Peeling the DEA Onion: Layering and Rank-Ordering DMUs Using Tiered DEA

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May, 1994 (Revised June, 2000)

Abstract

Presented herein is a methodology and justification for separating DMUs in a data envelopment analysis into a series of nested efficient-frontier layers, thus providing a new efficiency metric and explanation for the inefficiecy of decision units. When coupled with a within-layer ordering technique, a complete ranking is identified for all DMUs within a dataset, with results that can be more meaningful than the traditional sorted-efficiency-score approach. These models are illustrated with computational results on sample problems with up to 25,000 observations.

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Data envelopment analysis was originally developed for efficiency measurement. The primary focus to date has been on the development of a set of DEA models to identify the efficient and inefficient DMUs under different sets of assumptions designed to measure various types of inefficiencies. In spite of the tremendous growth and success of the DEA methodology, there has been little research into methodologies for ordering the DMUs according to the efficiency of their production and management practices. The inability to rank DMUs by the comparative degree of efficiency or inefficiency limits the potential for DEA to fully characterize successful and unsuccessful management practices.

Although inefficient DMUs receive a DEA score which reflects the degree of inefficiency, a direct comparison of DMUs is problematic unless both DMUs have the same *efficient reference set* (ERS). To illustrate, Fig. 1 depicts a unit isoquant plot of a set of firms with one output and two inputs. Since each point represents a firm and all firms produce the same level of output, the more efficient units are closer to the origin. The efficient frontier is formed by DMUs A, B ,C and D. DMUs E, F, and G are all inefficient. The efficiency of each point is determined by a ratio whose denominator is the length of the line segment from the point to the origin and the numerator is the length of the line segment from the origin to the efficient boundary. The line connecting an inefficient DMU's point and the origin will intersect one of the line segments forming the efficient frontier. The endpoints of this line segment, composed of efficient DMU points, form the efficient reference set for the inefficient DMU.

According to Charnes and Cooper [6], to compare DMUs with different efficient reference sets would require assumptions of the weighting (or pricing) scheme used by DEA. But it is precisely this lack of restrictions on the weighting scheme that makes the DEA methodology so attractive. Consequently, in general such assumptions are undesirable. In our example, DMU_F can be compared to DMU_G because they share the same ERS consisting of DMU_C and DMU_D. In this case DMU_G with an efficiency score of 0.800 is more efficient than DMU_F with an efficiency score of 0.774. However, neither of these DMUs should be compared with DMU_E, with an efficiency score of 0.733, which has a different ERS composed of DMUs B and C. Hence, for inefficient DMUs, a new approach is necessary to further discriminate and allow comparisons across all inefficient DMUs.

For efficient DMUs, the relative importance of each DMU is difficult to discern. Because each such unit has a DEA score of 1, there exists no variation in scores to determine a relative value. Charnes and Cooper [6, 7] suggested a tool which they called the *envelopment map* to characterize the magnitude of the importance of each efficient DMU. This method consisted of counting the number of times each efficient DMU occurred as a member of an ERS. Those DMUs occurring more often would be considered more "consistently efficient." However, there are at least two problems with this measure. First, to correctly count all occurrences of an efficient DMU in an ERS, all alternate optimal solutions of the DEA models would need to be identified. This can be computationally expensive and difficult to track. Secondly, this counting offers only a limited amount of useful information. An efficient DMU that occurs often in

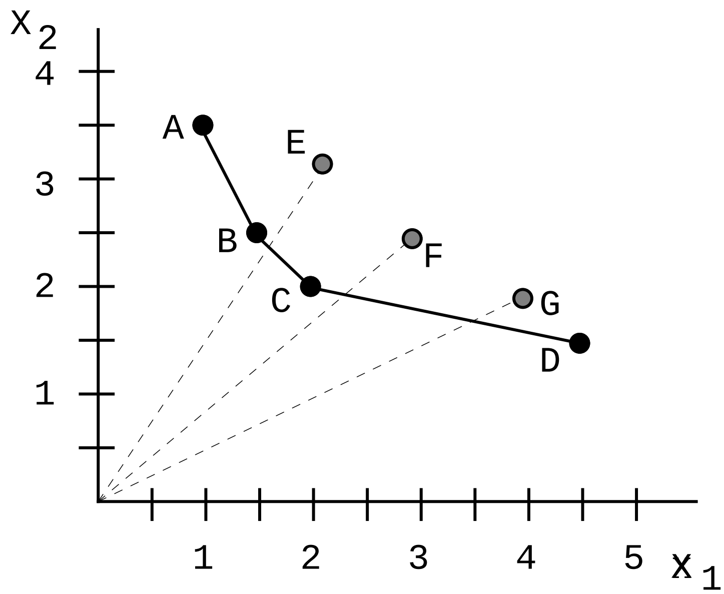

Figure 1: Normalized data isoquant plot

an ERS merely indicates that the DMU helps define part of the efficient surface which overshadows a high concentration of inefficient DMUs. Firms utilizing new production techniques may be extremely efficient, yet operate far from the "crowd" of other DMUs. As a result, these efficient firms do not occur often in the efficient reference sets of the inefficient units. Consequently, these *maverick* DMUs may not be deemed as important as they should be.

To discriminate between and identify the successful and unsuccessful production practices of the DMUs, a new procedure is necessary that provides a more detailed classification of DMUs than the ordinary DEA efficiency scores offer. This procedure should result in a rank ordering of DMUs which serves as a proxy measure for managerial efficiency. In this way, managers and analysts will have a useful tool to identify management practices that both accentuate and detract from productive efficiency by observing the practices of the higher and lower ranked DMUs. Additionally, experts may want to state, a priori to the DEA analysis, what they believe to be the most efficient firms in the industry. The rank ordering procedure can then be used to determine how the DEA results compare with the experts' opinions. As in standard DEA analysis, the ordering should allow either constant or variable returns-to-scale envelopments. The remainder of this chapter presents just such a rank ordering procedure which is easy to implement and meets all of the above criteria.

1 Need for a Ranking Procedure

Until recently, most modern writing on production theory assumed that all producers were efficient. The premise was that in a competitive market place the inefficient producers would realize the direct costs and indirect opportunity costs of continued production and would leave the market to pursue more profitable adventures. However, economic analysts have come to accept that inefficient production occurs in the market place and its causes vary. Inefficient production can occur because information on the most productive methods is neither perfect nor free. As a result, some firms may be slower to respond to changing market conditions than others. Along with imperfect information, market uncertainty influences the production process. The organization's (or the manager's) position towards risk will dictate the rapidity with which the firm will respond to change in the shadow of this uncertainty. Additionally, because perfect competition is rarely (if ever) seen, regulations, and other exogenous constraints may induce inefficiencies in the production process. Because of the social costs associated with inefficient conversion of input resources to output goods, there has been a growing interest in identifying and quantifying the inefficient processes. Data envelopment analysis has proven useful in measuring various types of the production inefficiencies which may be attributed to inefficient managerial practices.

