Chapter 1

Continuous-Time Signals

1.1 Introduction

What is a signal? Of course, we know that a signal can be a rather abstract notion, such as a flashing light on our car’s front bumper (turn signal), or an umpire’s gesture indicating that a pitch went over the plate during a baseball game (a strike signal). One of the definitions of signal in the Merrian-Webster dictionary is:

A detectable physical quantity or impulse (as a voltage, current, or magnetic field strength) by which messages or information can be transmitted.

These are the types of signals which will be of interest in this book. We will focus on two broad classes of signals, discrete-time and continuous-time. We will consider discrete-time signals later on in this book. For now, we will focus our attention on continuous-time signals. Fortunately, continuous-time signals have a very convenient mathematical representation. We represent a continuous-time signal as a function $x(t)$ of the real variable $t$. Here, $t$ represents continuous time and we can assign to $t$ any unit of time we deem appropriate (seconds, hours, years, etc.). We do not have to make any particular assumptions about $x(t)$ such as boundedness (a signal is bounded if it has a finite value). Some of the signals we will work with are in fact, not bounded (i.e. they take on an infinite value). However most of the continuous-time signals we will deal with in the real world are bounded. We actually encounter signals every day. Suppose we sketch a graph of the temperature...
outside the Jerry Junkins Electrical Engineering Building on the SMU campus as a function of time. The graph might look something like in Figure 1.1. This is an example of a signal which represents the physical quantity temperature as it changes with time during the course of a week. Figure 1.2 shows another common signal, the speech signal. Human speech signals are often measured by converting sound (pressure) waves into an electrical potential using a microphone. The speech signal therefore corresponds to the air pressure measured at the point in space where the microphone was located when the speech was recorded. The large deviations which the speech signal undergoes corresponds to vowel sounds such as “ahhh” or “eeeh” (voiced sounds) while the smaller portions correspond to sounds such as “th” or “sh” (unvoiced sounds). In Figure 1.3, we see yet another signal called an electrocardiogram (EKG). The EKG is a voltage which is generated by the heart and measured by subtracting the voltage recorded from two points on the human body as seen in Figure 1.4. Since the heart generates very low-level voltages, the difference signal must be amplified by a high-gain amplifier.
1.2 Signal Power, Energy, and Frequency

Signals can be characterized in several different ways. Audio signals (music, speech, and really, any kind of sound we can hear) are particularly useful because we can use our existing notion of “loudness” and “pitch” which we normally associate with an audio signal to develop ways of characterizing any kind of signal. In terms of audio signals, we use “power” to characterize the loudness of a sound. Audio signals which have greater power sound “louder” than signals which have lower power (assuming the pitch of the sounds are within the range of human hearing). Of course, power is related to the amplitude, or size of the signal. We can develop a more precise definition of power. The signal power is defined as:

\[
p_x = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t) dt
\]  

(1.1)

The energy of this signal is similarly defined

\[
e_x = \int_{-\infty}^{\infty} x^2(t) dt
\]  

(1.2)
We can see that power has units of energy per unit time. Strictly speaking, the units for energy depend on the units assigned to the signal. If $x(t)$ is a voltage, then the units for $e_x$ would be volts$^2$-seconds. Notice also that some signals may not have finite energy. As we will see shortly, periodic signals do not have finite energy. Signals having a finite energy are sometimes called energy signals. Some signals that have infinite energy however can have finite power. Such signals are sometimes called power signals.

We use the concept of “frequency” to characterize the pitch of audio signals. The frequency of a signal is closely related to the variation of the signal with time. Signals which change rapidly with time have higher frequencies than signals which are changing slowly with time as seen Figure 1.5. It turns out that signals can be represented in terms of their frequencies. Most of us have listened to music on a CD player which has a display that illustrates the frequency content of the signal. Figure 1.6 shows one of these frequency display found on the Microsoft Media player. In this display, frequency corresponds to the horizontal axis, while the vertical axis corresponds to the power present at each frequency.

Something to keep in mind is that the signals shown in Figures 1.1, 1.2, and 1.3 each have different units (degrees Fahrenheit, pressure, and voltage,
1.3 Basic Signal Operations

We will be considering the following basic operations on signals:

- **Time shifting:**
  \[ y(t) = x(t - \tau) \]
  The effect that a time shift has on the appearance of a signal is seen in Figure 1.7. If \( \tau \) is a positive number, the time shifted signal, \( x(t - \tau) \) gets shifted to the right, otherwise it gets shifted left.

- **Time reversal:**
  \[ y(t) = x(-t) \]
  Time reversal flips the signal about \( t = 0 \) as seen in Figure 1.7.

- **Addition:** any two signals can be added to form a third signal,
  \[ z(t) = x(t) + y(t) \]
Figure 1.5: The signal $y(t)$ contains a greater amount of high frequencies than $x(t)$.

- **Time scaling:**
  $$y(t) = x(\Omega t)$$

  Time scaling “compresses” the signal if $\Omega > 1$ or “stretches” it if $\Omega < 1$ (see Figure 1.8).

- **Multiplication by a constant, $\alpha$:**
  $$y(t) = \alpha x(t)$$

- **Multiplication of two signals, their product is also a signal.**
  $$z(t) = x(t)y(t)$$

  Multiplication of signals has many useful applications in wireless communications.

- **Differentiation:**
  $$y(t) = \frac{dx(t)}{dt}$$
• Integration:

\[ y(t) = \int x(t) dt \]

There is another very important signal operation called \textit{convolution} which we will look at in detail in Chapter 3. As we shall see, convolution is a combination of several of the above operations.
Figure 1.7: (a) original signal, (b) time-shift, (c) time-reversal.
Figure 1.8: (a) original signal, (b) $\Omega > 1$, (c) $\Omega < 1$. 
1.4 Complex Numbers and Complex Arithmetic

Before we begin studying signals, we need to review some basic aspects of complex numbers and complex arithmetic. The rectangular coordinate representation of a complex number $z$ is $z$ has the form:

$$z = a + jb$$  \hspace{1cm} (1.3)

where $a$ and $b$ are real numbers and $j = \sqrt{-1}$. The real part of $z$ is the number $a$, while the imaginary part of $z$ is the number $b$. We also note that $jb(jb) = -b^2$ (a real number) since $j(j) = -1$. Any number having the form $z = jb$ where $b$ is a real number is an imaginary number. A complex number can also be represented in polar coordinates

$$z = re^{j\theta}$$  \hspace{1cm} (1.4)

where

$$r = \sqrt{a^2 + b^2}$$  \hspace{1cm} (1.5)

is the magnitude and

$$\theta = \arctan \left( \frac{b}{a} \right)$$  \hspace{1cm} (1.6)

is the phase of the complex number $z$. The notation for the magnitude and phase of a complex number is given by $|z|$ and $\angle z$, respectively. Using Euler’s Identity:

$$e^{\pm j\theta} = \cos(\theta) \pm j \sin(\theta)$$

it follows that $a = r \cos(\theta)$ and $b = r \sin(\theta)$. Figure 1.9 illustrates how polar coordinates and rectangular coordinates are related. Rectangular coordinates and polar coordinates are each useful depending on the type of mathematical operation performed on the complex numbers. Often, complex numbers are easier to add in rectangular coordinates, but multiplication and division is easier in polar coordinates. If $z = a + jb$ is a complex number then its complex conjugate is defined by

$$z^* = a - jb$$
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Figure 1.9: Relationship between rectangular and polar coordinates.

in polar coordinates we have

\[ z^* = re^{-j\theta} \]

note that \( zz^* = |z|^2 = r^2 \) and \( z + z^* = 2a \). Also, if \( z_1, z_2, \ldots, z_N \) are complex numbers it can be easily shown that

\[
(z_1 + z_2 + \cdots + z_N)^* = z_1^* + z_2^* + \cdots + z_N^* \tag{1.7}
\]

and

\[
(z_1z_2\cdots z_N)^* = z_1^* z_2^* \cdots z_N^* \tag{1.8}
\]

Table 1.1 indicates how two complex numbers combine in terms of addition, multiplication, and division when expressed in rectangular and in polar coordinates.

<table>
<thead>
<tr>
<th>operation</th>
<th>rectangular</th>
<th>polar</th>
</tr>
</thead>
<tbody>
<tr>
<td>( z_1 + z_2 )</td>
<td>((a_1 + a_2) + j(b_1 + b_2))</td>
<td>( r_1r_2 e^{j(\theta_1 + \theta_2)} )</td>
</tr>
<tr>
<td>( z_1z_2 )</td>
<td>( a_1a_2 - b_1b_2 + j(a_1b_2 + a_2b_1) )</td>
<td>( \frac{r_1r_2 e^{j(\theta_1 + \theta_2)}}{a_1^2 + b_1^2} )</td>
</tr>
<tr>
<td>( z_1/z_2 )</td>
<td>( \frac{(a_1a_2 + b_1b_2) + j(b_1a_2 - a_1b_2)}{a_1^2 + b_1^2} )</td>
<td>( \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)} )</td>
</tr>
</tbody>
</table>

Table 1.1: Operations on two complex numbers, \( z_1 = a_1 + jb_1 = r_1e^{j\theta_1} \) and \( z_2 = a_2 + jb_2 = r_2e^{j\theta_2} \). The sum of two complex numbers is cumbersome to express in polar coordinates, and is not shown.
1.5 Periodic Signals

Periodic signals have the following property:

\[ x(t) = x(t + kT) \]  

(1.9)

where \( k \) is an integer and \( T \) is called the fundamental period. Periodic have the property that they “repeat” every \( T \) seconds. For periodic signals, the power can be defined as

\[ p_x = \frac{1}{T} \int_{t_0}^{t_0+T} x^2(t) \, dt \]  

(1.10)

Figure 1.10 shows an example of a periodic signal. We will study the frequency content of periodic signals in some detail in Chapter 2.

![Figure 1.10: General periodic signal.](image)

1.6 Sinusoidal Signals

Sinusoidal signals are perhaps the most important type of signal that we will encounter in signal processing. There are two basic types of signals, the cosine:

\[ x(t) = A \cos(\Omega t) \]  

(1.11)
and the sine:

\[ x(t) = A \sin(\Omega t) \]  \hspace{1cm} (1.12)

where \( A \) is a real constant. Plots of the sine and cosine signals are shown in Figure 1.11. Sinusoidal signals are periodic signals. The period of the cosine and sine signals shown above is given by \( T = 2\pi/\Omega \). The frequency of the signals is \( \Omega = 2\pi/T \) which has units of rad/sec. Equivalently, the frequency can be expressed as \( 1/T \), which has units of sec\(^{-1}\), cycles/sec, or Hz. The quantity \( \Omega t \) has units of radians and is often called the phase of the sinusoid. Recalling the effect of a time shift on the appearance of a signal, we can observe from Figure 1.11 that the sine signal is obtained by shifting the cosine signal by \( T/4 \) seconds, i.e.

\[ \sin(\Omega t) = \cos(\Omega(t - T/4)) \]
The quantity \( e^{j\Omega t} \) is called a complex sinusoid and can be expressed as

\[ e^{\pm j\Omega t} = \cos (j\Omega t) \pm j \sin (j\Omega t) \]  

There are a number of trigonometric identities which are sometimes useful. These are shown in Table 1.2. Table 1.3 shows some basic calculus operations on sine and cosine signals.
1.7 Exercises

1. Consider the signals shown in Fig. 1.12. Sketch the following signals:\(^1\):

(a) \( x_1(t) + 2x_2(t) \).
(b) \( x_1(-t) - x_2(t - 1) \).
(c) \( x_1(-t + 1) \).
(d) \( x_2(t - 1) \).
(e) \( x_1(2t) \).
(f) \( x_1(t/2) \).
(g) \( x_1(t)x_3(t + 1) \).
(h) \( x_3(2t - 4) \).
(i) \( x_3(-2t - 4) \).

2. For each of the signals in Figure 1.13:

(a) What is the period?

\(^1\)Assume that for step discontinuities, the signal takes on the greater of the two values.

\[ \frac{d}{dt} \cos(\Omega t) = -\Omega \sin(\Omega t) \]
\[ \frac{d}{dt} \sin(\Omega t) = \Omega \cos(\Omega t) \]
\[ \int \cos(\Omega t) dt = \frac{1}{\Omega} \sin(\Omega t) \]
\[ \int \sin(\Omega t) dt = -\frac{1}{\Omega} \cos(\Omega t) \]
\[ \int_0^T \sin(k\Omega_o t) \cos(n\Omega_o t) dt = 0 \]
\[ \int_0^T \sin(k\Omega_o t) \sin(n\Omega_o t) dt = 0, k \neq n \]
\[ \int_0^T \cos(k\Omega_o t) \cos(n\Omega_o t) dt = 0, k \neq n \]
\[ \int_0^T \sin^2(n\Omega_o t) dt = T/2 \]
\[ \int_0^T \cos^2(n\Omega_o t) dt = T/2 \]

Table 1.3: Derivatives and integrals of sinusoidal signals.
(b) Sketch $x(t - 0.25)$, and $x(t + 1)$.
(c) Find the power for each signal.

3. Suppose that $z_1 = 3 + j2$, $z_2 = 4 + j5$. Find:
   - $|z_1|, \angle z_1, |z_2|, \angle z_2$
   - $z_1 + z_2$ in rectangular coordinates.
   - $z_1z_2$ in rectangular and polar coordinates.
   - $z_1/z_2$ in polar coordinates.

4. What is the period, frequency, and power of the sinusoidal signal $x(t) = 2 \cos(5t)$.

5. Can you find a general formula for the power of the sinusoidal signal $x(t) = A \cos(\Omega t)$?

6. Express $2 \cos(10t) + 3 \sin(10t)$ as a single sinusoidal signal.

7. Sketch $x(t) = \sin(t - \theta)$ for $\theta = \pi/4, \pi/2, 3\pi/4$.

8. Sketch $x(t) = \sin(2t - \theta)$ for $\theta = \pi/4, \pi/2, 3\pi/4$.

9. Consider the motion of the second hand of a clock. Assume the length of the second hand is 1 meter. (a) what is the angular frequency of the second hand. (b) find an expression for the horizontal and vertical displacements of the tip of the second hand, assuming the origin is at the clock center.
Figure 1.12: Signals for problem 1.
Figure 1.13: Signals for problem 2.
Chapter 2

Fourier Series of Periodic Signals

2.1 Symmetry Properties of Periodic Signals

A signal has even symmetry if it satisfies:

\[ x(t) = x(-t) \]  \hspace{1cm} (2.1)

and odd symmetry if it satisfies

\[ x(t) = -x(-t) \]  \hspace{1cm} (2.2)

Figure 2.1 shows pictures of periodic even and odd symmetric signals. If \( x(t) \) is an odd symmetric periodic signal, then we must have:

\[ \int_{t_0}^{t_0+T} x(t)dt = 0 \]  \hspace{1cm} (2.3)

This is easy to see if we choose \( t_0 = -T/2 \).

We also note that the product of two even signals is also even while the product of an even signal and an odd signal must be odd. Finally, the product of two odd signals must be even. For example, suppose \( x_o(t) \) has odd symmetry and \( x_e(t) \) has even symmetry. Their product has odd symmetry because if \( y(t) = x_o(t)x_e(t) \), then \( y(-t) = x_o(-t)x_e(-t) = -y(t) \).
Figure 2.1: (a) Even-symmetric, and (b) odd-symmetric periodic signals. Note that the integral over any period of an odd-symmetric periodic signal is zero.
2.2 Trigonometric Form of the Fourier Series

An important goal of this book is to develop tools which will enable us to study the frequency content of signals. An important first step is the Fourier Series. The Fourier Series enables us to completely characterize the frequency content of any periodic signal. Any periodic signal \( x(t) \) can be expressed in terms of the Fourier Series, which is given by:

\[
x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\Omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\Omega_0 t) \quad (2.4)
\]

where

\[
\Omega_0 = \frac{2\pi}{T} \quad (2.5)
\]

is the fundamental frequency of the periodic signal. Examination of (2.4) suggests that periodic signals can be represented as a sum of suitably scaled cosine and sine waveforms at frequencies of \( \Omega_0, 2\Omega_0, 3\Omega_0, \ldots \). The cosine and sine terms at frequency \( n\Omega_0 \) are called the \( n^{th} \) harmonics. Evidently, periodic signals contain only the fundamental frequency and its harmonics. A periodic signal cannot contain a frequency that is not an integer multiple of its fundamental frequency.

