ESTIMATING BOX-JENKINS MODELS

1. ARMA(0,0) Model

$$y_t = \phi_0 + a_t$$

The **least squares estimator** of ϕ_0 is the sample mean of y, $\hat{\phi}_0 = \sum_{t=1}^{T} y_t / T = \bar{y}$.
This estimator is obtained by minimizing the least squares criterion

$$S = \sum_{t=1}^{T} a_t^2 = \sum_{t=1}^{T} (y_t - \phi_0)^2$$

with respect to ϕ_0. As it turns out, $\bar{y}$ is also the **method-of-moments estimator** of ϕ_0 since $E(y_t) = \phi_0$ and the sample mean of y can be used to estimate it.

2. AR(1) Model

$$y_t = \phi_0 + \phi_1 y_{t-1} + a_t$$

The **least squares estimators** of ϕ_1 and ϕ_0 are, respectively,

$$\hat{\phi}_1 = \frac{\sum_{t=2}^{T} (y_{t-1} - \bar{y}_{-1})(y_t - \bar{y})}{\sum_{t=2}^{T} (y_{t-1} - \bar{y}_{-1})}$$

$$\hat{\phi}_0 = \bar{y} - \hat{\phi}_1 \bar{y}_{-1}$$

where $\bar{y} = \sum_{t=2}^{T} y_t /(T-1)$ and $\bar{y}_{-1} = \sum_{t=2}^{T} y_{t-1} /(T-1)$. These estimators are obtained by minimizing the least squares criterion

$$S = \sum_{t=1}^{T} a_t^2 = \sum_{t=1}^{T} (y_t - \phi_0 - \phi_1 y_{t-1})^2$$

with respect to ϕ_0 and ϕ_1.

Alternatively, one could use the **method-of-moments** to estimate the parameters ϕ_0 and ϕ_1. Consider the following two moments.

$$E(y_t) = \frac{\phi_0}{1-\phi_1} \tag{1}$$

and

$$\text{Corr}(y_t, y_{t-1}) = \rho_1 = \phi_1 \tag{2}$$

Therefore, a consistent **method-of-moments estimate** of ϕ_1 is

$$\hat{\phi}_1 = r_1, \tag{3}$$

where r_1 is the first-order sample autocorrelation coefficient. From (1) we see that the sample mean of y, $\bar{y}$, can be used to estimate $\phi_0/(1-\phi_1)$. That is,

$$\frac{\hat{\phi}_0}{1-\hat{\phi}_1} = \bar{y} \ . \tag{4}$$

Substituting $\hat{\phi}_1 = r_1$ into (4) allows us to determine an **method-of-moments estimator** of ϕ_0, namely,

$$\hat{\phi}_0 = \bar{y}(1-r_1) \ . \tag{5}$$

Although the least squares and method-of-moments estimators of ϕ_0 and ϕ_1 are not the same in finite samples, they equal each other in infinite samples.

3.MA(1) Model

$$y_t = \phi_0 + a_t - \theta_1 a_{t-1}$$

Unfortunately, the least squares method cannot be used to estimate ϕ_0 and ϕ_1 in this model since the “data” a_{t-1} is not observable. However, we can use the method of moments to estimate these parameters. Consider that

$$E(y_t) = \phi_0 \tag{6}$$

and

$$\rho_1 = \frac{-\theta_1}{1+\theta_1^2} \ . \tag{7}$$

Replacing these moments with their sample estimates, we have

$$\hat{\phi}_0 = \bar{y} \tag{8}$$

and $\hat{\theta}_1$ so as to satisfy the moment condition

$$r_1 = \frac{-\hat{\theta}_1}{1+\hat{\theta}_1^2} \tag{9}$$

and, at the same time, the invertibility condition $|\hat{\theta}_1| < 1$. Again, r_1 is the first-order sample autocorrelation coefficient of the time series y_t. The two roots that will satisfy (9) are

$$\hat{\theta}_1 = \frac{-1 \pm \sqrt{1-4r_1^2}}{2r_1} \tag{10}$$

as long as $r_1 \leq 1/2$. One then just chooses the root $\hat{\theta}_1$ that satisfies the invertibility condition.