The original DEA models focused on identifying technical inefficiency or scale inefficiency [2]. Färe, Grosskopf, and Lovell [9], relaxed the usual DEA assumption of strong disposability of inputs and outputs to further identify inefficiencies due to congestion of resources.[1] In addition, Färe, Grosskopf, and Lovell, following the lead of Farrell [8], introduced prices of inputs and outputs to identify allocative inefficiencies. A firm demonstrates allocative inefficiency when it departs from its predefined goal such as maximizing profits or minimizing costs.

Even though technical, scale, and allocative inefficiencies can be measured, little has been written to formulate a means of ranking the DMUs based on the types of inefficiencies they demonstrate. As a result, even though the inefficiencies can be identified, they have not fully been related to various aspects of producer behavior. With a ranking system, the DMUs exhibiting the best production processes, in terms of efficiency, could be compared to those characterized by the worst production techniques. The management practices of the best producing DMUs could then be compared to the worst DMUs in order to identify the underlying managerial inefficiencies. Once identified, the less efficient firms could adopt the practices of the best firms to improve the productivity of their operations.

The purpose of this study is to present a new approach to rank order or stratify the DMUs to more clearly relate the efficiency (or inefficiency) of a given DMU to all others in the set. The intent is not to suggest this approach as the only valid rank ordering scheme. Indeed, any set of items can be ranked by

[1] Congestion of inputs implies that as at least one input increases, at least one output is reduced. That is, there is not a positive correlation between all outputs and inputs.

any subjective means. However, the purpose is to present a methodology, with theoretical underpinnings, which can result in a meaningful ordering system. It is hoped that this methodology may stimulate research into other possible ranking procedures so that the richness of the DEA methodology can be more fully utilized.

2 Basis of the Ranking Procedure

Data envelopment analysis defines efficiency based on empirical observations of each evaluation unit's behavior. Each DMU consumes multiple inputs in order to produce one or more outputs. The implicit assumption in DEA is that the DMUs transform the inputs into outputs by means of a well-behaved production technology. According to Koopmans, a feasible input-output vector for a given DMU is technically efficient if it is technologically impossible to increase any output and/or to reduce any input without simultaneously reducing at least one other output and/or increasing at least one other input. The empirical models of DEA use observed data to identify the DMUs which form the Pareto-Koopman efficient surface, and hence, are best at using the current technology to convert the input resources to output goods. The level of technological achievement revealed by the efficient surface will be highly dependent upon the choice of DMUs used in the study since the methodology only measures relative (not absolute) efficiency.

The ranking method presented in this study separates DMUs into groups based on the level of technological achievement which they demonstrate. It will be shown that this aggregation reveals additional aspects of inefficiencies not available with traditional DEA measures. For example, a DMU that appears to be very inefficient by the standard DEA measures, may rank well, compared to other firms, when viewed in terms of the DMU's ability to employ the most recent technological advances. Once the DMUs are separated into these achievement levels, a procedure will be presented to rank the DMUs within each level. The ranking within a level will be determined by the contribution the DMU makes to defining the shape of the efficient surface of that level.

An attractive feature of the proposed ranking procedure is that it can be used across the many formulations of the DEA models. In this paper, the ranking procedure is introduced and applied to both the input- and output-oriented BCC (variable-returns-to-scale) models. The input- and output-oriented models achieve identical stratification of DMUs into tiers of common technological achievement levels. However, the ranking within each tier will differ according to the orientation used. Next, the ranking procedure is applied to the CCR (constant returns to scale) model. In this case, the input- and output-oriented schemes result in identical rankings. The CCR model adds interesting interpretations of the most productive scale size to achieve each technological level.

Procedure TDEA

1. Initialize: $t \leftarrow 1, \mathrm{D}^{[1]} \leftarrow \mathrm{D}$.
2. While $\mathrm{D}^{[t]} \neq \emptyset$ do:
   (a) Apply a DEA model to the DMUs in set $\mathrm{D}^{[t]}$ to identify $\mathrm{E}^*_{[t]}$.
   (b) $\mathrm{I}^*_{[t]} = \mathrm{D}^{[t]} - \mathrm{E}^*_{[t]}$.
   (c) $t \leftarrow t + 1$.
   (d) $\mathrm{D}^{[t]} = \mathrm{I}^*_{[t]}$.

where t is a tier index and $\mathrm{E}^*_{[t]}$ and $\mathrm{I}^*_{[t]}$ are the sets of efficient and inefficient DMUs on tier t, respectively, relative to set $\mathrm{D}^{[t]}$.

Figure 2: Tiered DEA Algorithm

3 Tiered DEA

At the heart of data envelopment analysis is the separation of evaluation units into (relatively) efficient and inefficient sets. The models' objective function values, θ or z, have been used as metrics for the *degree* of inefficiency for comparative and predictive purposes [4, 5]. Since these values may be incompatible, from an economic point-of-view, we present a different approach to comparing DMUs that has significant appeal, from both intuitive and economic-theoretic standpoints.

3.1 Tiering Algorithm

The *tiered DEA* (TDEA) procedure given in Figure 2 stratifies decision units into tiers, or layers, of comparable productive efficiency, as measured by any standard DEA model. The TDEA procedure begins as with a traditional data envelopment analysis, then progressively strips away production surfaces, revealing a series of frontiers of decreasing productivity.

Specifically, at tier 1 all of the DMUs in the dataset are analyzed using a standard DEA model, thus separating them into efficient and inefficient sets. The efficient units are then assigned to the current tier and the inefficient ones become the dataset of interest for the next higher tier; this process is applied recursively until all DMUs are assigned to a tier. In this way, the tier levels represent successive layers of relatively efficient production surfaces where the DMUs at any given tier are less productively efficient than those of "outer" (lower-numbered) tiers and more efficient than DMUs at "inner" (higher-numbered) tiers.

An example application of the TDEA procedure can be seen in 3 where 10

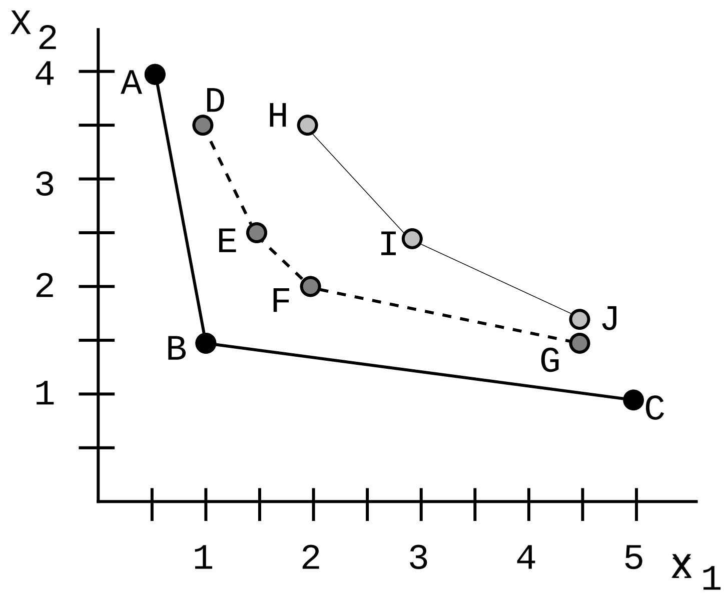

Figure 3: Example tiered results – two inputs, one output

DMUs, each with 1 output and 2 inputs, are plotted. All DMUs have the same output level so only the inputs are shown. DMUs A, B, and C are DEA efficient since the line segments joining these DMUs envelop the other DMUs from below. If firms A, B, and C were removed from the data set, a new efficient frontier would be formed by DMUs D, E, F, and G. The TDEA procedure reveals that the data contains three production surface layers.

