leading indicator, say, z_t.

<u>Estimated transfer function model for series M data set.</u>

y_t = sales of carpet store at time t

x_t = housing permits issued in the county at time t

stationary form of y_t: Δy_t d=1

stationary form of x_t: Δx_t d^*=1

Box-Jenkins model for the leading indicator x_t (using the M sample dataset of obs 1-120)

$\Delta x_t =$

Therefore, $p^* = 0$, $q^* = 1$.