In order to find the Fourier Series, we must compute the Fourier Series coefficients. These are given by

\[
a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x(t)dt \quad (2.6)
\]

\[
a_n = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \cos(n\Omega_0 t)dt, \quad n = 1, 2, \ldots \quad (2.7)
\]

\[
b_n = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \sin(n\Omega_0 t)dt, \quad n = 1, 2, \ldots \quad (2.8)
\]

From our discussion of even and odd symmetric signals, it is clear that if \( x(t) \) is even, then \( x(t) \sin(n\Omega_0 t) \) must be odd and so \( b_n = 0 \). Also if, \( x(t) \) has odd symmetry, then \( x(t) \cos(n\Omega_0 t) \) also has odd symmetry and hence \( a_n = 0 \) (see exercise 1). Moreover, if a signal is even, since \( x(t) \cos(n\Omega_0 t) \) is also even, if we use the fact that for any even symmetric periodic signal \( v(t) \),

\[
\int_{-T/2}^{T/2} v(t)dt = 2 \int_{0}^{T/2} v(t)dt \quad (2.9)
\]
then setting \( t_0 = -T/2 \) in (2.7) gives,

\[
a_n = \frac{4}{T} \int_{0}^{T/2} x(t) \cos(n\Omega_0 t) dt, \quad n = 1, 2, \ldots
\]  

(2.10)

This can sometimes lead to a savings in the number of integrals that must be computed. Similarly, if \( x(t) \) has odd symmetry, we have

\[
b_n = \frac{4}{T} \int_{0}^{T/2} x(t) \sin(n\Omega_0 t) dt, \quad n = 1, 2, \ldots
\]  

(2.11)

**Example 2.2.1** Consider the signal in Figure 2.2. This signal has even symmetry, hence all of the \( b_n = 0 \). We compute \( a_0 \) using,

\[
a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) dt
\]  

(2.12)

which we recognize as the area of one period, divided by the period. Hence, \( a_0 = \tau/T \). Next, using (2.7) we get

\[
a_n = \frac{2}{T} \int_{-\tau/2}^{\tau/2} \cos(n\Omega_0 t) dt
\]  

(2.13)

Note how the limits of integration only go from \(-\tau/2\) to \(\tau/2\) since \( x(t) \) is zero everywhere else. Evaluating this integral leads to

\[
a_n = \frac{2\tau \sin(n\Omega_0 \tau/2)}{n\Omega_0 \tau/2}, \quad n = 1, 2, \ldots
\]  

(2.14)

Figure 2.3 shows the first few Fourier Series coefficients for \( \tau = 1/2 \) and \( T = 1 \). If we attempt to reconstruct \( x(t) \) based on only a limited number (say, \( N \)) of Fourier Series coefficients, we have

\[
\hat{x}(t) = a_0 + \sum_{n=0}^{N} a_n \cos(n\Omega_0 t)
\]  

(2.15)

Figures 2.4 and 2.5 show \( \hat{x}(t) \) for \( N = 10 \), and \( N = 50 \), respectively. The ringing characteristic is known as Gibb’s phenomenon and disappears only as \( N \) approaches \( \infty \).
2.3. **HALF-WAVE SYMMETRY**

The following example looks at the Fourier series of an odd-symmetric signal, a *sawtooth* signal.

**Example 2.2.2** Now let’s compute the Fourier series for the signal in Figure 2.6. The signal is odd-symmetric, so all of the $a_n$ are zero. The period is $T = 3/2$, hence $\Omega_0 = 4\pi/3$. Using (2.8), the $b_n$ coefficients are found by computing the following integral,

$$b_n = \frac{8}{3} \int_{-1/2}^{1/2} t \sin\left(\frac{4\pi nt}{3}\right) dt$$  \hspace{1cm} (2.16)

After integrating by parts, we get

$$b_n = 3 \frac{\sin\left(\frac{2\pi n}{3}\right)}{\left(\frac{\pi n}{3}\right)^2} - 2 \frac{\cos\left(\frac{2\pi n}{3}\right)}{\pi n}, \hspace{0.5cm} n = 1, 2, \ldots$$ \hspace{1cm} (2.17)

These are plotted in Figure 2.7 and approximations of $x(t)$ using $N = 10$ and $N = 50$ coefficients are shown in Figures 2.8 and 2.9, respectively.

### 2.3 Half-Wave Symmetry

Periodic signals having half-wave symmetry have the property

$$x(t) = -x(t - T/2)$$ \hspace{1cm} (2.18)

$$x(t) = -x(t + T/2)$$ \hspace{1cm} (2.19)

It turns out that signals with this type of symmetry only have odd-numbered harmonics, the even harmonics are zero. To see this, let’s look at the formula for the coefficients $a_n$:  

![Figure 2.2: Example 2.2.1. This signal is sometimes called a pulse train.](image-url)
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Figure 2.3: Fourier Series coefficients for Example 2.2.1.

Figure 2.4: Approximation to $x(t)$ based on the first 10 Fourier Series coefficients for Example 2.2.1.
2.3. HALF-WAVE SYMMETRY

Figure 2.5: Fourier Series coefficients for Example 2.2.1.

Figure 2.6: Example 2.2.2. Sawtooth signal.
CHAPTER 2. FOURIER SERIES OF PERIODIC SIGNALS

Figure 2.7: Fourier Series coefficients for Example 2.2.2.

Figure 2.8: Approximation to $x(t)$ based on the first 10 Fourier Series coefficients for Example 2.2.2.
2.3. HALF-WAVE SYMMETRY

Figure 2.9: Fourier Series coefficients for Example 2.2.2.

\[ a_n = \frac{2}{T} \int_{t_0-T/2}^{t_0+T/2} x(t) \cos(n\Omega_0 t) dt \quad (2.20) \]

\[ = \frac{2}{T} \left[ \int_{t_0-T/2}^{t_0} x(t) \cos(n\Omega_0 t) dt + \int_{t_0}^{t_0+T/2} x(t) \cos(n\Omega_0 t) dt \right] \quad (2.21) \]

\[ = \frac{2}{T} \left[ I_1 + I_2 \right] \quad (2.22) \]

Making the substitution \( \tau = t + T/2 \) in \( I_1 \) gives

\[ I_1 = \int_{t_0}^{t_0+T/2} x(\tau - T/2) \cos(n\Omega_0 (\tau - T/2)) d\tau \quad (2.23) \]

\[ = -\int_{t_0}^{t_0+T/2} x(\tau) \left[ \cos(n\Omega_0 \tau) \cos(n\Omega_0 T/2) + \sin(n\Omega_0 \tau) \sin(n\Omega_0 T/2) \right] d\tau \quad (2.24) \]

where in the second equality we have used the trigonometric identity

\[ \cos(u \pm v) = \cos(u) \cos(v) \mp \sin(u) \sin(v) \]
and (2.19). Now recognizing that \( \sin(n \Omega_0 T/2) = \sin(n \pi) = 0 \) gives

\[
I_1 = -\cos(n \pi) \int_{t_0}^{t_0+T/2} x(\tau) \cos(n \Omega_0 \tau) d\tau 
\]  

(2.25)

Since the integral in \( I_2 \) is the same as the integral in the expression for \( I_1 \), we can write:

\[
a_n = \frac{2}{T} \int_{t_0}^{t_0+T/2} x(t) \cos(n \Omega_0 t) dt 
\]  

(2.26)

From this expression we find that \( a_n = 0 \) whenever \( n \) is even. In fact, we have

\[
a_n = \begin{cases} 
\frac{4}{T} \int_{t_0}^{t_0+T/2} x(t) \cos(n \Omega_0 t) dt, & n, \text{ odd} \\
0, & n, \text{ even} 
\end{cases} 
\]  

(2.27)

A similar derivation leads to

\[
b_n = \begin{cases} 
\frac{4}{T} \int_{t_0}^{t_0+T/2} x(t) \sin(n \Omega_0 t) dt, & n, \text{ odd} \\
0, & n, \text{ even} 
\end{cases} 
\]  

(2.28)

A good choice of \( t_0 \) can lead to a considerable savings in time when calculating the Fourier Series of half-wave symmetric signals. Note that half-wave symmetric signals need not have odd or even symmetry for the above formulae to apply. If a signal has half-wave symmetry and in addition has odd or even symmetry, then some additional simplification is possible. Consider the case when a half-wave symmetric signal also has even symmetry. Then clearly \( b_n = 0 \), and (2.27) applies. However since the integrand in (2.27) is the product of two even signals, \( x(t) \) and \( \cos(n \Omega_0 t) \), it too has even symmetry. Therefore, instead of integrating from, say, \( -T/4 \) to \( T/4 \), we need only integrate from 0 to \( T/4 \) and multiply the result by 2. Therefore the formula for \( a_n \) for an even, half-wave symmetric signal becomes:

\[
a_n = \begin{cases} 
\frac{8}{T} \int_{0}^{T/4} x(t) \cos(n \Omega_0 t) dt, & n, \text{ odd} \\
0, & n, \text{ even} 
\end{cases} 
\]  

(2.29)

\[
b_n = 0 
\]  

(2.30)
For an odd half-wave symmetric signals, a similar argument leads to

\[ a_n = 0 \] (2.31)

\[ b_n = \begin{cases} 
\frac{8}{T} \int_0^{T/4} x(t) \sin(n\Omega_0 t) dt, & n, \text{ odd} \\
0, & n, \text{ even} 
\end{cases} \] (2.32)

### 2.4 Convergence of the Fourier Series

Consider the trigonometric form of the Fourier series

\[ x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\Omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\Omega_0 t) \] (2.33)

It is important to state under what conditions this series (the right-hand side of (2.33)) will actually converge to \( x(t) \). The nature of the convergence also needs to be specified. There are several ways of defining the convergence of a series.

1. **Uniform convergence**: define the finite sum:

\[ x_N(t) = a_0 + \sum_{n=1}^{N} a_n \cos(n\Omega_0 t) + \sum_{n=1}^{N} b_n \sin(n\Omega_0 t) \] (2.34)

where \( N \) is finite. Then the series converges uniformly if the absolute value of \( x(t) - x_N(t) \) satisfies

\[ |x(t) - x_N(t)| < \epsilon \] (2.35)

for all values of \( t \) and some small positive constant \( \epsilon \).

2. **Point-wise convergence**: as with uniform convergence, we require that

\[ |x(t) - x_{N(t)}(t)| < \epsilon \]

for all \( t \). The main difference between uniform and point-wise convergence is that for the former, the number of terms in the summation \( N(t) \) needed to get the error below \( \epsilon \) may vary for different values of \( t \).
3. **Mean-squared convergence:** here, the series converges if for all $t$:

$$\lim_{N \to \infty} \int_{t_0}^{t_0+T} \left| x(t) - x_{N(t)}(t) \right|^2 dt = 0$$

Gibb’s phenomenon, mentioned in some of the examples above, is an example of mean-squared convergence of the series. The overshoot in Gibb’s phenomenon occurs only at abrupt discontinuities. Moreover, the height of the overshoot stays the same independently of the number of terms in the series, $N$. The overshoot merely becomes less noticeable because it becomes more and more narrow as $N$ increases.

Dirichlet has given a series of conditions which are necessary for a periodic signal to have a Fourier series. If these conditions are met, then

- the Fourier series has point-wise convergence for all $t$ at which $x(t)$ is continuous.
- where $x(t)$ has a discontinuity, then the series converges to the midpoint between the two values on either side of the discontinuity.

The **Dirichlet Conditions** are:

1. $x(t)$ must be absolutely integrable over one period:

$$\int_{t_0}^{t_0+T} |x(t)| dt < \infty$$

2. $x(t)$ may have only a finite number of discontinuities over any one period.

3. $x(t)$ may have only a finite number of extrema.

Fortunately, most periodic signals of practical interest satisfy these conditions.

### 2.5 Complex Form of the Fourier Series

The trigonometric form of the Fourier Series, shown in 2.4 can be converted into a more convenient form by doing the following substitutions:
2.5. **COMPLEX FORM OF THE FOURIER SERIES**

\[
\cos(n\Omega_0 t) = \frac{e^{jn\Omega_0 t} + e^{-jn\Omega_0 t}}{2} 
\]

(2.36)

\[
\sin(n\Omega_0 t) = \frac{e^{jn\Omega_0 t} - e^{-jn\Omega_0 t}}{j2}
\]

(2.37)

After some straight-forward rearranging, we obtain

\[
x(t) = a_0 + \sum_{n=1}^{\infty} \left[ \frac{a_n - jb_n}{2} \right] e^{jn\Omega_0 t} + \sum_{n=1}^{\infty} \left[ \frac{a_n + jb_n}{2} \right] e^{-jn\Omega_0 t}
\]

(2.38)

Keeping in mind that \(a_n\) and \(b_n\) are only defined for positive values of \(n\), let’s sum over the negative integers in the second summation:

\[
x(t) = a_0 + \sum_{n=1}^{\infty} \left[ \frac{a_n - jb_n}{2} \right] e^{jn\Omega_0 t} + \sum_{n=-1}^{\infty} \left[ \frac{a_n - jb_n}{2} \right] e^{-jn\Omega_0 t}
\]

(2.39)

Next, let’s assume that \(a_n\) and \(b_n\) are defined for both positive and negative \(n\). In this case, we find that \(a_n = a_{-n}\) and \(b_n = -b_{-n}\), since

\[
a_{-n} = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \cos(-n\Omega_0 t) dt = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \cos(n\Omega_0 t) dt = a_n
\]

(2.40)

and

\[
b_{-n} = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \sin(-n\Omega_0 t) dt = \frac{-2}{T} \int_{t_0}^{t_0+T} x(t) \sin(n\Omega_0 t) dt = -b_n
\]

(2.41)

Using this fact, we can rewrite (2.39) as

\[
x(t) = a_0 + \sum_{n=1}^{\infty} \left[ \frac{a_n - jb_n}{2} \right] e^{jn\Omega_0 t} + \sum_{n=-1}^{\infty} \left[ \frac{a_n - jb_n}{2} \right] e^{-jn\Omega_0 t}
\]

(2.42)
If we define\(^1\) \(c_0 \equiv a_0\), and
\[
c_n \equiv \frac{a_n - jb_n}{2}
\]
then we can rewrite (2.42) as
\[
x(t) = \sum_{n=-\infty}^{\infty} c_ne^{jn\Omega_0t}
\]
which is called the complex form of the Fourier Series. Note that since \(a_{-n} = a_n\) and \(b_{-n} = -b_n\), we have
\[
c_{-n} = c_n^* \quad (2.45)
\]
This means that
\[
|c_{-n}| = |c_n| \quad (2.46)
\]
and
\[
\angle c_{-n} = -\angle c_n \quad (2.47)
\]
Next, we must find formulas for finding the \(c_n\) given \(x(t)\). We first look at a property of complex exponentials:
\[
\int_{t_0}^{t_0+T} e^{jk\Omega_0t} \, dt = \begin{cases} T, & k = 0 \\ 0, & \text{otherwise} \end{cases} \quad (2.48)
\]
To see this, we note that
\[
\int_{t_0}^{t_0+T} e^{jk\Omega_0t} \, dt = \int_{t_0}^{t_0+T} \cos(k\Omega_0t) + j \int_{t_0}^{t_0+T} \sin(k\Omega_0t) \, dt \quad (2.49)
\]
It’s easy to see that \(k\Omega_0\) also has period \(T\), hence the integral is over \(k\) periods of \(\cos(k\Omega_0t)\) and \(\sin(k\Omega_0t)\). Therefore, if \(k \neq 0\), then
\[
\int_{t_0}^{t_0+T} e^{jk\Omega_0t} \, dt = 0 \quad (2.50)
\]
otherwise
\[
\int_{t_0}^{t_0+T} e^{jk\Omega_0t} \, dt = \int_{t_0}^{t_0+T} dt = T \quad (2.51)
\]
\(^1\)The notation “\(\equiv\)” is often used instead of the “\(=\)” sign when defining a new variable.
We use (2.48) to derive an equation for \(c_n\) as follows. Consider the integral
\[
\int_{t_0}^{t_0+T} x(t)e^{-jn\Omega_0 t}dt
\] (2.52)