Fig. 4 shows 10 DMUs each with 1 input and 2 outputs. In this case, all DMUs use the same level of input so only the outputs are shown. Again, DMUs A, B, and C are efficient since they form a boundary that envelops the other DMUs from above. Again, TDEA reveals three production surface layers.

3.2 Example TDEA Applications

The TDEA procedure was applied to three 8,000-DMU data sets, one from industry and the others randomly generated. The "Banking" data represents a selection of banks from the Federal Reserve Bank's Southwest district, with 6 input and 3 output values, as described in [3]. A set of "Cobb-Douglas" data with 5 input and 4 output values per observation was created with DEA-GEN using constant returns to scale and the parameter set $\alpha = (.20, .20, .20, .20, .20)$. (See Appendix B for the generation procedure.) Also, a "Multi-Normal" data set, with 4 inputs and 3 outputs and the variance-covariance matrix given in Appendix A, was generated using the DRNMVN routine from the IMSL library.

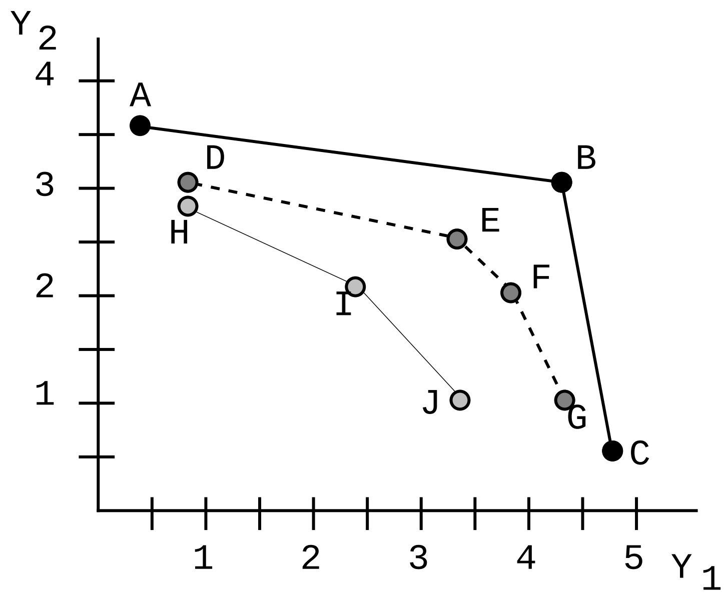

Figure 4: Example tiered results – one input, two outputs

For this test bed, the TDEA procedure was applied, and used the CCR_j^i model to assign DMUs to layers. Table 1 reports, for the first 20 tiers, the number of DMUs assigned to each layer and their maximum, minimum, and mean *tier 1* efficiency score (θ). Note that, in each case, that the mean and maximum efficiencies drop with each successive interior layer, as might be expected. However, the TDEA provides insight that DEA cannot provides. As noted by the minimum values, some DMUs possess a low tier 1 efficiency score, yet fall on an outer tier, indicating an efficient use of current technology. Conversely, the maximum values indicate that while other DMUs may seem fairly efficient, from a θ-standpoint, they fall on an inner tier because of less productive uses of current technology. It is precisely this behavior that should prove useful in identifying a DMU's true level of efficiency.

Of interest also are the differences between these three problems. The frequency counts reveal that the banking data has only 18 tiers, while the Cobb-Douglas data still has 3,286 of its 8,000 points still un-stratified after 20 tiers. The multi-normal problem has more populous tiers than the Cobb-Douglas, but less than most of the bank's. The banking and multi-normal data sets have tier-1-efficient DMUs on interior layers, but Cobb-Douglas does not. The banking and Cobb-Douglas data have much larger mean-efficiency drops between tiers 1 and 2, relative to the multi-normal. Each problem seems to have a very different structure from the others, as uncovered by the TDEA process.

Tier	Banking				Cobb-Douglas				Multi-Normal			
t	Max	Min	Mean	Freq.	Max	Min	Mean	Freq.	Max	Min	Mean	Freq.
1	1.00	1.00	1.00	77	1.00	1.00	1.00	70	1.00	1.00	1.00	104
2	1.00	0.60	0.87	242	0.98	0.62	0.81	91	1.00	0.58	0.93	215
3	0.92	0.39	0.78	373	0.80	0.34	0.60	109	0.96	0.64	0.86	312
4	0.88	0.11	0.73	552	0.66	0.25	0.47	114	0.95	0.51	0.82	410
5	0.84	0.42	0.70	694	0.52	0.20	0.40	106	0.92	0.61	0.80	471
6	0.80	0.41	0.68	826	0.43	0.16	0.30	95	0.88	0.52	0.77	526
7	0.78	0.39	0.65	855	0.34	0.10	0.24	118	0.86	0.53	0.74	556
8	0.76	0.25	0.64	872	0.31	0.09	0.19	133	0.84	0.52	0.73	579
9	0.73	0.40	0.62	812	0.25	0.08	0.15	150	0.82	0.49	0.71	612
10	0.72	0.35	0.60	739	0.20	0.06	0.12	198	0.80	0.43	0.69	543
11	0.69	0.40	0.59	619	0.17	0.05	0.10	233	0.78	0.48	0.68	555
12	0.67	0.41	0.57	477	0.15	0.04	0.09	285	0.76	0.45	0.66	560
13	0.66	0.42	0.57	372	0.12	0.04	0.08	323	0.75	0.45	0.64	463
14	0.64	0.40	0.55	242	0.10	0.04	0.07	363	0.73	0.40	0.63	410
15	0.61	0.36	0.53	151	0.09	0.03	0.06	388	0.72	0.37	0.61	384
16	0.59	0.40	0.51	60	0.08	0.03	0.06	395	0.69	0.40	0.60	283
17	0.56	0.38	0.47	31	0.07	0.03	0.05	424	0.67	0.44	0.58	270
18	0.49	0.28	0.42	6	0.06	0.02	0.05	391	0.64	0.42	0.56	240
19	0.00	0.00	0.00	0	0.06	0.02	0.04	361	0.63	0.39	0.55	15
20	0.00	0.00	0.00	0	0.05	0.02	0.04	367	0.61	0.19	0.52	12
¿20	0.00	0.00	0.00	0	0.05	0.00	0.02	3286	0.58	0.34	0.49	222

Table 1: Large dataset, tiered DEA results

3.3 Tiered DEA and the BCC Model

The examples of the figures above illustrate the result of applying the tiering procedure to the BCC_j^i and BCC_j^o models. These models, used to identify the Pareto-Koopmans efficient production surface, with no restrictions on returns to scale, were first proposed by Banker, Charnes, and Cooper [2]. The BCC_j^i model can be written as:

In Figs. 3 and 4, the outer layers designated by DMU_A, DMU_B, and DMU_C represent the efficient set which demonstrate best practice in the production process. However, this empirical production surface is dependent on the data observed.

Suppose the original set had not included B. The models would have identified a different, yet legitimate, production surface revealing the current technology consisting of DMUs A, E, F, and C. If data on B later became available and was added to the set of observations, it would markedly alter the shape of the efficient production surface. B now reveals a new production process with a corresponding level of technological achievement that was previously unseen. It is precisely this realization that motivates the tiered ranking procedure.

In DEA, all of the efficient DMUs share a common characteristic: they all demonstrate an ability to make the best use of the current technology to conduct their production process (i.e., they demonstrate best practice behavior). Once these DMUs have been identified, they can be temporarily removed, and the remaining DMUs form a new, valid DEA data set. When the BCC_o^i model is applied to this new data set a new efficient production surface is revealed. Had the data for DMUs of the outer tier not been available originally, this new production surface would legitimately characterize the best technological achievement level observed. Consequently, the DMUs comprising this new efficient surface share a common level of success of utilizing the currently revealed technology. Repeating this process groups the DMUs according to common achievement levels. DMUs on outer tiers reveal a technological advance not realized by DMUs on inner tiers.

Of primary significance is that the tiering procedure provides greater discriminatory power in determining managerial efficiency than previous DEA measures. To appreciate the importance of the new measure, an understanding of what causes a DMU to be inefficient is helpful. By stratifying across different tiers, the new measure provides a more complete description of the DMU's managerial efficiency relative to its contemporaries. One reason a DMU may be characterized as relatively inefficient in DEA is that it is dominated by a few highly efficient DMUs. A high concentration of other DMUs, with similar, but superior, management practices can cause the inferior DMUs to attain a low tier assignment. Fig. 5 depicts a set of DMUs with one output and one input. The tiering procedure produces three layers. In this case, J is enveloped by a concentration of DMUs using nearly the same level of inputs to produce a similar level of outputs. However, J consistently under-achieves compared to the other DMUs, consequently, it would receive a relatively low ranking. Even though the original DEA scores for DMUs F, G, and J are similar, TDEA further

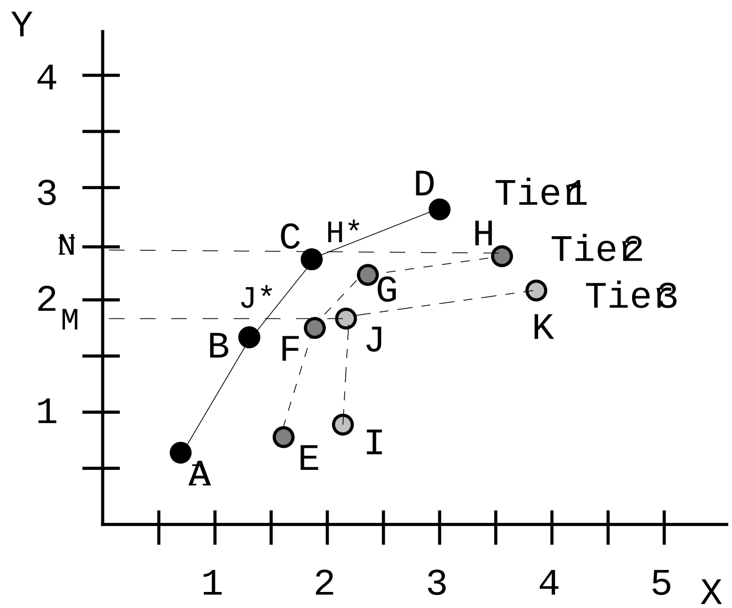

Figure 5: Example 1 tiered results – one input, one output

discriminates J as less productive given the available technology than DMUs F and G.

A second reason an inefficient DMU may result in a low efficiency score is that it operates in a production process "away from the crowd." These maverick DMUs may be leaders in the introduction of new production technologies or management methods into the market place. The transition to these new procedures is penalized under traditional DEA analysis because it appears that the DMUs introducing the new inputs consume more than the peers. The dominant DMU of this shift may be efficient, but the other DMUs can appear to be very inefficient. This situation can be seen with H. Notice, in this case H has a technical efficiency score seen as the ratio $\frac{NH^*}{NH}$. J has a higher technical efficiency score seen as the ratio $\frac{MJ^*}{MJ}$ but is ranked lower than H in terms of tier assignment. H may represent a risk-taker that shows a short-term reduction in its DEA efficiency score, to take advantage of new technology, by introducing new inputs. However, even with a short term loss of efficiency, the DMU could see a rise in its rank ordering by moving to a higher tier level. Once the new technology is fully integrated into the production process, the DMU may witness substantial increases in its DEA efficiency score. Traditional DEA analysis would penalize the management decision, in the short run, to introduce the new technology into the production process. Consequently, the decision to incur possible short term losses to achieve long term gains would appear unfavorable

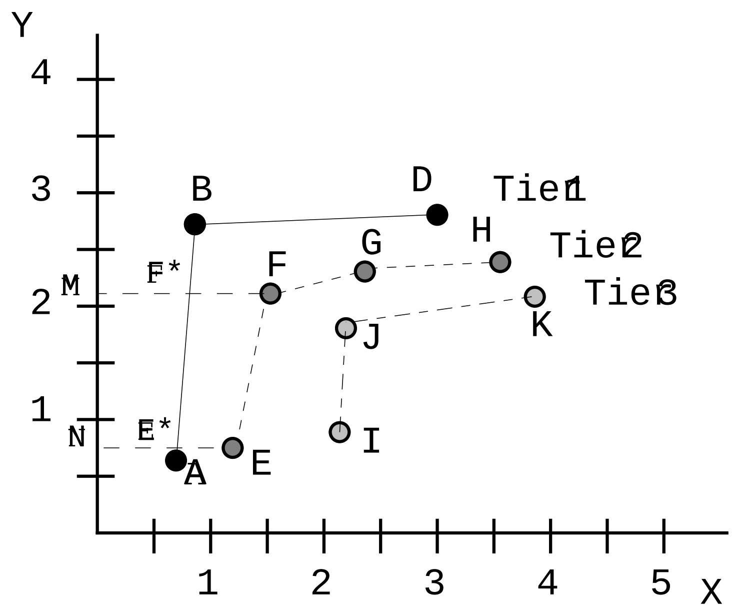

Figure 6: Example 2 tiered results – one input, one output

in a traditional DEA analysis. However, the stratified ranking of DMUs reveals the success of this management decision.