Substituting the complex form of the Fourier Series of \(x(t)\) in (2.52), (using \(k\) as the index of summation) we obtain
\[
\int_{t_0}^{t_0+T} \left[ \sum_{k=-\infty}^{\infty} c_k e^{j k \Omega_0 t} \right] e^{-jn\Omega_0 t}dt
\] (2.53)

Rearranging the order of integration and summation, combining the exponents, and using (2.48) gives
\[
\sum_{k=-\infty}^{\infty} c_k \int_{t_0}^{t_0+T} e^{j (k-n) \Omega_0 t}dt = Tc_n
\] (2.54)

Using this result, we find that
\[
c_n = \frac{1}{T} \int_{t_0}^{t_0+T} x(t)e^{-jn\Omega_0 t}dt
\] (2.55)

**Example 2.5.1** Let’s now find the complex form of the Fourier Series for the signal in Example 2.2.2 (see Figure 2.6). The integral to be evaluated is
\[
c_n = \frac{2}{3} \int_{-0.5}^{0.5} 2te^{-j\frac{2\pi n}{3}t}dt
\] (2.56)

Integrating by parts yields
\[
c_n = \frac{j}{n\pi} \cos(2\pi n/3) - \frac{j^3}{(n\pi)^2} \sin(2\pi n/3)
\] (2.57)

Figure 2.10 shows the magnitude of the coefficients, \(|c_n|\). Note that the complex Fourier Series coefficients have even symmetry as was mentioned earlier.
Can the basic formula for computing the $c_n$ in (2.55) be simplified when $x(t)$ has either even, odd, or half-wave symmetry? The answer is yes. We simply use the fact that

$$c_n = \frac{a_n - jb_n}{2}$$

and solve for $a_n$ and $b_n$ using the formulae given above for even, odd, or half-wave symmetric signals. This avoids having to integrate complex quantities. This can also be seen by noting that (setting $t_0 = T/2$ in (2.55)):

$$
\begin{align*}
    c_n &= \frac{1}{T} \int_{-T/2}^{T/2} x(t)e^{-jn\Omega_0 t}dt \\
    &= \frac{1}{T} \int_{-T/2}^{T/2} x(t)\cos(n\Omega_0 t)dt - \frac{j}{T} \int_{-T/2}^{T/2} x(t)\sin(n\Omega_0 t)dt \\
    &= \frac{1}{2}a_n - \frac{j}{2}b_n
\end{align*}
$$
Alternately, if \( x(t) \) has half-wave symmetry, we can use (2.59), (2.27), and (2.28) to get

\[
c_n = \begin{cases} 
\frac{2}{T} \int_{-T/4}^{T/4} x(t) e^{-jn\Omega_0 t} dt, & n \text{ odd} \\
0, & n \text{ even}
\end{cases}
\]

(2.60)

Unlike the trigonometric form, we cannot simplify this further if \( x(t) \) is even or odd symmetric since \( e^{-jn\Omega_0 t} \) has neither even nor odd symmetry.

**Example 2.5.2** In this example we will look at the effect of adjusting the period of a pulse train signal. Consider the signal depicted in Figure 2.11. The Fourier Series coefficients for this signal are given by

\[
c_n = \frac{1}{T} \int_{-\tau/2}^{\tau/2} e^{-jn\Omega_0 t} dt
\]

(2.61)

\[
= \frac{-1}{jn\Omega_0 T} \left( e^{-jn\Omega_0 \tau/2} - e^{jn\Omega_0 \tau/2} \right)
\]

\[
= \frac{\tau \sin(n\Omega_0 \tau/2)}{T \ n\Omega_0 \tau/2}
\]

\[
\equiv \frac{\tau}{T} \text{sinc}(n\Omega_0 \tau/2)
\]

Figure 2.12 shows the magnitude of \( |c_n| \), the amplitude spectrum, for \( T = 1 \) and \( \tau = 1/2 \) as well as the Fourier Series for the signal based on the first 30 coefficients

\[
\hat{x}(t) = \sum_{n=-30}^{30} c_n e^{n\Omega_0 t}
\]

(2.62)
Figure 2.12: Example 2.5.2, $T = 1, \tau = 1/2$: (top) Fourier Series coefficient magnitudes, (b) $\hat{x}(t)$. 
Figure 2.13: Example 2.5.2, $T = 4, \tau = 1/2$: (top) Fourier Series coefficient magnitudes, (b) $\hat{x}(t)$. 
Figure 2.14: Example 2.5.2, $T = 8, \tau = 1/2$: (top) Fourier Series coefficient magnitudes, (b) $\hat{x}(t)$. 
Similar plots are shown in Figures 2.13, and 2.14, for $T = 4$, and $T = 8$, respectively. This example illustrates several important points about the Fourier Series: As the period $T$ increases, $\Omega_0$ decreases in magnitude (this is obvious since $\Omega_0 = \frac{2\pi}{T}$). Therefore, as the period increases, successive Fourier Series coefficients represent more closely spaced frequencies. The frequencies corresponding to each $n$ are given by the following table:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\Omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\pm 1$</td>
<td>$\pm \Omega_0$</td>
</tr>
<tr>
<td>$\pm 2$</td>
<td>$\pm 2\Omega_0$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$\pm n$</td>
<td>$\pm n\Omega_0$</td>
</tr>
</tbody>
</table>

This table establishes a relation between $n$ and the frequency variable $\Omega$. In particular, if $T = 1$, we have $\Omega_0 = 2\pi$ and

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\Omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\pm 1$</td>
<td>$\pm 2\pi$</td>
</tr>
<tr>
<td>$\pm 2$</td>
<td>$\pm 4\pi$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$\pm n$</td>
<td>$\pm 2n\pi$</td>
</tr>
</tbody>
</table>

If $T = T$, then $\Omega_0 = \pi/2$ and
and if \( T = 8 \), we have \( \Omega_0 = \pi/4 \) and

\[
\begin{array}{c|c}
 n & \Omega \\
 0 & 0 \\
 \pm 1 & \pm \pi/2 \\
 \pm 2 & \pm \pi \\
 \vdots & \vdots \\
 \pm n & \pm n\pi/2 \\
\end{array}
\]

Note that in all three cases, the first zero coefficient corresponds to the value of \( n \) for which \( \Omega = 4\pi \). Also, as \( T \) gets bigger, the \( c_n \) appear to resemble more closely spaced samples of a continuous function of frequency (since the \( n\Omega \) are more closely spaced). Can you determine what this function is?

As we shall see, by letting the period \( T \) get large (infinitely large), we will derive the Fourier Transform in the next chapter.

### 2.6 Parseval’s Theorem

Recall that in Chapter 1, we defined the power of a periodic signal as (see (1.10))

\[
p_x = \frac{1}{T} \int_{t_0}^{t_0+T} x^2(t) \, dt
\]  

\[ (2.63) \]
where \( T \) is the period. Using the complex form of the Fourier series, we can write

\[
x(t)^2 = \left( \sum_{n=-\infty}^{\infty} c_n e^{j n \Omega_0 t} \right) \left( \sum_{m=-\infty}^{\infty} c_m^* e^{-j m \Omega_0 t} \right)^* \tag{2.64}
\]

where we have used the fact that \( x(t)^2 = x(t)x(t)^* \), i.e., since \( x(t) \) is real \( x(t) = x(t)^* \). Applying (1.7) and (1.8) gives

\[
x(t)^2 = \left( \sum_{n=-\infty}^{\infty} c_n e^{j n \Omega_0 t} \right) \left( \sum_{m=-\infty}^{\infty} c_m^* e^{-j m \Omega_0 t} \right)
\]

\[
= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_n c_m^* e^{j(n-m)\Omega_0 t}
\]

\[
= \sum_{n=-\infty}^{\infty} |c_n|^2 + \sum_{n \neq m} c_n c_m^* e^{j(n-m)\Omega_0 t}
\]

Substituting this quantity into (2.63) gives

\[
p_x = \frac{1}{T} \int_{t_0}^{t_0+T} \left[ \sum_{n=-\infty}^{\infty} |c_n|^2 + \sum_{n \neq m} c_n c_m^* e^{j(n-m)\Omega_0 t} \right] dt \tag{2.66}
\]

\[
= \sum_{n=-\infty}^{\infty} |c_n|^2 + \frac{1}{T} \int_{t_0}^{t_0+T} \sum_{n \neq m} c_n c_m^* e^{j(n-m)\Omega_0 t} dt
\]

It is straightforward to show that

\[
\frac{1}{T} \int_{t_0}^{t_0+T} \sum_{n \neq m} c_n c_m^* e^{j(n-m)\Omega_0 t} dt = 0 \tag{2.67}
\]

(see Exercise 9). This leads to Parseval’s Theorem for the Fourier series:

\[
p_x = \sum_{n=-\infty}^{\infty} |c_n|^2 \tag{2.68}
\]

which states that the power of a periodic signal is the sum of the magnitude of the complex Fourier series coefficients.
2.7 Exercises

1. Show that an even-symmetric periodic signal has Fourier Series coefficients $b_n = 0$ while an odd-symmetric signal has $a_n = 0$.

2. Find the trigonometric form of the Fourier Series of the periodic signal shown in Figure 2.15.

3. Find the trigonometric form of the Fourier Series for the periodic signal shown in Figure 2.16.

4. Find the trigonometric form of the Fourier Series for the periodic signal shown in Figure 2.17 for $\tau = 1$, $T = 10$.

5. Suppose that $x(t) = 5 + 3 \cos(5t) - 2 \sin(3t) + \cos(45t)$.
   (a) Find the period of this periodic signal.
   (b) Find the trigonometric form of the Fourier Series.

6. Find the complex form of the Fourier Series of the periodic signal shown in Figure 2.15.

7. Find the complex form of the Fourier Series of the periodic signal shown in Figure 2.16.

8. Find the complex form of the Fourier Series for the signal in Figure 2.17 using:
   (a) $\tau = 1$, $T = 10$.
   (b) $\tau = 1$, $T = 100$.
   For each case plot the magnitude of the Fourier Series coefficients. You may use Matlab or some other programming language to do this.

9. Show that
   $$\frac{1}{T} \int_{t_0}^{t_0+T} \sum_{n \neq m} c_n c^* m e^{j(n-m)\Omega_0 t} dt = 0$$
2.7. EXERCISES

Figure 2.15: Signal for problems 2 and 6.

Figure 2.16: Signal for problem 3 and 7.

Figure 2.17: Pulse train signal for problems 4 and 7.
Chapter 3

The Fourier Transform

3.1 Derivation of $x(t) \leftrightarrow X(j\Omega)$

Let’s begin by writing down the formula for the complex form of the Fourier Series:

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\Omega_0 t}$$

(3.1)

as well as the corresponding Fourier Series coefficients:

$$c_n = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) e^{-jn\Omega_0 t} dt$$

(3.2)

As was mentioned in the Example 2.5.2, as the period $T$ gets large, the Fourier Series coefficients represent more closely spaced frequencies. Let’s take the limit as the period $T$ goes to infinity. We first note that the fundamental frequency approaches a differential

$$\Omega_0 = \frac{2\pi}{T} \to d\Omega$$

(3.3)

consequently

$$\frac{1}{T} = \frac{\Omega_0}{2\pi} \to \frac{d\Omega}{2\pi}$$

(3.4)

The $n$th harmonic, $n\Omega_0$, in the limit approaches the frequency variable $\Omega$

$$n\Omega_0 \to \Omega$$

(3.5)
From equation (3.2), we have

\[ c_n T \rightarrow \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt \]  

(3.6)

The right hand side of (3.6) is called the Fourier Transform of \( x(t) \):

\[ X(j\Omega) \equiv \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt \]  

(3.7)

Now, using (3.6), (3.4), and (3.5) in equation (3.1) gives

\[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega \]  

(3.8)

which corresponds to the inverse Fourier Transform. Equations (3.7) and (3.8) represent what is known as a transform pair. The following notation is used to denote a Fourier Transform pair

\[ x(t) \leftrightarrow X(j\Omega) \]  

(3.9)

We say that \( x(t) \) is a time domain signal while \( X(j\Omega) \) is a frequency domain signal. Some additional notation which is sometimes used is

\[ X(j\Omega) = F\{x(t)\} \]  

(3.10)

and

\[ x(t) = F^{-1}\{X(j\Omega)\} \]  

(3.11)

### 3.2 Properties of the Fourier Transform

The Fourier Transform (FT) has several important properties which will be useful:

1. **Linearity:**

\[ \alpha x_1(t) + \beta x_2(t) \leftrightarrow \alpha X_1(j\Omega) + \beta X_2(j\Omega) \]  

(3.12)

where \( \alpha \) and \( \beta \) are constants. This property is easy to verify by plugging the left side of (3.12) into the definition of the FT.
2. **Time shift:**

\[ x(t - \tau) \leftrightarrow e^{-j\Omega \tau} X(j\Omega) \quad (3.13) \]

To derive this property we simply take the FT of \( x(t - \tau) \)

\[
\int_{-\infty}^{\infty} x(t - \tau) e^{-j\Omega t} dt \quad (3.14)
\]

using the variable substitution \( \gamma = t - \tau \) leads to

\[ t = \gamma + \tau \quad (3.15) \]

and

\[ d\gamma = dt \quad (3.16) \]

We also note that if \( t = \pm \infty \) then \( \tau = \pm \infty \). Substituting (3.15), (3.16), and the limits of integration into (3.14) gives

\[
\int_{-\infty}^{\infty} x(\gamma) e^{-j\Omega \gamma} d\gamma = e^{-j\Omega \tau} \int_{-\infty}^{\infty} x(\gamma) e^{-j\Omega \gamma} d\gamma = e^{-j\Omega \tau} X(j\Omega) \]

which is the desired result.

3. **Frequency shift:**

\[ x(t)e^{j\Omega_0 t} \leftrightarrow X(j(\Omega - \Omega_0)) \quad (3.18) \]

Deriving the frequency shift property is a bit easier than the time shift property. Again, using the definition of FT we get:

\[
\int_{-\infty}^{\infty} x(t)e^{j\Omega_0 t} e^{-j\Omega t} dt = \int_{-\infty}^{\infty} x(t)e^{-j(\Omega - \Omega_0)t} dt = X(j(\Omega - \Omega_0)) \]

4. **Time reversal:**

\[ x(-t) \leftrightarrow X(j\Omega)^* \quad (3.20) \]

To derive this property, we again begin with the definition of FT:

\[
\int_{-\infty}^{\infty} x(-t) e^{-j\Omega t} dt \quad (3.21)
\]
and make the substitution \( \gamma = -t \). We observe that \( dt = -d\gamma \) and that if the limits of integration for \( t \) are \( \pm \infty \), then the limits of integration for \( \gamma \) are \( \mp \gamma \). Making these substitutions into (3.21) gives

\[
- \int_{-\infty}^{\infty} x(\gamma)e^{j\Omega \gamma} d\gamma = \int_{-\infty}^{\infty} x(\gamma)e^{j\Omega \gamma} d\gamma
\]

\[
= \left[ \int_{-\infty}^{\infty} x(\gamma)e^{-j\Omega \gamma} d\gamma \right]^*
\]

\[
= X(j\Omega)^*
\]

where we have assumed that \( x(t) \) is a real signal.

5. **Convolution:** The convolution integral is given by

\[
y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau) d\tau
\]

The convolution property is given by

\[
Y(j\Omega) \leftrightarrow X(j\Omega)H(j\Omega)
\]

To derive this important property, we again use the FT definition:

\[
Y(j\Omega) = \int_{-\infty}^{\infty} y(t)e^{-j\Omega t} dt
\]

\[
= \int_{-\infty}^{\infty} x(\tau)h(t - \tau)e^{-j\Omega t} d\tau dt
\]

\[
= \int_{-\infty}^{\infty} x(\tau) \left[ \int_{-\infty}^{\infty} h(t - \tau)e^{-j\Omega t} dt \right] d\tau
\]

Using the time shift property, the quantity in the brackets is \( e^{-j\Omega \tau} H(j\Omega) \), giving

\[
Y(j\Omega) = \int_{-\infty}^{\infty} x(\tau)e^{-j\Omega \tau} H(j\Omega) d\tau
\]

\[
= H(j\Omega) \int_{-\infty}^{\infty} x(\tau)e^{-j\Omega \tau} d\tau
\]

\[
= H(j\Omega)X(j\Omega)
\]

Therefore, convolution in the time domain corresponds to multiplication in the frequency domain.
6. Multiplication (Modulation):

\[ w(t) = x(t)y(t) \leftrightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j(\Omega - \Theta))Y(j\Theta)d\Theta \quad (3.27) \]

Notice that multiplication in the time domain corresponds to convolution in the frequency domain. This property can be understood by applying the inverse Fourier Transform (3.8) to the right side of (3.27)

\[ w(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j(\Omega - \Theta))Y(j\Theta)e^{j\Omega t}d\Theta d\Omega \quad (3.28) \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(j\Theta) \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j(\Omega - \Theta))e^{j\Omega t}d\Omega \right] d\Theta \quad (3.29) \]

The quantity inside the brackets is the inverse Fourier Transform of a frequency shifted Fourier Transform,

\[ w(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(j\Theta) \left[ x(t)e^{j\Theta t} \right] d\Theta \quad (3.30) \]

\[ = x(t) \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(j\Theta)e^{j\Theta t}d\Theta \]

\[ = x(t)y(t) \]

The properties associated with the Fourier Transform are summarized in Table 3.1.