The above scenario can occur often in a free competitive market. Under dynamic conditions, the firms in the competitive market must adapt to maintain market share. A DMU that adopts a management style and production process similar to other DMUs, but consistently under performs, may result in a seemingly high relative efficiency score, but with a low rank ordering. The inability of this competitive firm to find its niche and distinguish itself from the "competition" may result in its failure in the market place. Consequently it is vital that these managerial effects are revealed. Current DEA methodologies are inadequate to reveal such conditions; TDEA offers an attractive means to discriminate between these managerial behaviors.

The example should not imply that the phenomena of a low efficient DMU achieving a relatively high rank is restricted to DMUs at "fringe" production levels. Data outliers can strongly affect the shape of the efficient surface. Fig. 6 shows the effect that outlier B has on the production surface. The outlier significantly distorts the surface making F seem relatively inefficient. Had B not been in the data set, F would be efficient. In spite of F's low efficiency score, TDEA ranks F relatively high. Consequently, the performance of F may not be as poor as indicated by the DEA score.

A key advantage of stratifying DMUs into tiers is that it allows the DEA methodology to more closely describe true managerial efficiency that may be masked by traditional DEA analysis. As a result, managers of inefficient DMUs have increased flexibility in improving production operations. The manager's long term goal may be to achieve efficiency. The DMU can strive to accomplish this by improving the short run rank ordering without a myopic focus on its DEA efficiency score.

3.4 Tiered DEA and the CCR Model

The ranking procedure can also be applied to the CCR models. For these models, which assume constant returns to scale, the efficient DMUs not only are operating most efficiently (with the greatest level of technological achievement) but they are also operating at the most productive scale size (MPSS). For the single input and single output model, the MPSS is determined by the DMUs yielding the highest ratio of quantity output to quantity input. In economics terms, this equates to the DMUs yielding the highest average product. Banker [1] demonstrated that the CCR model revealed the efficient DMUs operating at the MPSS in multi-dimensional processes.

The MPSS units which form the frontier for the CCR models are dependent on the observed set of data. In a traditional DEA analysis, the CCR scores for the inefficient DMUs reflect both technical and scale inefficiencies. To separate the technical from the scale inefficiency, the BCC model must also be run. The tiered procedure, when applied to the CCR model, presents a different picture of scale inefficiency. As each outer tier is removed from the data set, the set of DMUs which composes the most productive scale size changes. As a result, the values of the scale inefficiencies for the other DMUs also change. Therefore traditional DEA measures may overstate the amount of scale inefficiency that a DMU demonstrates.

Fig. 7 shows two tiers of scale-efficient DMUs for the same set of data used in Fig. 3.4, where DMUs A, B, C, and D were shown to be technically efficient. However, Fig. 7 indicates B and C are scale efficient while A and D are both scale inefficient. Because D falls far from the most productive scale boundary, the CCR_j^i DEA measure would indicate it is more scale inefficient than A. Yet, with the tiering procedure, both become scale efficient at a common level. Consequently, the two DMUs may not be as different, in terms of scale, as first suggested by the DEA scores.

4 Ranking Within Tiers

The TDEA approach extends traditional DEA to allow for a stratification of DMUs reflecting different productive efficiency layers. However, a ranking system within each layer is still needed. This section describes a new procedure that can provide a useful ranking.

Each tier defines a set of DMUs that forms a production surface (layer). To determine the relative rank ordering among the DMUs at each tier, one could measure the contribution of the DMU to the shape of its layer. DMUs that

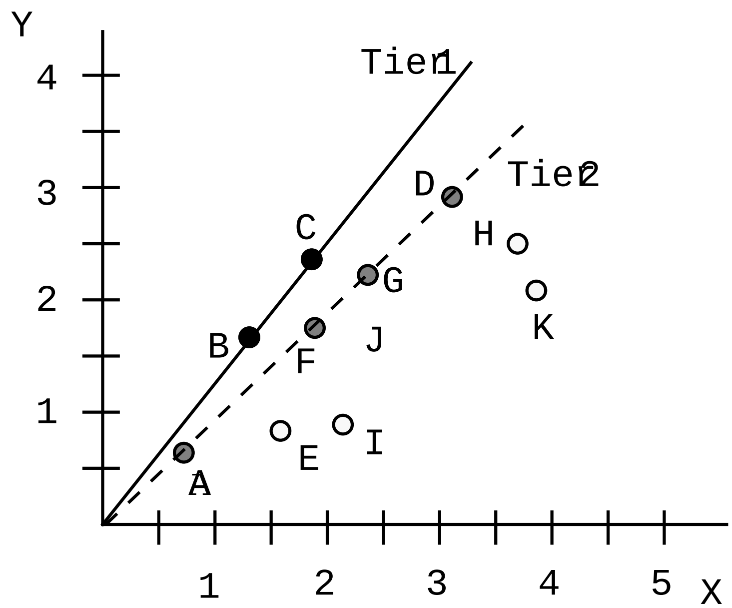

Figure 7: Example CCR tiered results – one input, one output

significantly distort the production surface layer on which they are assigned play a more prominent role in determining the shape of the production surface. Consequently, if these distortions can be measured, the DMUs could be ranked by the degree of distortion they contribute to the production layer.

The following model describes an *extremal DEA* (EDEA) approach for the CCR_j^i model. However, the formulation also applies to the output-oriented models and to either constant- or variable-returns-to-scale formulations. With this methodology, the DEA formulation is modified slightly to measure how far a DMU is from the resulting production surface when the DMU itself is removed from the data set of interest. The EDEA procedure can be described as follows. Let $\mathbf{X}$ be an $(m \times n)$ matrix and $\mathbf{Y}$ be an $(s \times n)$ matrix of the observed input and output values, respectively, of all the DMUs of interest. Select a DMU to be observed, in this case DMU$_o$. Let $\mathbf{X}^{[o]}$ and $\mathbf{Y}^{[o]}$ be $\mathbf{X}$ and $\mathbf{Y}$, respectively, with the observations of DMU$_o$ removed. Then the extremal DEA measure $\hat{\theta}$ is computed by the following model.

In contrast to a traditional DEA analysis, $\hat{\theta}$ can now be greater than one. For $\hat{\theta} \geq 1.0$ the DMU will be efficient and $\hat{\theta}$ measures the allowable proportional increase in inputs for the DMU to remain efficient. If $\hat{\theta} = 1$, the corresponding DMU is weakly efficient from a DEA standpoint. However, for the inefficient DMUs, the EDEA scores will be identical to traditional DEA scores, (e.g., $\hat{\theta} = \theta$), since the inefficient DMU itself will never be a member of an optimal basis (this can be shown in the same manner as theorem 1 of chapter II).

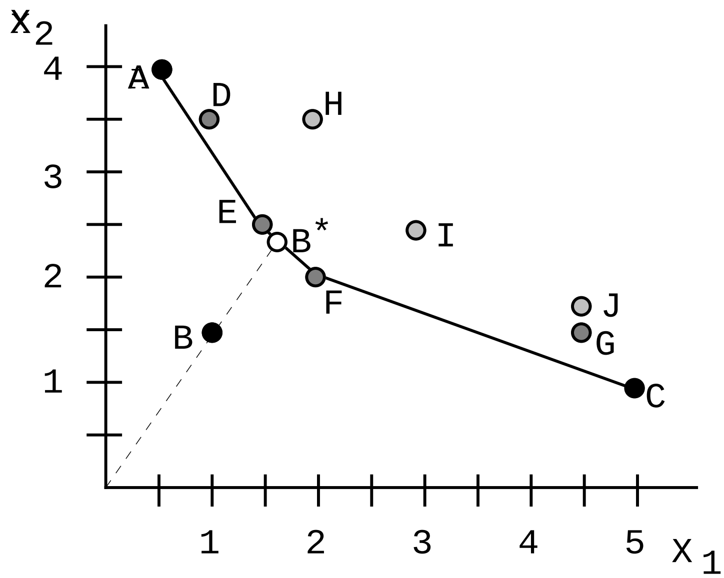

Figure 8: Extremal DEA example

Consequently, EDEA has an advantage over traditional DEA models in that it provides greater meaning to the scores for the efficient DMUs by allowing additional variability in the efficient score values.

Fig. 8 illustrates the EDEA procedure for B when applied to the data in Fig. 3. Here, B is projected onto the "new" efficient surface using EDEA. The resultant objective value, $\hat{\theta}^* = \frac{0B^*}{0B}$ measures how "far" B is from the efficient surface. Viewing the problem from a different perspective, $\hat{\theta}^*$ reflects the degree to which B would contribute to the shape of the new efficiency surface if it was added to the data set.

DMUs that cause significant and important distortions of the efficiency surface will result in a high EDEA objective value. Those DMUs that have little influence on the shape of the production surface will have objective values close to 1. Consequently, the DMUs can be ranked by order of influence at each tier level based on the EDEA scores.

An overall ranking of all DMUs can be achieved by: (1) using TDEA to to stratify all DMUs into tiers, (2) applying EDEA to each tier level, (3) ranking each tier's DMUs by $\hat{\theta}$, (4) then finding each unit's overall rank by ranking those DMUs on outer tiers as more important than those on inner tiers. The following section presents an illustrative example of this procedure and provides some numerical interpretations.

	DMU									
	A	B	C	D	E	F	G	H	I	J
y	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
x_1	0.50	1.00	5.00	1.00	1.50	2.00	4.50	2.00	3.00	4.50
x_2	4.00	1.50	1.00	3.50	2.50	2.00	1.50	3.50	2.50	1.80

Table 2: Tiered DEA Example Data

	DMU									
	A	B	C	D	E	F	G	H	I	J
DEA	1.00	1.00	1.00	.765	.650	.722	.788	.482	.565	.688
TDEA	1	1	1	2	2	2	2	3	3	3
TEDEA1	2.00	1.60	1.50	.765	.650	.722	.788	.482	.565	.688
TEDEA2	—	—	—	1.50	1.05	1.13	1.33	.733	.774	.889
TEDEA3	—	—	—	—	—	—	—	1.50	1.13	1.39
RANK1	1	2	3	—	—	—	—	—	—	—
RANK2	—	—	—	1	4	3	2	—	—	—
RANK3	—	—	—	—	—	—	—	1	3	2
RANK	1	2	3	4	7	6	5	8	10	9
DEARANK	1	1	1	5	8	6	4	10	9	7

Table 3: Results of the tiered rank-ordering procedure

5 Example of Tiered DEA

As illustrated in Fig. 3, based on the data in Table 2, the TDEA approach yields three production surfaces. Those DMUs on tier 1 are the same efficient units found in a standard envelopment analysis. The DMUs of succeeding inner tiers compose less efficient production surface layers. Notice though, the inefficient DMUs can now be further distinguished by the production surface on which they fall. DMUs on outer tier levels represent more successful production processes. Consequently, the DMUs on outer tiers can be ranked more efficient than DMUs on inner tiers. As will be demonstrated, this does not necessarily coincide with the traditional DEA score. The results of the TDEA coupled with the EDEA are shown in Table 3 and labeled as TEDEA1 through TEDEA3.

The DEA row lists the CCR^i θ scores for each unit in the data set. The TDEA row indicates the tier level to which each DMU is assigned as a result of the TDEA procedure. TEDEAt rows give the EDEA $\hat{\theta}$ values for all DMUs on tier t or higher. DMUs in tier t are ranked within tier in the RANKt rows. The overall rank for the entire set of DMUs is given in the RANK row. This can be compared to the ranking the DMUs would have been given, listed in DEARANK, had they been ordered by the CCR^i θ value.

Some important observations can be made concerning these results. Notice, the DEA and the TEDEA1 results are identical for the inefficient DMUs.

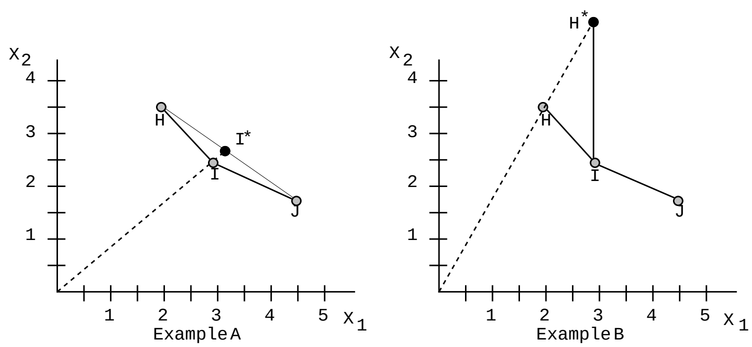

Figure 9: TEDEA example

Likewise, the efficient DMUs are appropriately identified with values greater or equal to 1. Notice that the higher inefficiency scores do not necessarily indicate on what tier level a DMU may fall. For example J has a higher θ than E but falls on a lower tier level. By observing the results at TEDEA3 one notices that the ranking of H, I, and J does not correspond with the ranking the DMUs would receive if the DEA efficiency scores were used. In fact, H has the lowest DEA efficiency score, but the highest rank of tier 3. Fig. 9 illustrates why this is so. I does not significantly distort the efficiency surface that exists when the I is not present. This is not true for H which significantly distorts the shape of the production surface; H is more influential than I and thus is ranked higher.

A major advantage of TEDEA, besides ranking each DMU by its influence, is that it can help paint a numerical picture of the environment in which a DMU operates. As each tier level and associated production surface is removed, the new DEA scores for the remaining DMUs can be calculated. In this way, the *migration* of the DEA scores for a particular DMU can be traced through a series of tiers. A rapid rise in the scores may indicate that the DMU is in a region with a relative low density of other DMUs but is dominated by a few highly efficient ones. A slow rise in the score may indicate that the DMU is surrounded by a larger density of other DMUs which may have similar management styles or environments but are operating more efficiently. This information can assist the analyst and managers in determining appropriate courses of actions to improve either the rank ordering or the efficiency score.