### 3.3 The Unit Impulse Function

The unit impulse is an often maligned signal which is actually very useful in the analysis of signals, linear systems, and sampling (the latter two subjects will be taken up in the near future). Consider the plot of a rectangular pulse in Figure 3.1. Note the height of the pulse is \(1/\tau\) and the width of the pulse is \(\tau\). So we can write

\[ \int_{-\infty}^{\infty} x_p(t)dt = 1 \quad (3.31) \]
CHAPTER 3. THE FOURIER TRANSFORM

<table>
<thead>
<tr>
<th>Property</th>
<th>$y(t)$</th>
<th>$Y(j\Omega)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linearity</td>
<td>$\alpha x_1(t) + \beta x_2(t)$</td>
<td>$\alpha X_1(j\Omega) + \beta X_2(j\Omega)$</td>
</tr>
<tr>
<td>Time Shift</td>
<td>$x(t - \tau)$</td>
<td>$X(j\Omega)e^{-j\Omega\tau}$</td>
</tr>
<tr>
<td>Frequency Shift</td>
<td>$x(t)e^{j\Omega_0 t}$</td>
<td>$X(j(\Omega - \Omega_0))$</td>
</tr>
<tr>
<td>Time Reversal</td>
<td>$x(-t)$</td>
<td>$X(j\Omega)^*$</td>
</tr>
<tr>
<td>Convolution</td>
<td>$x(t) \ast h(t)$</td>
<td>$X(j\Omega)H(j\Omega)$</td>
</tr>
<tr>
<td>Modulation</td>
<td>$x(t)w(t)$</td>
<td>$\frac{1}{2\pi} \int_{-\infty}^{\infty} X(j(\Omega - \Theta))W(j\Theta)d\Theta$</td>
</tr>
</tbody>
</table>

Table 3.1: Fourier Transform properties.

Now as we let $\tau$ get smaller and smaller, then clearly the width of the pulse gets successively narrower and its height gets progressively higher. In the limit (as $\tau$ approaches zero), we have a pulse which has infinite height, and zero width, yet its area is still one. We define the unit impulse function in just this manner

$$\delta(t) \equiv \lim_{\tau \to 0} x_p(t)$$  \hspace{1cm} (3.32)

We observe that the area under $\delta(t)$ is one, in general, we can write

$$\int_{-\infty}^{\infty} \delta(t - \tau) dt = 1$$  \hspace{1cm} (3.33)

We can multiply the unit impulse by a constant, $K$, and that changes its area to that constant, i.e.

$$\int_{-\infty}^{\infty} K\delta(t - \tau) dt = K$$  \hspace{1cm} (3.34)

The area of the unit impulse is usually indicated by the number shown next to the arrow as seen in Figure 3.2. Suppose we multiply the signal $x(t)$ with a time-shifted unit impulse, $\delta(t - \tau)$. The product is a single unit impulse, having an area of $x(\tau)$. This is illustrated in Figure 3.3. In other words,

$$\int_{-\infty}^{\infty} x(t)\delta(t - \tau) dt = x(\tau)$$  \hspace{1cm} (3.35)

Equation (3.35) is called the sifting property of the unit impulse. As we will see, the sifting property of the unit impulse will be very useful.
3.4 The Unit Step Function, $u(t)$

The unit step function is defined as

$$u(t) = \begin{cases} 
1, & t \geq 0 \\
0, & t < 0 
\end{cases}$$  \hspace{1cm} (3.36)

This function is useful for defining signals which we wish to “begin” at $t = 0$. In other words, often, we would like for signals to be zero for negative values of $t$. We can force this situation by simply multiplying by $u(t)$. 

Figure 3.1: Rectangular pulse, $x_p(t)$ approaches the unit impulse function, $\delta(t)$, as $\tau$ approaches zero.

Figure 3.2: $K\delta(t - \tau)$. 

Example diagram: 

- Two axes with $x$-axis ranging from $-\tau/2$ to $\tau/2$ and $t$-axis ranging from $0$ to $0$. 
- A pulse $x_p(t)$ is shown with amplitude $1/\tau$ and duration $\tau$. 
- A signal $\delta(t)$ is shown with an impulse at $t = 0$. 
- The pulse is shown approaching the impulse as $\tau$ approaches zero. 

Diagram notes: 

- The pulse diagram includes labels for $x_p(t)$ and $\delta(t)$, with an arrow indicating the direction of $\tau \to 0$. 
- The diagram is labeled to illustrate the movement of the pulse towards the impulse function as $\tau$ decreases.
Figure 3.3: Sifting property of unit impulse, the product of the two signals, $x(t)$ and $\delta(t - \tau)$, is $x(\tau)\delta(t - \tau)$. Consequently, the area under $x(\tau)\delta(t - \tau)$ is $x(\tau)$.

### 3.5 Fourier Transform of Common Signals

Next, we’ll derive the FT of some basic continuous-time signals. Table 3.2 summarizes these transform pairs.

#### 3.5.1 Rectangular pulse

Let’s begin with the rectangular pulse

$$\text{rect}(t, \tau) \equiv \begin{cases} 1, & t \leq \tau/2 \\ 0, & t > \tau/2 \end{cases} \quad (3.37)$$
3.5. FOURIER TRANSFORM OF COMMON SIGNALS

The pulse function, \( \text{rect}(t, \tau) \) is shown in Figure 3.4. Substituting \( x(t) = \text{rect}(t, \tau) \) into (3.7) gives

\[
X(j\Omega) = \int_{-\tau/2}^{\tau/2} e^{-j\Omega t} dt = \frac{-1}{j\Omega} e^{-j\Omega \tau/2} \bigg|_{-\tau/2}^{\tau/2} = \frac{1}{j\Omega} \left[ e^{j\Omega \tau/2} - e^{-j\Omega \tau/2} \right] = \frac{\tau \sin(\Omega \tau/2)}{\Omega \tau/2} = \tau \text{sinc}(\Omega \tau/2\pi)
\]

A plot of \( X(j\Omega) \) is shown in Figure 3.4. Note that when \( \tau = 0 \), \( X(j\Omega) = 1 \).

![Figure 3.4: Fourier transform pair showing the rectangular pulse signal (left) and its Fourier Transform, the sinc function (right).](image-url)

We now have the following transform pair:

\[
\text{rect}(t, \tau) \leftrightarrow \tau \frac{\sin(\Omega \tau/2)}{\Omega \tau/2}
\]
3.5.2 Impulse

The unit impulse function was described in Section 3.3. From the sifting property of the impulse function we find that

\[ X(j\Omega) = \int_{-\infty}^{\infty} \delta(t - \tau)e^{-j\Omega t} dt \]  
\[ = e^{-\Omega \tau} \quad (3.39) \]

or

\[ \delta(t - \tau) \leftrightarrow e^{-\Omega \tau} \quad (3.40) \]

3.5.3 Complex Exponential

The complex exponential function, \( x(t) = e^{j\Omega_0 t} \), has a Fourier Transform which is difficult to evaluate directly. It is easier to start with the Fourier Transform itself and work backwards using the inverse Fourier Transform. Suppose we want to find the time-domain signal which has Fourier Transform \( X(j\Omega) = \delta(\Omega - \Omega_0) \). We can begin by using the definition of the inverse Fourier Transform in (3.8)

\[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\Omega - \Omega_0)e^{j\Omega t} d\Omega \]  
\[ = \frac{1}{2\pi} e^{\Omega t} \quad (3.41) \]

This result follows from the sifting property of the impulse function. By linearity, we can then write

\[ e^{j\Omega t} \leftrightarrow 2\pi \delta(\Omega - \Omega_0) \quad (3.42) \]

3.5.4 Cosine

The cosine signal can be expressed in terms of complex exponentials using Euler’s Identity

\[ \cos(\Omega_0 t) = \frac{1}{2} \left( e^{j\Omega_0 t} + e^{-j\Omega_0 t} \right) \]  
\[ \quad (3.43) \]

Applying linearity and the Fourier Transform of complex exponentials to the right side of (3.43), we quickly get:

\[ \cos(\Omega_0 t) \leftrightarrow \pi \delta(\Omega - \Omega_0) + \pi \delta(\Omega + \Omega_0) \quad (3.44) \]
3.5.5 Real Exponential

The real exponential function is given by \( x(t) = Ke^{-\alpha t}u(t) \), where \( K \) and \( \alpha \) are real constants. To find its FT, we start with the definition

\[
X(j\Omega) = K \int_{0}^{\infty} e^{-\alpha t}e^{-j\Omega t} dt
\]

\[
= K \int_{0}^{\infty} e^{-(\alpha + j\Omega)t} dt
\]

\[
= \frac{-K}{\alpha + j\Omega} e^{-(\alpha + j\Omega)t} \bigg|_{0}^{\infty}
\]

\[
= \frac{-K}{\alpha + j\Omega} (0 - 1)
\]

\[
= \frac{K}{\alpha + j\Omega}
\]

therefore,

\[
Ke^{-\alpha t}u(t) \leftrightarrow \frac{K}{\alpha + j\Omega}
\]  

(3.46)

When working problems involving finding the Fourier Transform, it is often preferable to use a table of transform pairs rather than to recalculate the Fourier Transform from scratch. Often, transform pairs in can be combined with known Fourier Transform properties to find new Fourier Transforms.

<table>
<thead>
<tr>
<th>( x(t) )</th>
<th>( X(j\Omega) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>rect((t, \tau))</td>
<td>( \tau \frac{\sin(\Omega\tau/2)}{\Omega^2\tau/2} )</td>
</tr>
<tr>
<td>( \delta(t - \tau) )</td>
<td>( e^{-j\Omega\tau} )</td>
</tr>
<tr>
<td>( e^{j\Omega_0 t} )</td>
<td>( 2\pi \delta(\Omega - \Omega_0) )</td>
</tr>
<tr>
<td>( \cos(\Omega_0 t) )</td>
<td>( \pi \delta(\Omega - \Omega_0) + \pi \delta(\Omega + \Omega_0) )</td>
</tr>
<tr>
<td>( Ke^{-\alpha t}u(t) )</td>
<td>( \frac{K}{\alpha + j\Omega} )</td>
</tr>
</tbody>
</table>

Table 3.2: Some common Fourier Transform pairs.
Example 3.5.1 Find the Fourier Transform of: $y(t) = 2e^{5t}u(-t)$. Clearly, we can write $y(t) = x(-t)$ where $x(t) = 2e^{-5t}u(t)$. Therefore, we can combine the known transform of $x(t)$ from Table 3.2, namely,

$$X(j\Omega) = \frac{2}{5+j\Omega}$$

with the time shift property found in Table 3.1:

$$x(-t) \leftrightarrow X(j\Omega)^*$$

to get the answer:

$$Y(j\Omega) = \frac{2}{5-j\Omega}$$

3.6 Fourier Transform of Periodic Signals

If the signal of interest is periodic with period $T$, then it has a Fourier Series:

$$x(t) = \sum_{n=-\infty}^{\infty} c_ne^{j\Omega_0nt} \tag{3.47}$$

Using the linearity of the Fourier Transform, we have

$$X(j\Omega) = \sum_{n=-\infty}^{\infty} c_nF\{e^{j\Omega_0nt}\} \tag{3.48}$$

$$= 2\pi \sum_{n=-\infty}^{\infty} c_n\delta(\Omega - n\Omega_0) \tag{3.49}$$

where $F\{\}$ corresponds to the Fourier Transform of the signal within the brackets.

3.7 Filters

Filters are devices which are commonly found in electronic gadgets. When you adjust the bass (low frequency) or treble (high frequency) settings on your MP3 player, you are adjusting the characteristics of a filter. A more technical name for a filter is a linear system. A filter is represented by a box
having a single input (usually $x(t)$) and a single output (say, $y(t)$) as seen in Figure 3.5. We can denote the operation the filter has on the input using the following notation:

$$y(t) = L[x(t)] \quad (3.50)$$

The types of filters we will consider in this book are linear and time-invariant. A filter is time-invariant if given that $y(t) = L[x(t)]$, then $y(t - \tau) = L[x(t - \tau)]$. In other words, if the input to the filter is delayed by $\tau$, then the output is also delayed by $\tau$. A filter is linear if given that $y_1(t) = L[x_1(t)]$ and $y_2(t) = L[x_2(t)]$ then

$$\alpha y_1(t) + \beta y_2(t) = L[\alpha x_1(t) + \beta x_2(t)] \quad (3.51)$$

Equation 3.51 is often referred to as the superposition principle. We can use linearity and time invariance to derive the mathematical operation which the filter performs on the input, $x(t)$. To do this we begin with the assumption that

$$h(t) = L[\delta(t)] \quad (3.52)$$

The signal $h(t)$ is called the impulse response of the filter. From time invariance, we have

$$h(t - \tau) = L[\delta(t - \tau)] \quad (3.53)$$

Now we can use linearity to find the filter output when the input is $x(\tau)\delta(t - \tau)$, where $x(\tau)$ is a constant

$$x(\tau)h(t - \tau) = L[x(\tau)\delta(t - \tau)] \quad (3.54)$$

We can extend the linearity property further by noting that

$$\sum_n x(\tau_n)\Delta_n h(t - \tau_n) = L[\sum_n x(\tau_n)\Delta_n \delta(t - \tau_n)] \quad (3.55)$$

where we can assume that the constants $\tau_n$ are ordered so that $\tau_i < \tau_k, i < k$ and $\Delta_n = \tau_n - \tau_{n-1}$. In (3.55), we are simply multiplying each $\delta(t - \tau_n)$ by
the constant \( x(\tau_n) \Delta_n \), so once again linearity should prevail. Now if we take
the limit \( \Delta_n \to 0 \), we obtain

\[
\int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau = L \left[ \int_{-\infty}^{\infty} x(\tau) \delta(t-\tau) d\tau \right]
\] (3.56)

Using the sifting property of the unit impulse in the right side of (3.56) gives

\[
\int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau = L \left[ x(t) \right]
\] (3.57)

So it follows that the filter performs the following operation on the input, \( x(t) \):

\[
y(t) = L \left[ x(t) \right] = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau
\] (3.58)

The integral in (3.58) is called the convolution integral. A change of variables can be used to show that

\[
\int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau
\]

which means that the order in which two signals are convolved is unimportant. A short-hand notation for convolution is

\[
\int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau \equiv x(t) * h(t)
\]

### 3.8 Properties of Convolution Integrals

We list several important properties and their proofs.

1. **Commutative Property:**

\[
x(t) * h(t) = h(t) * x(t)
\] (3.59)

Let’s start with

\[
x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau
\] (3.60)
3.8. PROPERTIES OF CONVOLUTION INTEGRALS

and make the substitution \( \gamma = t - \tau \). It follows that

\[
x(t) * h(t) = \int_{-\infty}^{\infty} x(t - \gamma)h(\gamma)d\gamma d\tau \tag{3.61}
\]

\[
= h(t) * x(t) \tag{3.62}
\]

2. **Associative Property:**

\[
[x(t) * h_1(t)] * h_2(t) = x(t) * [h_1(t) * h_2(t)] \tag{3.63}
\]

To prove this property we begin with an expression for the left-hand side of (3.63)

\[
\int_{-\infty}^{\infty} x(\tau)h_1(t - \tau)d\tau * h_2(t) \tag{3.64}
\]

where we have expressed \( x(t) * h_1(t) \) as a convolution integral. Expanding the second convolution gives

\[
\int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} x(\tau)h_1(\gamma-\tau)d\tau \right] h_2(t-\gamma)d\gamma \tag{3.65}
\]

Reversing the order of integration gives

\[
\int_{-\infty}^{\infty} x(\tau) \left[ \int_{-\infty}^{\infty} h_1(\gamma-\tau)h_2(t-\gamma)d\gamma \right] d\tau \tag{3.66}
\]

Using the variable substitution \( \phi = \gamma - \tau \) and integrating over \( \phi \) in the inner integral gives the final result:

\[
\int_{-\infty}^{\infty} x(\tau) \left[ \int_{-\infty}^{\infty} h_1(\phi)h_2(t-\tau-\phi)d\gamma \right] d\tau \tag{3.67}
\]

where the inner integral is recognized as \( h_1(t) * h_2(t) \) evaluated at \( t = t - \tau \), which is required for the convolution with \( x(t) \).