The rank ordering procedure may also prove useful in window analysis. If DMU behavior is tracked over time, the changes in rank ordering should reflect the relative effectiveness of on-going managerial decision-making. These managerial changes may remain hidden from traditional DEA analysis unless the changing practices result in a change in status of the DMU as being efficient or inefficient. Consequently, the rank ordering methodology may provide prompt managerial feedback as to how a DMU compares with the competition as a result of implemented changes.

6 Computational Considerations

It is important to note, unlike CCR or BCC models, with the EDEA method the basic elements in the optimal solution need not represent efficient DMUs. Fig. 8 depicts such a situation. When B is removed from the data set, the resulting efficient surface includes E and F, both of which are inefficient according to a traditional DEA. Therefore, the advantage of computational efficient techniques for DEA, such as restricted basis entry and early identification of efficient DMUs, can not be maintained for the EDEA models. However, other computational efficient procedures are possible for the EDEA model. Computational efficiency can be maintained by combining TDEA and EDEA into a single formulation, TEDEA*, based on the following observation.

When a DMU is removed from tier t's data set, the resulting production surface will consist of DMUs belonging to either tier t or $t+1$.

In this way, the stratification of DMUs proves to be a valuable computational tool for the EDEA method. Since the number of DMUs at *tier* t and $t+1$ is typically small compared to the entire set, the linear programming problems remain small when determining the EDEA scores. As a result, TDEA and EDEA can be combined to form a computationally efficient rank ordering method. Let $\mathbf{X}_{t^*}$ and $\mathbf{Y}_{t^*}$ be the matrices of input and output vectors of DMUs belonging to *tier* t or $t+1$. Choose a DMU from tier t for analysis, let this be DMU$_o$. The TEDEA* model can be written as:

$$
\begin{aligned}
&\text{(TEDEA*)} \qquad \min \hat{\theta} && && (1)\\
&\text{s.t.} \qquad \mathbf{Y}_{t^*}^{[o]}\boldsymbol{\lambda} - \mathbf{s}^o &=& \; Y_o && (2)\\
&\hat{\theta} X_o - \mathbf{X}_{t^*}^{[o]}\boldsymbol{\lambda} - \mathbf{s}^i &=& \; \mathbf{0} && (3)\\
&\boldsymbol{\lambda}, \mathbf{s}^i, \mathbf{s}^o &\geq& \; \mathbf{0} && (4)\\
&\hat{\theta} && \text{free} && (5)
\end{aligned}
$$

Because the TEDEA* problem consists of LPs that are much smaller than EDEA, computational efficiency is maintained.

7 Additional Benefits of Ranking

As mentioned previously, the empirical production surface of the DEA model is greatly influenced by possible outliers in the data. In 1971, Timmer [11] assumed a Cobb-Douglas form to develop a probabilistic frontier production function for observed data. Using the Cobb-Douglas from, Timmer translated the problem into a linear programming model with a striking similarity to the DEA model. Since he assumed a single output, multi-input case, he was able to compare the frontier analysis with the traditional econometric model. He showed by eliminating the top 2% of observations which appeared to be outliers, the linear programming frontier model yielded results that could be supported by the econometric models.

In DEA, no functional form is assumed. Consequently, potential outliers have been difficult to identify. In this case, the efficient DMUs may not represent normal production activities and therefore may not serve as a legitimate basis of comparison. Consequently, DEA efficiency scores of outliers could distort the true picture of managerial efficiency of the inefficient DMUs. As a result, identification of possible outlier or extremely dominant DMUs is essential. With TEDEA*, the outliers will be located on outer tiers. If possible outliers are revealed, the problematic DMUs can be removed from the analysis. By following this procedure, a better estimation of managerial efficiency can be achieved.

8 Conclusions

This chapter has outlined a rank ordering of DMUs based on the influence they serve in forming empirical production layers for the data set. The intent is not to present the only valid system or rank ordering, but to stimulate thought as to how DEA methods may be modified to make the results more meaningful to practitioners. One concern of the rank ordering approach is the computational requirements to achieve the results. On some layers the density of efficient DMUs will be high and other layers the density may be low. The code must be computationally efficient across this wide variety of conditions. In addition, the code must be flexible to allow for constant or variable returns to scale in either the input or output models. Flexibility to switch between models at different layers may also be desirable. The TEDEA* approach meets the requirements to effectively and efficiently perform such tasks.

The rank order procedure presented in this discussion classifies DMUs by their ability to use the current technology to efficiently conduct the production process. This classification can serve as a proxy measure of management efficiency that provides more detailed information than offered by standard DEA models. By identifying the best and worst performing DMUs, analysts can pinpoint which DMUs should serve as a basis of comparison from which to determine which management practices enhance and which deter from productive efficiency. In addition, the rank ordering methodology presented here reveals the need to further characterize the computational issues of DEA and provide efficient codes to accomplish these new tasks.

A Multinormal Data Generator

Since few large-scale DEA problems are currently available, randomly generated data sets were needed to simulate possible real life data. One procedure to generate such data is to draw the samples from a multinormal distribution. To this end, the DRNMVN routine from the IMSL library was used to generate data for 10,000 DMUs. The variance covariance matrix of Table 4 was used to generate the data for the input and output variables for each DMU. The code to generate this data is described by Hickman [10].

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
x_1	303.4							
x_2	2.5	2.5						
x_3	12493.9	6093.9	112750625.0					
x_4	4062.6	484.2	-2890405.7	31033992.2				
x_5	265.2	45.3	-1346076.8	-455158.7	7629765.5			
x_6	19597.2	-195.0	3436081.9	-270405.4	-674642.7	86492323.0		
x_7	36418.8	6428.4	111950224.4	27415022.3	5153887.3	88983356.8	233502490.7	
x_8	3647.0	1001.0	10815783.1	112293.8	-64824.7	- 319883.0	10543369.3	7812141.0

Table 4: Variance-Covariance Matrix for Multinormal Data

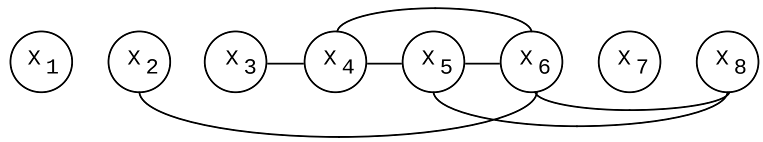

Figure 10: Correlations of multinormal data variables

From the randomly generated data, the variables representing inputs and outputs needed to be carefully chosen. To coincide with sound economic theory, all outputs should be positively correlated with all the inputs. In Figure 10, all negatively correlated variables are connected with a line.

By eliminating variable x_4, variables x_1, x_3, and x_7 could represent outputs since they would be positively correlated with all the other remaining variables that would represent inputs. In this way, each DMU would be comprised of 3 outputs and 4 inputs. Notice also, inputs positively correlated would be compliments in the production process; those negatively correlated are substitutes. Since the input variables cover a wide range of cases of compliment and substitute relationships, the randomly generated data further simulates possible real life data.