3. **Distributive Property:**

\[
x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t) \tag{3.68}
\]

This property is easily proven from the definition of the convolution integral.

4. **Time-Shift Property:** If \( y(t) = x(t) * h(t) \) then \( x(t-t_0) * h(t) = y(t-t_0) \)

Again, the proof is trivial.
3.9 Evaluation of Convolution Integrals

The key to evaluating a convolution integral such as

\[ x(t) \ast h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \]  

(3.69)

is to realize that as far as the integral is concerned, the variable \( t \) is a constant and the integral is over the variable \( \tau \). Therefore, for each \( t \), we are finding the area of the product \( x(\tau)h(t-\tau) \). Let’s look at an example that illustrates how this works.

**Example 3.9.1** Find the convolution of \( x(t) = u(t) \) and \( h(t) = e^{-t}u(t) \). The convolution integral is given by

\[ h(t) \ast x(t) = \int_{-\infty}^{\infty} e^{-\tau}u(\tau)u(t-\tau)d\tau \]

Figure 3.9.1 shows the graph of \( e^{-\tau}u(\tau) \), \( e^{-t}u(t) \), and their product. From the graph of the product, it is easy to see the convolution integral becomes

\[ \int_0^t e^{-\tau}d\tau = \begin{cases} 
1 - e^{-t}, & t \geq 0 \\
0, & t < 0 
\end{cases} \]

Signals which can be expressed in functional form should be convolved as in the above example. Other signals may not have an easy functional representation but rather may be piece-wise linear. In order to convolve such signals, one must evaluate the convolution integral over different intervals on the \( t \)-axis so that each distinct interval corresponds to a different expression for \( x(t) \ast h(t) \). The following example illustrates this:

**Example 3.9.2** Suppose we attempt to convolve the unit step function \( x(t) = u(t) \) with the trapezoidal function

\[ h(t) = \begin{cases} 
t, & 0 \leq t < 1 \\
1, & 1 \leq t < 2 \\
0, & \text{elsewhere} 
\end{cases} \]
Figure 3.6: Graphs of signals used in Example 3.9.1.
From Figure 3.9.2, it can be seen that on the interval $0 \leq t < 1$, the product $x(t - \tau)h(\tau)$ is an equilateral triangle with area $t^2/2$. On the interval $1 \leq t < 2$, the area of $x(t - \tau)h(\tau)$ is $t - 1/2$. This latter area results by adding the area of an equilateral triangle having a base of 1, and the area of a rectangle having a base of $t - 1$ and a height of 1. For all values of $t$ greater than 2, the convolution is 1.5 since $x(t - \tau)h(\tau) = h(\tau)$ and $h(\tau)$ is a trapezoid having an area of 1.5. Finally, for $t < 0$, the convolution is zero since $x(t - \tau)h(\tau) = 0$.

Figure 3.7: Graphs of signals used in Example 3.9.2.

3.10 Frequency Response

Recall from the section 3.2 that the convolution integral

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau) d\tau$$

has the Fourier Transform:

$$Y(j\Omega) = H(j\Omega)X(j\Omega)$$
3.11. THE SINUSOIDAL STEADY STATE RESPONSE

where \( H(j\Omega) \) and \( X(j\Omega) \) are the Fourier Transforms of \( h(t) \) and \( x(t) \), respectively. Solving for \( H(j\Omega) \) gives the frequency response:

\[
H(j\Omega) = \frac{Y(j\Omega)}{X(j\Omega)}
\]

The frequency response, the Fourier Transform of the impulse response of a filter, is useful since it gives a highly descriptive representation of the properties of the filter. The frequency response can be described as the “gain” of the filter, expressed as a function of frequency. The magnitude of the frequency response evaluated at \( \Omega = \Omega_0 \), \(|H(j\Omega_0)|\) gives the factor the frequency component of \( x(t) \) at \( \Omega = \Omega_0 \) would be scaled by. The phase of the frequency response at \( \Omega = \Omega_0 \), \( \angle H(j\Omega_0) \) gives the phase shift the component of \( x(t) \) at \( \Omega = \Omega_0 \) would undergo. This idea will be discussed in greater detail in Section 3.11. A lowpass filter is a filter which only passes low frequencies, while attenuating or filtering out higher frequencies. A highpass filter would do just the opposite, it would filter out low frequencies and allow high frequencies to pass. Figure 3.10 shows examples of these various filter types.

3.11 The Sinusoidal Steady State Response

It is useful to see what the effect of the filter is on a sinusoidal signal, say \( x(t) = \cos(\Omega_0 t) \). If \( y(t) \) is the output of the filter, then we can write

\[
y(t) = \int_{-\infty}^{\infty} \cos(\Omega_0 (t - \tau)) h(\tau) d\tau
\]

(3.70)

Using the Euler formula for \( \cos(\Omega_0 t) \), right hand side of (3.70) can be written as:

\[
\frac{1}{2} \int_{-\infty}^{\infty} \left( e^{j(\Omega_0 (t-\tau))} + e^{-j(\Omega_0 (t-\tau))} \right) h(\tau) d\tau
\]

(3.71)

This integral can be split into two separate integrals, and written as:

\[
\frac{e^{j\Omega_0 t}}{2} \int_{-\infty}^{\infty} e^{-j\Omega_0 \tau} h(\tau) d\tau + \frac{e^{-j\Omega_0 t}}{2} \int_{-\infty}^{\infty} e^{j\Omega_0 \tau} h(\tau) d\tau
\]

(3.72)

The first of the two integrals can be recognizes as the Fourier Transform of the impulse response evaluated at \( \Omega = \Omega_0 \). The second integral is just the
Figure 3.8: Different filter types: (a) lowpass, (b) bandpass, (c) highpass.
3.12. PARALLEL AND CASCADED FILTERS

complex conjugate of the first integral. Therefore (3.72) can be written as:

\[
\frac{e^{j\Omega_0 t}}{2} H(j\Omega_0) + \frac{e^{-j\Omega_0 t}}{2} H^*(j\Omega_0)
\]  

(3.73)

Since the second term in (3.73) is the complex conjugate of the first term, we can express (3.73) as:

\[
\text{Re} \left\{ e^{j\Omega_0 t} H(j\Omega_0) \right\}
\]

(3.74)

or expressing \( H(j\Omega_0) \) in terms of polar coordinates:

\[
\text{Re} \left\{ e^{j\Omega_0 t} |H(j\Omega_0)| e^{j\angle H(j\Omega_0)} \right\} = \text{Re} \left\{ |H(j\Omega_0)| e^{j(\Omega_0 t + \angle H(j\Omega_0))} \right\}
\]

(3.75)

Therefore, we find that the filter output is given by

\[
y(t) = |H(j\Omega_0)| \cos(\Omega_0 t + \angle H(j\Omega_0))
\]

(3.76)

This is called the sinusoidal steady state response. It tells us that when the input to a linear, time-invariant filter is a cosine, the filter output is a cosine whose amplitude has been scaled by \( |H(j\Omega_0)| \) and that has been phase shifted by \( \angle H(j\Omega_0) \). The same result applies to an input that is an arbitrarily phase shifted cosine (e.g. a sine wave).

**Example 3.11.1** Find the output of a filter whose impulse response is \( h(t) = e^{-5t} u(t) \) and whose input is given by \( x(t) = \cos(2t) \). It can be readily seen that the frequency response of the filter is

\[
H(j\Omega) = \frac{1}{5 + j\Omega}
\]

(3.77)

and therefore \( |H(j2)| = 0.1857 \) and \( \angle H(j2) = -0.3805 \). Therefore, using (3.76):

\[
y(t) = 0.1857 \cos(2t - 0.3805)
\]

(3.78)

3.12 Parallel and Cascaded Filters

In some applications, such as graphic equalizers, it is useful to place filters in parallel as shown in Figure 3.12. Can the parallel combination of filters
be characterized by a single equivalent filter \( h_{eq}(t) \)? The answer is yes and results by noting that

\[
y(t) = \sum_{i=1}^{N} x(t) * h_i(t) \tag{3.79}
\]

\[
= \sum_{i=1}^{N} \int_{-\infty}^{\infty} x(t - \tau)h_i(\tau)d\tau
\]

\[
= \int_{-\infty}^{\infty} x(t - \tau) \sum_{i=1}^{N} h_i(\tau)d\tau
\]

Therefore, the last equation in (3.79) shows that

\[
h_{eq}(t) = \sum_{i=1}^{N} h_i(t) \tag{3.80}
\]

The equivalent transfer function for the parallel filter structure is given by

\[
H_{eq}(j\Omega) = \sum_{i=1}^{N} H_i(j\Omega) \tag{3.81}
\]

Figure 3.9: Parallel filter structure. We wish to find an equivalent filter with impulse response \( h_{eq}(t) \).
3.12. PARALLEL AND CASCADED FILTERS

Next we wish to find an equivalent filter for the cascaded structure shown in Figure 3.12. This can be done by finding an expression for the intermediate value $y_1(t)$:

$$y_1(t) = \int_{-\infty}^{\infty} x(t - \tau) h_1(\tau) d\tau \quad (3.82)$$

The output of the cascaded structure is given by

$$y(t) = \int_{-\infty}^{\infty} y_1(t - \gamma) h_2(\gamma) d\gamma \quad (3.83)$$

substituting (3.82) into (3.83) gives

$$y(t) = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} x(t - \gamma - \tau) h_1(\tau) d\tau \right] h_2(\gamma) d\gamma \quad (3.84)$$

Reversing the order of integration and rearranging slightly gives

$$y(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(t - \gamma - \tau) h_1(\tau) h_2(\gamma) d\gamma d\tau \quad (3.85)$$

Now let $\xi = \gamma + \tau$, solving for $\tau$ gives $\tau = \xi - \gamma$ and $d\xi = d\tau$. Substituting these quantities into (3.85) leads to

$$y(t) = \int_{-\infty}^{\infty} x(t - \xi) \left[ \int_{-\infty}^{\infty} h_1(\xi - \gamma) h_2(\gamma) d\gamma \right] d\xi \quad (3.86)$$

Notice that we can factor $x(t - \xi)$ from the inner integral since $x(t - \xi)$ does not depend on $\gamma$. The integral in the brackets is recognized as $h_1(t) * h_2(t)$.
evaluated at $\xi$. Therefore for the cascaded system, the equivalent impulse response is given by

$$h_{eq}(t) = \int_{-\infty}^{\infty} h_1(t-\gamma) h_2(\gamma) d\gamma$$  \hspace{1cm} (3.87)

This can be generalized to any number of cascaded filters giving

$$h_{eq}(t) = h_1(t) * h_1(t) * \cdots * h_N(t)$$  \hspace{1cm} (3.88)

Figure 3.11: Frequency response magnitude and phase for a first-order low-pass filter ($\Omega_c = 1 \text{ rad/sec}$).

### 3.13 First Order Filters

A first-order lowpass filter has the frequency response

$$H_{LP}(j\Omega) = \frac{1}{1 + j\frac{\Omega}{\Omega_c}}$$  \hspace{1cm} (3.89)
3.13. **FIRST ORDER FILTERS**

The frequency at which the frequency response magnitude has dropped to \(1/\sqrt{2}\) is called the *corner frequency*\(^1\). The frequency response magnitude and phase are plotted in Figure 3.12. It is common to express the frequency response magnitude in units of *decibels* (dB) using the formula

\[
20 \log_{10} |H(j\Omega)|
\]  

(3.90)

At the corner frequency for a first order lowpass filter, the frequency response magnitude is \(1/\sqrt{2}\) or roughly -3 dB. From Table 3.2, it can easily be seen that the impulse response for the first-order lowpass filter is given by

\[
h_{LP}(t) = \Omega_c e^{-\Omega_c t} u(t)
\]  

(3.91)

A first-order highpass filter is given by

\[
H_{HP}(j\Omega) = \frac{j \frac{\Omega}{\Omega_c}}{1 + j \frac{\Omega}{\Omega_c}}
\]  

(3.92)

---

\(^1\)This term most likely originates from it’s role in Bode plots, a shortcut method for sketching the graph of a frequency response.
Notice that

\[ H_{HP}(j\Omega) = 1 - \frac{1}{1 + j\Omega \frac{1}{\Omega_c}} \]  \hspace{1cm} (3.93)

This makes sense since a highpass filter can be constructed by taking the filter input \( x(t) \) and subtracting from it a lowpass filtered version of \( x(t) \). The impulse response of the first-order highpass filter therefore becomes:

\[ h_{HP}(t) = \delta(t) - \Omega_c e^{-\Omega_c t} u(t) \]  \hspace{1cm} (3.94)

First order filters can be easily implemented using linear circuit elements like resistors, capacitors, and inductors. Figure 3.13 shows a first order filter based on a resistor and a capacitor. Since the impedance for a resistor and capacitor are \( R \) and \( 1/j\Omega C \), respectively, voltage division leads to a frequency response of

\[ H_{LP}(j\Omega) = \frac{1}{1 + j\Omega R_1 C_1} \]  \hspace{1cm} (3.95)

Therefore the corner frequency for this filter is \( \Omega_c = \frac{1}{R_1 C_1} \). Similarly, a first-order highpass filter can be implemented using a resistor and capacitor as shown in Figure 3.13. This filter has a frequency response of

\[ H_{HP}(j\Omega) = \frac{j\Omega R_2 C_2}{1 + j\Omega R_2 C_2} \]  \hspace{1cm} (3.96)

The corner frequency for the highpass filter is seen to be \( \Omega_c = \frac{1}{R_2 C_2} \). Now

\[ H_{eq}(j\Omega) = H_{LP}(j\Omega)H_{HP}(j\Omega) \]

Figure 3.13: Circuit implementation of a first-order lowpass filter having \( \Omega_c = 1/R_1 C_1 \).

one might be tempted to apply the results of Section 3.12 to build a band-pass filter by cascading the lowpass and highpass circuits in Figures 3.13 and 3.13, respectively. Theory would predict that the equivalent frequency response of this circuit is given by

\[ H_{eq}(j\Omega) = H_{LP}(j\Omega)H_{HP}(j\Omega) \]
Unfortunately, this is not possible since the circuit elements in the lowpass and highpass filters interact with one another and therefore affect the overall behavior of the circuit. This interaction between the two circuits is called *loading* will be studied in greater detail in the exercises. To get theoretical behavior, it is necessary to use a *voltage follower* circuit, between the lowpass filter from the highpass circuits. The voltage follower circuit is usually an active circuit (requires external power supply) that has very high input impedance and very low output impedance. This eliminates any loading effects which would normally occur between the lowpass and highpass filter circuits.

### 3.14 Parseval’s Theorem for the Fourier Transform

In Chapter 2, we looked at a version of Parseval’s theorem for the Fourier series. Here, we will look at a similar version of this theorem for the Fourier transform. Recall that the energy of a signal is given by

\[ e_x = \int_{-\infty}^{\infty} x(t)^2 dt \quad (3.97) \]

If the energy is finite then \( x(t) \) is an energy signal, as described in Chapter 1. Suppose \( x(t) \) is an energy signal, then the *autocorrelation* function is defined as

\[ r_x(\tau) = x(t) * x(-t) \quad (3.98) \]

It can be shown that \( r_x(\tau) \) is an even function of \( \tau \) and that \( r_x(0) = e_x \) (see Exercises). The Fourier transform of \( r_x(t) \) is given by \( X(j\Omega)X(j\Omega)^* = \)
\[ |X(j\Omega)|^2. \] If follows that

\[
e_x = \frac{1}{2\pi} \left[ \int_{-\infty}^{\infty} |X(j\Omega)|^2 e^{j\Omega\tau} d\Omega \right]_{\tau=0} = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\Omega)|^2 \, d\Omega
\]

Which is Parseval’s theorem for the Fourier transform.
3.15 Exercises

1. Find the Fourier Transform of the following signals, for each case sketch the magnitude of the Fourier Transform:

(a) \( x(t) = 4e^{-0.2t}u(t) \)
(b) \( x(t) = 4e^{0.2t}u(-t) \)
(c) \( x(t) = 4e^{-0.2(t-10)}u(t-10) \)
(d) \( x(t) = \delta(t - 5) \)
(e) \( x(t) = \begin{cases} 
1, & |t| \leq 0.5 \\
0, & |t| > 0.5 
\end{cases} \)
(f) \( x(t) = 4e^{-j0.2t} \)
(g) \( x(t) = \cos(10\pi t) \)
(h) \( x(t) = 6 \)
(i) \( x(t) = \begin{cases} 
\cos(100t), & |t| \leq 0.5 \\
0, & |t| > 0.5 
\end{cases} \)

2. Find the convolution of the following pairs of signals:

\( x(t) \) \hspace{1cm} \( h(t) \)

- (a) \( x(t) \) \hspace{1cm} \( h(t) \)
- (b) \( x(t) \) \hspace{1cm} \( h(t) \)
CHAPTER 3. THE FOURIER TRANSFORM

(b)

3. Find the output of the filter whose transfer function is

\[ H(j\Omega) = \frac{2\pi}{2\pi + j\Omega} \]

and whose input is \( x(t) = u(t) \). Hint, find the impulse response \( h(t) \) corresponding to \( H(j\Omega) \) and convolve it with the input.

4. Show that if \( v(t) = L[u(t)] \), then

\[ \int_{-\infty}^{\infty} v(t) dt = L \left[ \int_{-\infty}^{\infty} u(t) dt \right] \quad (3.101) \]

Hint: Integrate both sides of \( v(t) = L[u(t)] \). Then express the right hand integral as the limit of a sum (as in a calculus textbook). Then by linearity, you can exchange the sum and the \( L[\cdot] \).

5. Find an expression for the convolution of \( x(t) = u(t) \) and \( h(t) = \sin(8t)u(t) \), you can check your answer using this Java applet.

6. Find an expression for the convolution of \( x(t) = \text{rect}(t - 0.5, 1) \) and \( h(t) = e^{-t}u(t) \), you can check your answer using this Java applet.

7. Find the Fourier transform of the periodic signal in problem 2.16.

8. Consider a filter having the impulse response

\[ h(t) = e^{-2t}u(t) \]

Sketch the frequency response (both magnitude and phase) of the filter and find the output of the filter when the input is \( x(t) = \cos(10t) \).

9. Repeat the previous problem for the impulse response given by

\[ h(t) = \begin{cases} 
1, & 0 \leq t < 1 \\
0, & \text{otherwise}
\end{cases} \]

10. Suppose that two filters having impulse responses \( h_1(t) \) and \( h_2(t) \) are cascaded (i.e. connected in series). Find the impulse response of the equivalent filter assuming \( h_1(t) = 10e^{-10t}u(t) \) and \( h_2(t) = 5e^{-5t}u(t) \).
11. Design a first-order lowpass filter having a corner frequency of 100 Hz. Use a 100\(k\Omega\) resistor. Plot both the magnitude and phase of the filter’s frequency response.

12. Design a first-order highpass filter having a corner frequency of 1000 Hz. Use a 0.01\(\mu F\) capacitor. Plot both the magnitude and phase of the filter’s frequency response.

13. The following problems are associated with the circuits in Figure 3.15:

![Figure 3.15: Problem 13](image)

(a) Find the frequency response of the circuit in Figure 3.15(a), and sketch its magnitude and phase.

(b) Find the frequency response of the circuit in Figure 3.15(b) and sketch its magnitude and phase.

(c) Find the frequency response of the filter in Figure 3.15(c), sketch its magnitude and phase and show that it is not the product of the frequency responses for problems 13a and 13b.
Chapter 4

The Laplace Transform

4.1 The Double-Sided Laplace Transform

You may have noticed that we avoided taking the Fourier transform of a number of signals. For example, we did not try to compute the Fourier Transform of the unit step function, \( x(t) = u(t) \), or the ramp function \( x(t) = tu(t) \). The reason for this is that these signals do not have a Fourier transform that converges to a finite value for all \( \Omega \). Recall that in order to deal with periodic signals such as \( \cos(\Omega_0 t) \) we had to settle for Fourier transforms having impulse functions in them. The Laplace transform gives us a mechanism for dealing with signals that do not have finite-valued Fourier Transforms.

The double-sided Laplace Transform of \( x(t) \) is defined as follows:

\[
X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \tag{4.1}
\]

where we define

\[
s = \sigma + j\Omega \tag{4.2}
\]

We observe that the Laplace transform is a generalization of the Fourier transform since

\[
X(j\Omega) = X(s)|_{s=j\Omega} \tag{4.3}
\]

Therefore, we can write

\[
X(s) = F \{e^{-\sigma t}x(t)\} \tag{4.4}
\]
The inverse Laplace transform can be derived using this idea. Applying the inverse Fourier transform to $F \{e^{-\sigma t}x(t)\}$ gives

$$e^{-\sigma t}x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F \{e^{-\sigma t}x(t)\} e^{j\Omega t} d\Omega \quad (4.5)$$

Using (4.2) leads to

$$d\Omega = \frac{ds}{j} \quad (4.6)$$

Substituting $s$ for $\Omega$ and solving for $x(t)$ in (4.5) gives

$$x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds \quad (4.7)$$

Equation (4.7) is called the inverse Laplace transform. The integration is along a straight line in the complex $s$-plane corresponding to a fixed value of $\sigma$. This is illustrated in Fig. 4.2. It is important that this line exist in a region of the $s$-plane that corresponds to the region of convergence for the Laplace transform. The region of convergence is defined as that region in the $s$-plane for which

$$\int_{-\infty}^{\infty} |x(t)e^{-st}| dt < \infty \quad (4.8)$$

Note that since $s = \sigma + j\Omega$ this is equivalent to

$$\int_{-\infty}^{\infty} |x(t)e^{-\sigma t}| dt < \infty \quad (4.9)$$

Figure 4.1: The inverse Laplace transform integrates along a line having a constant $\sigma$ in the complex $s$-plane.
4.2 The Single-Sided Laplace Transform

We define the single-sided Laplace transform as

\[ X(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt \]  

(4.10)

where the lower limit of integration tacitly includes the point \( t = 0 \). That is, \( 0^- \) represents a point just to the negative side of \( t = 0 \). This allows for the integral to take into account signal features that occur at \( t = 0 \) such as a step or impulse function. The single-sided Laplace transform is motivated by the fact that most signals are turned on at some point. Moreover, if the region of convergence is not specified then different signals can yield identical bilateral Laplace transforms, as the following example illustrates.

Example 4.2.1 Consider the signal

\[ x_1(t) = e^{-\alpha t}u(t) \]  

(4.11)

The bilateral Laplace transform is given by

\[ X_1(s) = \int_{0}^{\infty} e^{-\alpha t}e^{-st}dt \]  

(4.12)

\[ = \int_{0}^{\infty} e^{-(s+\alpha)t}dt \]

\[ = \left. \frac{-1}{s+\alpha}e^{-(\sigma+j\Omega+\alpha)t} \right|_{0}^{\infty} \]

\[ = \frac{-1}{s+\alpha}(e^{-(\sigma+j\Omega+\alpha)\infty}e^{j\Omega\infty} - 1) \]

the magnitude of \( e^{-(\sigma+j\Omega)\infty} \) is zero only if \( \sigma > -\alpha \) which establishes the region of convergence. Therefore we have

\[ e^{-\alpha t}u(t) \leftrightarrow \frac{1}{s+\alpha}, \sigma > -\alpha \]  

(4.13)

Now consider the Laplace transform of the signal

\[ x_2(t) = -e^{-\alpha t}u(-t) \]  

(4.14)
We have
\[ X_2(s) = -\int_{-\infty}^{0} e^{-\alpha t} e^{-st} dt \]  
\[ = \int_{-\infty}^{0} e^{-(s+\alpha)t} dt \]  
\[ = \frac{-1}{s+\alpha} e^{-(\sigma+j\Omega+\alpha)t} \bigg|_{-\infty}^{0} \]  
\[ = \frac{1}{s+\alpha} (1 - e^{(\sigma+\alpha)\infty} e^{j\Omega\infty}) \]

Here, the quantity \( e^{(\sigma+\alpha)\infty} \) is zero only if \( \sigma < -\alpha \) so we have
\[ -e^{-\alpha t} u(-t) \leftrightarrow \frac{1}{s+\alpha}, \sigma < -\alpha \]  
(4.16)

which is identical to \( X_1(s) \) except for the region of convergence.

To avoid scenarios where two different signals have the same bilateral Laplace transform, we restrict our signals to those which are assumed to be zero for \( t < 0 \), for which the single-sided Laplace transform applies. Such signals are sometimes called causal signals.

The region of convergence for the single-sided Laplace transform is a region in the \( s \)-plane satisfying \( \sigma > \sigma_{\text{min}} \) as shown in Fig. 4.2. To see this, we observe that if
\[ \int_{0^-}^{\infty} |x(t)e^{-\sigma_{\text{min}} t}| dt < \infty \]  
(4.17)

then it must be the case that
\[ \int_{0^-}^{\infty} |x(t)e^{-\sigma t}| dt < \infty \]  
(4.18)

for \( \sigma > \sigma_{\text{min}} \), since \( e^{-\sigma t} \) decreases faster than \( e^{-\sigma_{\text{min}} t} \). Finally, the inverse single-sided Laplace transform is the same as the inverse double-sided Laplace transform (see (4.7)), since a single sided Laplace transform can be interpreted as the double-sided Laplace transform of a signal satisfying \( x(t) = 0, t < 0 \). From here on, we will work exclusively with the single-sided Laplace transform. Unless we need to specifically differentiate between the single or double-sided transforms, we will refer to the single-sided Laplace transform as simply the “Laplace transform”.

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4.3 Properties of the Laplace Transform

The properties associated with the Laplace transform are similar to those of the Fourier transform. First, let’s set define some notation, we will use the notation \( \mathcal{L}\{\} \) to denote the Laplace transform operation. Therefore we can write \( X(s) = \mathcal{L}\{x(t)\} \) and \( x(t) = \mathcal{L}^{-1}\{X(s)\} \) for the forward and inverse Laplace transforms, respectively. We can also use the transform pair notation used earlier:

\[
x(t) \leftrightarrow X(s)
\] (4.19)

With this notation defined, let’s now look at some properties.

4.3.1 Linearity

Given that \( x_1(t) \leftrightarrow X_1(s) \) and \( x_2(t) \leftrightarrow X_2(s) \) then for any constants \( \alpha \) and \( \beta \), we have

\[
\alpha x_1(t) + \beta x_2(t) \leftrightarrow \alpha X_1(s) + \beta X_2(s)
\] (4.20)

The linearity property follows easily using the definition of the Laplace transform.

4.3.2 Time Delay

The reason we call this the time delay property rather than the time shift property is that the time shift must be positive, i.e. if \( \tau > 0 \), then \( x(t - \tau) \)
corresponds to a delay. If $\tau < 0$ then we would not be able to use the single-sided Laplace transform because we would have a lower integration limit of $\tau$, which is less than zero. To derive the property, let’s evaluate the Laplace transform of the time-delayed signal

$$\mathcal{L}\{x(t-\tau)\} = \int_{0}^{\infty} (t-\tau)e^{-st}dt \tag{4.21}$$

Letting $\gamma = t-\tau$ leads to $t = \gamma + \tau$ and $dt = d\gamma$. Substituting these quantities into (4.21) gives

$$\mathcal{L}\{x(t-\tau)\} = \int_{-\tau}^{\infty} x(\gamma)e^{-s(\gamma+\tau)}d\gamma \tag{4.22}$$

$$= e^{-s\tau} \int_{-\tau}^{\infty} x(\gamma)e^{-s\gamma}d\gamma$$

$$= e^{-s\tau} \int_{-\tau}^{0} x(\gamma)e^{-s\gamma}d\gamma + e^{-s\tau} \int_{0}^{\infty} x(\gamma)e^{-s\gamma}d\gamma$$

where we note that the first integral in the last line is zero since $x(t) = 0, t < 0$. Therefore the time delay property is given by

$$\mathcal{L}\{x(t-\tau)\} = e^{-s\tau}X(s) \tag{4.23}$$

### 4.3.3 s-Shift

This property is the Laplace transform corresponds to the frequency shift property of the Fourier transform. In fact, the derivation of the s-shift property is virtually identical to that of the frequency shift property.

$$\mathcal{L}\{e^{-at}x(t)\} = \int_{0}^{\infty} e^{-at}x(t)e^{-st}dt \tag{4.24}$$

$$= \int_{0}^{\infty} x(t)e^{-(a+s)t}dt$$

$$= \int_{0}^{\infty} x(t)e^{-(a+\sigma+j\Omega)t}dt$$

$$= X(s + a)$$

The $s$-shift property also alters the region of convergence of the Laplace transform. If the region of convergence for $X(s)$ is $\sigma > \sigma_{\text{min}}$, then the region of convergence for $\mathcal{L}\{e^{-at}x(t)\}$ is $\sigma > \sigma_{\text{min}} - a$. 
4.3.4 Multiplication by $t$

Let’s begin by taking the derivative of the Laplace transform:

$$\frac{dX(s)}{ds} = \frac{d}{ds} \int_{0}^{\infty} x(t)e^{-st}dt$$  \hspace{1cm} (4.25)

$$= \int_{0}^{\infty} x(t) \frac{d}{ds}e^{-st}dt$$

$$= -\int_{0}^{\infty} tx(t)e^{-st}dt$$

So we can write

$$\mathcal{L}\{tx(t)\} = -\frac{dX(s)}{ds}$$  \hspace{1cm} (4.26)

This idea can be extended to multiplication by $t^n$. Letting $y(t) = tx(t)$, if follows that

$$ty(t) \leftrightarrow -\frac{dY(s)}{ds}$$  \hspace{1cm} (4.27)

$$\leftrightarrow \frac{d^2X(s)}{ds^2}$$

Proceeding in this manner, we find that

$$t^n x(t) \leftrightarrow (-1)^n \frac{d^{n+1}X(s)}{ds^{n+1}}$$  \hspace{1cm} (4.28)

4.3.5 Time Scaling

The time scaling property for the Laplace transform is similar to that of the Fourier transform:

$$\mathcal{L}\{x(\alpha t)\} = \int_{0}^{\infty} x(\alpha t)e^{-\gamma t}dt$$  \hspace{1cm} (4.29)

$$= \frac{1}{\alpha} \int_{0}^{\infty} x(\gamma)e^{-\frac{\gamma}{\alpha}d\gamma}$$

$$= \frac{1}{\alpha} X\left(\frac{s}{\alpha}\right)$$

where in the second equality, we made the substitution $t = \frac{\gamma}{\alpha}$ and $dt = \frac{d\gamma}{\alpha}$. 
4.3.6 Convolution

The derivation of the convolution property for the Laplace transform is virtually identical to that of the Fourier transform. We begin with

\[ \mathcal{L} \left\{ \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau \right\} = \int_{-\infty}^{\infty} x(\tau) \mathcal{L} \{ h(t - \tau) \} d\tau \]  

(4.30)

Applying the time-delay property of the Laplace transform gives

\[ \int_{-\infty}^{\infty} x(\tau) \mathcal{L} \{ h(t - \tau) \} d\tau = H(s) \int_{-\infty}^{\infty} x(\tau)e^{-st} d\tau \]

\[ = H(s)X(s) \]  

(4.31)

If \( h(t) \) is the impulse response of a linear time-invariant system, then we call \( H(s) \) the transfer function of the system. The frequency response results by setting \( s = j\Omega \) in \( H(s) \). The transfer function provides us with a very powerful means of determining the output of a linear time-invariant filter given the input signal. It will also enable us to determine a means of establishing the stability\(^1\) of a linear-time invariant filter, something which was not possible with the frequency response.