B DEA-GEN

While random data generators can provide large problems, a more systematic approach is need to represent real life scenarios. To this end, economic theory was used to generate large-scale DEA problem sets. In production economics, the most widely used functional form is known as the Cobb-Douglas production function. This function is written as:

$$y_j = a_o \prod_{i=1}^{m} x_{ij}^{\alpha_i}, \; x_{ij} > 0, \; j = 1, \ldots, n$$

Here, y_j is the single aggregate output produced by DMU_j, x_{ij} is the values of the input variables used by DMU_j in the production process, α_i is the factor elasticity for input i, and a_o is a constant scale factor. If $\sum_{i=1}^{m} \alpha_i = 1$ then only constant returns to scale exist in the production process. For $\sum_{i=1}^{m} \alpha_i < 1$ decreasing returns to scale are present, while $\sum_{i=1}^{m} \alpha_i > 1$ indicates increasing returns to scale. In the DEA studies, increasing returns to scale are not used because the function results in only a few DEA efficient points. Although this does not pose a problem, it does not realistically represent true life data.

For the single output model, this production function has many desirable properties. If the inputs are randomly generated, the function generates output values that will always lie on the production possibility frontier, i.e., they will be DEA efficient for $\sum_{i=1}^{m} \alpha_i \leq 1$. This frontier is central to the theory of economic growth and measures the rate of technological progress. The Cobb-Douglas frontier represents the best use of technology as well as the best management practices to achieve efficient production. This coincides with the practical use of DEA. Unlike DEA, the quest in economic studies is to attempt to estimate the values of α_i by fitting the function to the observed data. By choosing the α_i values a priori, then randomly generating the input values, the output values can be determined so that they coincide with widely accepted economic theory. To insure that not all generated data sets fall on the efficient frontier surface, the a_o scale factor can be randomly generated. The efficient DMUs will consist of those generated where a_o takes on it maximum value. Control over the number of DMUs that are efficient in the data set can be maintained by limiting the number of DMUs generated with $\max a_o$.

In DEA analysis, the single-output, multiple-input scenario is not of primary interest. DEA was developed to analyze the case of multiple-output, multiple-input studies. Färe and Grosskopf [9] point out that the existence of a joint production function has not been established. That is, the multiple-output, multiple-input model does not produce values strictly on the efficient production frontier. To do so, would require strict assumptions that would not provide the desired realism for the DEA study. However, a joint model without the strict assumptions would simulate economically sound production processes. Consequently, a joint model was developed for the DEA-GEN problem generator.

For each of the problem sets, the input values were generated from a uniform distribution. The α_i values were chosen to simulate constant returns to scale processes as well as a variety of cases representing different levels of decreasing returns to scale. The a_o value was randomly generated, but with a modest control of the number of efficient DMUs that could be present in the problem data.

Once the single aggregate output level was calculated, the individual output levels were determined by assigning each individual output as a percentage of the aggregate. The percentages for each individual output were drawn from normal distributions with predetermined means and standard deviations. The means of the normal distributions were chosen so that the percentages sum to one. Table 5 lists the α_i values as well as the means and standard deviations that were used to generate twelve different Cobb-Douglas data sets.

Because of the economic foundations of the DEA-GEN code, the data generated resembles a class of problems that should more closely simulate realistic economic data than what would be possible from the data sampled from a multinormal distribution.

References

[1] A. Ali and L. Seiford. Computational accuracy and infinitesimas in data envelopment analysis. *INFOR*, 31(4):290–297, 1993.

[2] R. D. Banker. Estimating most productive scale size using data envelopment analysis. *European Journal of Operational Research*, 17:35–44, 1984.

	DMUs	α Values	Mean Values	Standard Dev.
DEA-GENa	1,000	.3,.2,.2,.3	.12,.13,.13,.14,.11	.03,.04,.02,.01,.01
DEA-GENa	2,000	.3,.2,.2,.2,.1	.20,.30,.15	.03,.04,.02
DEA-GENa	4,000	.3,.3,.2,.2	.12,.12,.15,.14	.03,.04,.02,.01
DEA-GENa	8,000	.4,.2,.1,.1,.1	.5	.1
DEA-GENb	1,000	.03,.07,.12,.03,.12,.1,.02	.2,.3,.15	.01,.05,.01
DEA-GENb	2,000	.1,.1,.1,.1,.1,.1,.1	.52,.13	.03,.04
DEA-GENb	4,000	.4,.3,.1	.4,.3,.15	.13,.08,.02
DEA-GENb	8,000	.3,.17,.12,.03,.12,.2	.2,.3,.15,.14	.01,.05,.01,.01
DEA-GENc	1,000	.18,.2,.14,.1,.1,.1	.2,.12,.14,.11,.11	.03,.04,.02,.01,.01
DEA-GENc	2,000	.13,.18,.2,.14,.1,.1	.2,.18,.15,.19	.03,.04,.02,.01
DEA-GENc	4,000	.13,.2,.12,.13	.2,.3,.15,.14,.11	.03,.04,.02,.01,.01
DEA-GENc	8,000	.08,.12,.21,.07,.11	.3,.3,.3,.1	.04,.04,.1,.03
DEA-GENd	25,000	.3,.3,.3	.3,.3	.01,.01

Table 5: Parameter Values for DEA-GEN Code

[3] R. S. Barr and B. L. Hickman. Reporting computational experiments with parallel algorithms: issues, measures, and experts' opinions. *ORSA Journal on Computing*, 5:2–18, 1993.

[4] R. S. Barr, L. M. Seiford, and T. Siems. An envelopment-analysis approach to measuring the managerial quality of banks. *Annals of Operations Research*, 42:1–19, 1993.

[5] Richard S. Barr and Thomas M. Siems. Predicting bank failure using DEA to quantify management quality. In R. Barr, R. Helgason, and J. Kennington, editors, *Interfaces in Computer Science and Operations Research*, pages 341–366. Kluwer Academic Publishers, Boston, 1997.

[6] A. Charnes and W. W. Cooper. *Management Models and Industrial Applications of Linear Programming.* John Wiley & Sons, New York, 1961.

[7] A. Charnes and W. W. Cooper. Preface to topics in data envelopment analysis. *Annals of Operations Research*, 2:59–94, 1985.

[8] T.H. Cormen, C.E. Leiverson, and R.L. Rivest. *Introduction to Algorithms.* MIT Press, Cambridge, MA, 1990.

[9] M. J. Farrell. The measurement of productive efficiency. *Journal of the Royal Statistical Society*, 120:253–290, 1957.

[10] B. L. Hickman. *Parallel Algorithms for Pure network Probles and Related Applications.* PhD thesis, Southern Methodist University, Dallas, TX, 1988.

[11] C.P. Timmer. Using a probabilistic frontier production function to measure technical efficiency. *Journal of Political Economy*, 79:776–794, 1971.