4.3.7 Differentiation

The Laplace transform of the derivative of a signal will be used widely. Consider

\[ \mathcal{L} \left\{ \frac{d}{dt} x(t) \right\} = \int_{0^-}^{\infty} x'(t)e^{-st} dt \]  

(4.32)

this can be integrated by parts:

\[ u = e^{-st} \quad v' = x'(t) \]
\[ u' = -se^{-st} \quad v = x(t) - x(0^-) \]

\(^1\)We will discuss stability shortly
which gives

\[ \mathfrak{L} \left\{ \frac{d}{dt} x(t) \right\} = uv|_{0}^{\infty} - \int_{0}^{\infty} u'vdt \]

\[ = e^{-st} \left[ x(t) - x(0^-) \right] |_{0}^{\infty} + \int_{0}^{\infty} s \left[ x(t) - x(0^-) \right] e^{-st}dt \]

\[ = 0 + s \int_{0}^{\infty} x(t)e^{-st}dt - sx(0^-) \int_{0}^{\infty} e^{-st}dt \]

\[ = sX(s) + sx(0^-) \frac{e^{-st}}{s} |_{0}^{\infty} \]

\[ = sX(s) - x(0^-) \]

therefore we have,

\[ \frac{d}{dt} x(t) \leftrightarrow sX(s) - x(0^-) \]

4.3.8 Higher Order Derivatives

The previous derivation can be extended to higher order derivatives. Consider

\[ y(t) = \frac{dx(t)}{dt} \leftrightarrow sX(s) - x(0^-) \]

it follows that

\[ \frac{dy(t)}{dt} \leftrightarrow sY(s) - y(0^-) \] (4.34)

which leads to

\[ \frac{d^2}{dt^2} x(t) \leftrightarrow s^2 X(s) - s x(0^-) - \frac{dx(0^-)}{dt} \] (4.35)

This process can be iterated to get the Laplace transform of arbitrary higher order derivatives, giving

\[ \frac{d^n x(t)}{dt^n} \leftrightarrow s^n X(s) - s^{n-1} x(0^-) - \sum_{k=2}^{n} s^{n-k} \frac{d^{k-1} x(0^-)}{dt^{k-1}} \] (4.36)

where it should be understood that

\[ \frac{d^m x(0^-)}{dt^m} \equiv \frac{d^m x(t)}{dt^m} \bigg|_{t=0^-}, m = 1, \ldots, n-1 \]
4.3.9 Integration

Let

\[ g(t) = \int_{0^-}^{t} x(\tau) d\tau \]  \hspace{1cm} (4.37)

it follows that

\[ \frac{dg(t)}{dt} = x(t) \]  \hspace{1cm} (4.38)

and \( g(0^-) = 0 \). Moreover, we have

\[ X(s) = \mathcal{L}\left\{ \frac{dg(t)}{dt} \right\} \]

\[ = sG(s) - g(0^-) \]

\[ = sG(s) \]

therefore

\[ G(s) = \frac{X(s)}{s} \]  \hspace{1cm} (4.40)

but since

\[ G(s) = \mathcal{L}\left\{ \int_{0^-}^{t} x(\tau) d\tau \right\} \]  \hspace{1cm} (4.41)

we have

\[ \int_{0^-}^{t} x(\tau) d\tau \leftrightarrow \frac{X(s)}{s} \]  \hspace{1cm} (4.42)

Now suppose \( x(t) \) has a non-zero integral over negative values of \( t \). We have

\[ \int_{\infty}^{t} x(\tau) d\tau = \int_{-\infty}^{0^-} x(\tau) d\tau + \int_{0^-}^{t} x(\tau) d\tau \]  \hspace{1cm} (4.43)

The quantity \( \int_{-\infty}^{0^-} x(\tau) d\tau \) is a constant for positive values of \( t \), and can be expressed as

\[ u(t) \int_{-\infty}^{0^-} x(\tau) d\tau \]

it follows that

\[ \int_{\infty}^{t} x(\tau) d\tau \leftrightarrow \frac{\int_{-\infty}^{0^-} x(\tau) d\tau}{s} + \frac{X(s)}{s} \]  \hspace{1cm} (4.44)

where we have used the fact that \( u(t) \leftrightarrow \frac{1}{s} \).
4.3. PROPERTIES OF THE LAPLACE TRANSFORM

4.3.10 Initial and Final Value Theorems

The initial and final value theorems make it possible to determine \( x(t) \) at \( t = 0^+ \) and at \( t \to \infty \). We look first at the initial value theorem, from the derivative property of the Laplace transform, we can write

\[
sX(s) - x(0^-) = \int_{0^-}^{\infty} \frac{dx(t)}{dt} e^{-st} dt
\]

\[
= \int_{0^-}^{0^+} \frac{dx(t)}{dt} e^{-st} dt + \int_{0^+}^{\infty} \frac{dx(t)}{dt} e^{-st} dt
\]

\[
= e^{-st} x(t)|_{0^+} + \int_{0^+}^{\infty} \frac{dx(t)}{dt} e^{-st} dt
\]

\[
= x(0^+) - x(0^-) + \int_{0^+}^{\infty} \frac{dx(t)}{dt} e^{-st} dt
\]

(4.45)

We therefore have

\[
sX(s) = x(0^+) + \int_{0^+}^{\infty} \frac{dx(t)}{dt} e^{-st} dt
\]

(4.46)

Taking the limit as \( s \to \infty \) on both sides gives

\[
\lim_{s \to \infty} sX(s) = x(0^+) + \int_{0^+}^{\infty} \lim_{s \to \infty} \frac{dx(t)}{dt} e^{-st} dt
\]

(4.47)

The limit on the right hand side is zero if \( \sigma \to +\infty \), which leads to the initial value theorem

\[
\lim_{s \to \infty} sX(s) = x(0^+)
\]

(4.48)

To prove the final value theorem, we take the limit as \( s \to 0 \) in the derivative property of the Laplace transform

\[
\lim_{s \to 0} [sX(s) - x(0^-)] = \int_{0^-}^{\infty} \frac{dx(t)}{dt} \lim_{s \to 0} [e^{-st}] dt
\]

\[
= \int_{0^-}^{\infty} \frac{dx(t)}{dt} dt
\]

\[
= x(t)|_{0^-}^{\infty}
\]

\[
= x(\infty) - x(0^-)
\]

(4.49)

So the final value theorem is given by

\[
\lim_{s \to 0} sX(s) = x(\infty)
\]

(4.50)
### Table 4.1: Laplace Transform properties.

<table>
<thead>
<tr>
<th>Property</th>
<th>$y(t)$</th>
<th>$Y(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linearity</td>
<td>$\alpha x_1(t) + \beta x_2(t)$</td>
<td>$\alpha X_1(s) + \beta X_2(s)$</td>
</tr>
<tr>
<td>Time Delay</td>
<td>$x(t - \tau)$</td>
<td>$X(s)e^{-\tau}$</td>
</tr>
<tr>
<td>s-Shift</td>
<td>$x(t)e^{-at}$</td>
<td>$X(s + a)$</td>
</tr>
<tr>
<td>Multiplication by $t$</td>
<td>$tx(t)$</td>
<td>$-\frac{dX(s)}{ds}$</td>
</tr>
<tr>
<td>Multiplication by $t^n$</td>
<td>$t^n x(t)$</td>
<td>$(-1)^n \frac{d^{n+1}X(s)}{ds^{n+1}}$</td>
</tr>
<tr>
<td>Convolution</td>
<td>$x(t) \ast h(t)$</td>
<td>$X(j\Omega)H(j\Omega)$</td>
</tr>
<tr>
<td>Differentiation</td>
<td>$\frac{dx(t)}{dt}$</td>
<td>$sX(s) - x(0^-)$</td>
</tr>
<tr>
<td></td>
<td>$\frac{d^2x(t)}{dt^2}$</td>
<td>$s^2X(s) - sx(0^-) - \frac{dx(0^-)}{dt}$</td>
</tr>
<tr>
<td></td>
<td>$\frac{d^n x(t)}{dt^n}$</td>
<td>$s^n X(s) - s^{n-1}x(0^-) - \sum_{k=2}^{n} s^{n-k} \frac{d^{k-1} x(0^-)}{dt^{k-1}}$</td>
</tr>
<tr>
<td>Integration</td>
<td>$\int_{t}^{\infty} x(\tau)d\tau$</td>
<td>$\int_{-\infty}^{0^-} x(\tau)d\tau + \frac{X(s)}{s}$</td>
</tr>
<tr>
<td>Initial Value Theorem</td>
<td>$\lim_{s \to \infty} sX(s) = x(0^+)$</td>
<td></td>
</tr>
<tr>
<td>Final Value Theorem</td>
<td>$\lim_{s \to 0} sX(s) = x(\infty)$</td>
<td></td>
</tr>
</tbody>
</table>
4.4 Laplace Transforms of Some Common Signals

We’ll next build up a collection of Laplace transform pairs which we will include in a table. It’s important to keep in mind that once the transform pair has been derived, the focus should be on utilizing the transform pair found in the table rather than in recalculating the transform.

### 4.4.1 Exponential Signal

Consider the Laplace transform of \( x(t) = e^{\alpha t}u(t) \):

\[
\mathcal{L}\{e^{-\alpha t}u(t)\} = \int_0^\infty e^{-\alpha t}e^{-st}dt
\]

\[
= \int_0^\infty e^{-\alpha t}e^{-st}dt
\]

\[
= \int_0^\infty e^{-(\alpha+s)t}dt
\]

\[
\left. \frac{-1}{\alpha+s} e^{-(\alpha+s)t} \right|_0^\infty
\]

\[
= \frac{1}{\alpha+s}, \quad \sigma > -\alpha
\]

where \( \sigma > -\alpha \) defines the region of convergence. Notice also that if \( \alpha < 0 \), \( X(s) \) still exists provided \( \sigma > -\alpha \). Therefore,

\[
e^{-\alpha t}u(t) \leftrightarrow \frac{1}{\alpha+s}
\]

### 4.4.2 Unit Step Function

Recall we did not attempt to compute the Fourier transform of \( u(t) \) since the Fourier transform does not converge for this signal. Fortunately, the Laplace transform easily converges. In fact, we find that since \( u(t) \) is a special case of the exponential function with \( \alpha = 0 \), the simply have

\[
u(t) \leftrightarrow \frac{1}{s}
\]

The region of convergence is \( \sigma > 0 \).
4.4.3 Ramp Signal

This signal is given by \( x(t) = tu(t) \) and also does not have a Fourier transform. The Laplace transform is given by

\[
\mathcal{L} \{tu(t)\} = \int_0^\infty te^{-st} dt
\]  

(4.55)

Setting \( u = t, u' = 1, v' = e^{-st}, v = -\frac{1}{s}e^{-st} \) and integrating by parts gives

\[
\mathcal{L} \{tu(t)\} = -\frac{t}{s}e^{-st} \bigg|_0^\infty + \frac{1}{s} \int_0^\infty e^{-st} dt
\]

(4.56)

\[
= 0 - \frac{1}{s^2}e^{-st} \bigg|_0^\infty
\]

\[
= \frac{-1}{s^2} \left[ e^{-(\sigma+j\Omega)} - 1 \right]
\]

\[
= \frac{1}{s^2}, \quad \sigma > 0
\]

Here, the region of convergence is \( \sigma > 0 \), sometimes referred to as the right-half plane.

4.4.4 Cosine Signal

Even though we computed the Fourier transform of the cosine signal, \( x(t) = \cos(\Omega_0 t) \), the Fourier transform technically does not converge for this signal. That is why \( X(j\Omega) \) involves impulse functions. The Laplace transform produces quite a different result. First we use the fact that

\[
\cos(\Omega_0 t)u(t) = \frac{e^{j\Omega_0 t}u(t) + e^{-j\Omega_0 t}u(t)}{2}
\]  

(4.57)

Since each of the two terms is an exponential function we have

\[
\mathcal{L} \{\cos(\Omega_0 t)u(t)\} = \frac{\frac{1}{2}}{s - j\Omega_0} + \frac{\frac{1}{2}}{s + j\Omega_0}
\]

(4.58)

\[
= \frac{s}{s^2 + \Omega_0^2}
\]  

(4.59)

Here, the region of convergence corresponds to \( \sigma > 0 \) or right-half plane.
4.4. LAPLACE TRANSFORMS OF SOME COMMON SIGNALS

4.4.5 More Transform Pairs

We can use the existing transform pairs along with the properties of the Laplace transform to derive many new transform pairs. Consider the exponentially weighted cosine signal. This signal is given by

\[ x(t) = e^{-\alpha t} \cos(\Omega_0 t)u(t) \]

We can use the \( s \)-shift property of the Fourier transform (4.24) along with the Laplace transform of the cosine signal (4.58) to get

\[ e^{-\alpha t} \cos(\Omega_0 t)u(t) \leftrightarrow \frac{s + \alpha}{(s + \alpha)^2 + \Omega_0^2} \] (4.60)

Another common signal is

\[ x(t) = te^{-\alpha t}u(t) \]

Here, we use the Laplace transform of the exponential signal (4.13) and the \( t \) multiplication property (4.26) to get

\[ te^{-\alpha t}u(t) \leftrightarrow \frac{1}{(s + \alpha)^2} \] (4.61)

Extending this idea one step further, we have

\[ x(t) = t^2 e^{-\alpha t}u(t) \]

So (4.28) applies, giving

\[ t^2 e^{-\alpha t}u(t) \leftrightarrow \frac{2}{(s + \alpha)^3} \] (4.62)

Example 4.4.1 Consider the signal \( x(t) = te^{-2t}u(t) \). Therefore, we get

\[ te^{-2t}u(t) \leftrightarrow \frac{a}{(s + 2)^2} \] (4.63)

Example 4.4.2 Consider the signal \( x(t) = e^{-2t} \cos(5t)u(t) \). As seen in Table 4.2,

\[ e^{-2t} \cos(5t)u(t) \leftrightarrow \frac{s + 2}{(s + 2)^2 + 25} \] (4.64)
<table>
<thead>
<tr>
<th>$x(t)$</th>
<th>$X(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e^{-\alpha t}u(t)$</td>
<td>$\frac{1}{s + \alpha}$</td>
</tr>
<tr>
<td>$u(t)$</td>
<td>$\frac{1}{s}$</td>
</tr>
<tr>
<td>$\delta(t)$</td>
<td>$1$</td>
</tr>
<tr>
<td>$tu(t)$</td>
<td>$\frac{1}{s^2}$</td>
</tr>
<tr>
<td>$\cos(\Omega_0 t)u(t)$</td>
<td>$\frac{s}{s^2 + \Omega_0^2}$</td>
</tr>
<tr>
<td>$\sin(\Omega_0 t)u(t)$</td>
<td>$\frac{\Omega_0}{s^2 + \Omega_0^2}$</td>
</tr>
<tr>
<td>$e^{-\alpha t}\cos(\Omega_0 t)u(t)$</td>
<td>$\frac{s + \alpha}{(s + \alpha)^2 + \Omega_0^2}$</td>
</tr>
<tr>
<td>$e^{-\alpha t}\sin(\Omega_0 t)u(t)$</td>
<td>$\frac{\Omega_0}{(s + \alpha)^2 + \Omega_0^2}$</td>
</tr>
<tr>
<td>$te^{-\alpha t}u(t)$</td>
<td>$\frac{1}{(s + \alpha)^2}$</td>
</tr>
<tr>
<td>$t^2 e^{-\alpha t}u(t)$</td>
<td>$\frac{2}{(s + \alpha)^3}$</td>
</tr>
<tr>
<td>$t^n e^{-\alpha t}u(t)$</td>
<td>$\frac{n!}{(s + \alpha)^{n+1}}$</td>
</tr>
</tbody>
</table>

Table 4.2: Some common Laplace Transform pairs.
4.5 Finding the Inverse Laplace Transform

4.5.1 Using Transform Tables

The inverse Laplace transform, given by

\[ x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds \]  (4.65)

can be found by directly evaluating the above integral. However since this requires a background in the theory of complex variables, which is beyond the scope of this book, we will not be directly evaluating the inverse Laplace transform. Instead, we will utilize the Laplace transform pairs and properties found in Tables 4.1 and 4.2. Consider the following examples:

**Example 4.5.1** Find the inverse Laplace transform of

\[ X(s) = \frac{e^{-10s}}{s + 5} \]

By looking at Table 4.1 we find that multiplication by \(e^{-10s}\) corresponds to a time delay of 10 sec. Then from Table 4.2, we see that

\[ \frac{1}{s + 5} \]

corresponds to the Laplace transform of the exponential signal \(e^{-5t}u(t)\). Therefore we must have

\[ x(t) = e^{-5(t-10)}u(t-10) \]

**Example 4.5.2** Find the inverse Laplace transform of

\[ X(s) = \frac{1}{(s + 2)^2} \]

First we note that from Table 4.2, the Laplace transform of \(tu(t)\) is

\[ \frac{1}{s^2} \]

Then using the s-shift property in Table 4.1 gives

\[ x(t) = te^{-2t}u(t) \]

Also, the same answer may be arrived at by combining the Laplace transform of \(e^{-2t}u(t)\) with the multiplication by \(t\) property.
4.5.2 Partial Fraction Expansions

Partial fraction expansions are useful when we can express the Laplace transform in the form of a rational function,

\[ X(s) = \frac{b_q s^q + b_{q-1} s^{q-1} + \cdots + b_1 s + b_0}{a_p s^p + a_{p-1} s^{p-1} + \cdots + a_1 s + a_0} \]  

A rational function is a ratio of two polynomials. The numerator polynomial \( B(s) \) has order \( q \), i.e., the largest power of \( s \) in this polynomial is \( q \), while the denominator polynomial has order \( p \). The partial fraction expansion also requires that the Laplace transform be a proper rational function, which means that \( q < p \). Since \( B(s) \) and \( A(s) \) can be factored, we can write

\[ X(s) = \frac{(s - \beta_1)(s - \beta_2) \cdots (s - \beta_q)}{(s - \alpha_1)(s - \alpha_2) \cdots (s - \alpha_p)} \]  

The \( \beta_i, i = 1, 2, \ldots, q \) are the roots of \( B(s) \), and are called the zeros of \( X(s) \). The roots of \( A(s) \), are \( \alpha_i, i = 1, \ldots, p \) and are called the poles of \( X(s) \). If we evaluate \( X(s) \) at one of the zeros we get \( X(\beta_i) = 0, i = 1, \ldots, q \). Similarly, evaluating \( X(s) \) at a pole gives \( X(\alpha_i) = \pm\infty, i = 1, \ldots, p \). The partial fraction expansion of a Laplace transform will usually involve relatively simple terms whose inverse Laplace transforms can be easily determined from a table of Laplace transforms. We must consider several different cases which depend on whether the poles are distinct.

**Distinct Poles:**

When all of the poles are distinct (i.e. \( \alpha_i \neq \alpha_j, i \neq j \)) then we can use the following partial fraction expansion:

\[ X(s) = \frac{A_1}{s - \alpha_1} + \frac{A_2}{s - \alpha_2} + \cdots + \frac{A_p}{s - \alpha_p} \]  

The coefficients, \( A_i, i = 1, \ldots, p \) can then be found using the following formula

\[ A_i = X(s)(s - \alpha_i)|_{s=\alpha_i}, i = 1, \ldots, p \]  

\(^2\)The actual sign would need to be evaluated at some value of \( s \) that is sufficiently close to the pole.
4.5. FINDING THE INVERSE LAPLACE TRANSFORM

Equation (4.69) is easily derived by clearing fractions in (4.68). The inverse Fourier transform of \( X(s) \) can then be easily found since each of the terms in the right-hand side of (4.68) is the Laplace transform of an exponential signal. This method is sometimes called the cover up method.

**Example 4.5.3** Find the inverse Laplace transform of

\[
X(s) = \frac{2s - 10}{s^2 + 3s + 2}
= \frac{2s - 10}{(s + 1)(s + 2)}
\]

Since the poles are \( \alpha_1 = -1 \) and \( \alpha_2 = -2 \) are distinct, we have the expansion

\[
X(s) = \frac{A_1}{s + 1} + \frac{A_2}{s + 2}
\]

Using (4.68) then gives:

\[
A_1 = X(s)(s + 1)|_{s = -1} = \frac{2s - 10}{s + 2} \bigg|_{s = -1} = -12
\]

and

\[
A_2 = X(s)(s + 2)|_{s = -2} = \frac{2s - 10}{s + 1} \bigg|_{s = -2} = 14
\]

Therefore, we get:

\[
X(s) = \frac{-12}{s + 1} + \frac{14}{s + 2}
\]

The inverse Laplace transform of \( X(s) \) can be found by looking up the inverse transform of each of the terms in the right-hand side of (4.74) giving

\[
x(t) = -12e^{-t}u(t) + 14e^{-2t}u(t)
\]
Repetitd Poles:

Let’s consider the case when each pole is repeated,

\[ X(s) = \frac{B(s)}{(s - \alpha_1)^{p_1}(s - \alpha_2)^{p_2} \cdots (s - \alpha_r)^{p_r}} \]  (4.76)

where \( p_1 + p_2 + \cdots + p_r = p \). In this case the partial fraction expansion goes like this:

\[
X(s) = \frac{A_{1,1}}{s - \alpha_1} + \frac{A_{1,2}}{(s - \alpha_1)^2} + \cdots + \frac{A_{1,p_1}}{(s - \alpha_1)^{p_1}} \\
+ \frac{A_{2,1}}{s - \alpha_2} + \frac{A_{2,2}}{(s - \alpha_2)^2} + \cdots + \frac{A_{2,p_2}}{(s - \alpha_2)^{p_2}} \\
+ \cdots \\
+ \frac{A_{r,1}}{s - \alpha_r} + \frac{A_{r,2}}{(s - \alpha_r)^2} + \cdots + \frac{A_{r,p_r}}{(s - \alpha_r)^{p_r}} 
\]  (4.77)

We’ll look at two methods. In the first method, the coefficients can be found using the following formula

\[
A_{i,p_i-k} = \left. \frac{1}{k!} \frac{d^k}{ds^k} X_i(s) \right|_{s=\alpha_i} 
\]  (4.78)

where \( i = 1, 2, \ldots, r \), \( k = 0, 1, \ldots, p_i - 1 \) and

\[
X_i(s) = X(s)(s - \alpha_i)^{p_i} 
\]  (4.79)

Note that the computation of \( A_{i,p_i} \) does not require any differentiation, since \( k = 0 \).

**Example 4.5.4** Find the inverse Laplace transform of

\[
X(s) = \frac{s - 1}{(s + 2)^2} 
\]  (4.80)

Here we have a single repeated pole at \( s = -2 \). The expansion is therefore given by

\[
X(s) = \frac{A_{1,1}}{s + 2} + \frac{A_{1,2}}{(s + 2)^2} 
\]  (4.81)
4.5. FINDING THE INVERSE LAPLACE TRANSFORM

Using \((4.78)\), we begin with \(k = 0\) which corresponds to

\[
A_{1,2} = X(s)(s + 2)^2 \bigg|_{s=-2} = s - 1 \bigg|_{s=-2} = -3 \tag{4.82}
\]

Next, we set \(k = 1\) in \((4.78)\)

\[
A_{1,1} = \frac{d}{ds} \left[ X(s)(s + 2)^2 \right] \bigg|_{s=-2} = \frac{d}{ds} [s - 1] \bigg|_{s=-2} = 1 \tag{4.83}
\]

The partial fraction expansion is then given by

\[
X(s) = \frac{1}{s + 2} - \frac{3}{(s + 2)^2} \tag{4.84}
\]

Therefore,

\[
x(t) = e^{-2t}u(t) - 3te^{-2t}u(t) \tag{4.85}
\]

In the second method, the coefficients \(A_{i,p}, i = 1, \ldots, r\) can be found via the cover up method. The remaining coefficients, \(A_{k,p}, i = 1, \ldots, r, k = 1, \ldots, p_i - 1\) can be found by substituting values of \(s\) that are not equal to one of the poles in \(4.77\). This leads to a system of linear equations which can be used to solve for the remaining coefficients. This method is generally preferable if the order of each repeated pole as well as the number of poles is sufficiently small so that the number of unknown coefficients is at most two for hand calculations.

**Example 4.5.5** Find the inverse Laplace transform of:

\[
X(s) = \frac{s}{(s + 1)^3}
\]

\[
= \frac{A_{1,1}}{s + 1} + \frac{A_{1,2}}{(s + 1)^2} + \frac{A_{1,3}}{(s + 1)^3} \tag{4.86}
\]

Using the cover-up method we can find \(A_{1,3}\) as follows

\[
A_{1,3} = s \bigg|_{s=-1} = -1 \tag{4.87}
\]
So we are left with

\[
X(s) = \frac{s}{(s+1)^3} = \frac{A_{1,1}}{s+1} + \frac{A_{1,2}}{(s+1)^2} - \frac{1}{(s+1)^3}
\] (4.88)

Setting \( s = 0 \) in (4.88) leads to

\[
A_{1,1} + A_{1,2} = 1 \tag{4.89}
\]

and setting \( s = -2 \) in (4.88) gives

\[
-A_{1,1} + A_{1,2} = 1 \tag{4.90}
\]

These choices of \( s \) were used to simplify the linear equations to the greatest extent possible. The solution to (4.89) and (4.90) is easily found to be \( A_{1,1} = 0 \) and \( A_{1,2} = 1 \). The partial fraction expansion is given by

\[
X(s) = \frac{1}{(s+1)^2} - \frac{1}{(s+1)^3}
\]

Using the corresponding transform pairs in Table 4.2 leads to

\[
x(t) = te^{-t}u(t) - \frac{1}{2}t^2e^{-t}u(t)
\]

Distinct and Repeated Poles:

If a Laplace transform contains both distinct and repeated poles, then we would combine the expansions in (4.68) and (4.77). Perhaps the easiest way to indicate this is by way of an example:

Example 4.5.6 Find the inverse Laplace transform of

\[
X(s) = \frac{s+2}{(s+1)(s+3)(s+5)^2} = \frac{A_1}{s+1} + \frac{A_2}{s+3} + \frac{A_{3,1}}{s+5} + \frac{A_{3,2}}{(s+5)^2}
\] (4.91)
The coefficients corresponding to the distinct poles can be found using (4.69):

\[ A_1 = X(s)(s + 1)|_{s=-1} \]
\[ = \frac{s + 2}{(s + 3)(s + 5)^2}|_{s=-1} \]
\[ = \frac{1}{32} \quad (4.92) \]

\[ A_2 = X(s)(s + 3)|_{s=-3} \]
\[ = \frac{s + 2}{(s + 1)(s + 5)^2}|_{s=-3} \]
\[ = \frac{1}{8} \quad (4.93) \]

The coefficient \( A_{3,2} \) corresponding to the double pole at \( s = -5 \) can be found using (4.78) with \( k = 0 \):

\[ A_{3,2} = X(s)(s + 5)^2|_{s=-5} \]
\[ = \frac{s + 2}{(s + 1)(s + 3)}|_{s=-5} \]
\[ = -\frac{3}{8} \quad (4.94) \]

The remaining coefficient, \( A_{3,1} \) can be found using (4.78) with \( k = 1 \):

\[ A_{3,1} = \frac{d}{ds} X(s)(s + 5)^2|_{s=-5} \]
\[ = \frac{d}{ds} \left[ \frac{s + 2}{(s + 1)(s + 3)} \right]|_{s=-5} \]
\[ = \frac{(s^2 + 4s + 3) - (s + 2)(2s + 4)}{(s^2 + 4s + 3)^2}|_{s=-5} \]
\[ = -\frac{5}{32} \quad (4.95) \]

Alternately, \( A_{3,1} \) can be computed by substituting the values obtained for \( A_1, A_2 \) and \( A_{3,2} \) back into (4.91) and then substituting an arbitrary value for \( s \) that does not equal one of the poles as indicated earlier, like \( s = 0 \).
This leads to a simple equation whose only unknown is \( A_{3,1} \). The partial fraction of \( X(s) \) is then given by:

\[
X(s) = \frac{s + 2}{(s + 1)(s + 3)(s + 5)^2} = \frac{\frac{1}{32}}{s + 1} + \frac{\frac{1}{8}}{s + 3} - \frac{\frac{5}{32}}{s + 5} - \frac{\frac{3}{8}}{(s + 5)^2}
\]  

(4.96)

Applying the inverse Laplace transform to each of the individual terms in (4.96) and using linearity gives:

\[
x(t) = \frac{1}{32}e^{-t}u(t) + \frac{1}{8}e^{-3t}u(t) - \frac{5}{32}e^{-5t}u(t) - \frac{3}{8}te^{-5t}u(t)
\]  

(4.97)

The following example looks at a case where \( X(s) \) is a rational function, but is not proper.

**Example 4.5.7** Find the inverse Laplace transform of

\[
X(s) = \frac{s^2 + 6s + 1}{s^2 + 5s + 6}
\]  

(4.98)

Here since \( q = p = 2 \), we cannot perform a partial fraction expansion. First we must perform a long division, this leads to:

\[
X(s) = 1 + \frac{s - 5}{s^2 + 5s + 6} = 1 + \frac{s - 5}{(s + 2)(s + 3)}
\]  

(4.99)

where \( s - 5 \) is the remainder resulting from the long division. The quotient of 1 is called a direct term. In general, the direct term corresponds to a polynomial in \( s \). The partial fraction expansion is performed on the quotient term, which is always proper:

\[
\frac{s - 5}{(s + 2)(s + 3)} = \frac{A_1}{s + 2} + \frac{A_2}{s + 3}
\]  

(4.100)

Using (4.68) gives

\[
A_1 = \left. \frac{s - 5}{s + 3} \right|_{s = -2} = -7
\]  

(4.101)
4.5. FINDING THE INVERSE LAPLACE TRANSFORM

\[ A_2 = \frac{s - 5}{s + 2} \bigg|_{s = -3} \]
\[ = 8 \quad (4.102) \]

So we have

\[ X(s) = 1 - \frac{7}{s + 2} + \frac{8}{s + 3} \quad (4.103) \]

and

\[ x(t) = \delta(t) - 7e^{-2t}u(t) + 8e^{-3t}u(t) \quad (4.104) \]

Complex Conjugate Poles:

Some poles occur in complex conjugate pairs as in the following example:

**Example 4.5.8** Find the output of a filter whose impulse response is \( h(t) = e^{-5t}u(t) \) and whose input is given by \( x(t) = \cos(2t)u(t) \). Since the output is given by \( y(t) = x(t) * h(t) \), its Laplace transform is \( Y(s) = X(s)H(s) \). Therefore using the table of Laplace transform pairs (Table 4.2) we have

\[ X(s) = \frac{s}{s^2 + 4} \quad (4.105) \]

and

\[ H(s) = \frac{1}{s + 5} \quad (4.106) \]

which leads to

\[ Y(s) = \frac{s}{(s^2 + 4)(s + 5)} \]
\[ = \frac{s}{(s + j2)(s - j2)(s + 5)} \quad (4.107) \]
\[ = \frac{A_1}{s + j2} + \frac{A_2}{s - j2} + \frac{A_3}{s + 5} \]

The poles are at \( s = j2, -j2 \) and -5, all of which are distinct, so equation (4.68) applies:

\[ A_1 = Y(s)(s + j2)|_{s = -j2} \]
\[ = \frac{s}{(s - j2)(s + 5)} \bigg|_{s = -j2} \]
\[ = -j2 \]
\[ = \frac{5 + j2}{58} \quad (4.108) \]
The second coefficient is

\[ A_2 = Y(s)(s - j2)|_{s=j2} = \frac{5 - j2}{58} \]  

(4.109)

The calculations for \( A_2 \) where omitted but it is easy to see that \( A_2 \) will be the complex conjugate of \( A_1 \) since all of the terms in \( A_2 \) are the complex conjugates of those in \( A_1 \). Therefore, when there are a pair of complex conjugate poles, we need only calculate one of the two coefficients and the other will be its complex conjugate. The last coefficient corresponding to the pole at \( s = -5 \) is found using

\[ A_3 = Y(s)(s + 5)|_{s=-5} = \frac{s}{(s^2 + 4)}|_{s=-5} = -\frac{5}{29} \]  

(4.110)

This gives

\[ Y(s) = \frac{5+j2}{58} e^{-j2t}u(t) + \frac{5-j2}{58} e^{j2t}u(t) - \frac{5}{29} e^{-5t}u(t) \]  

(4.111)

We can now easily find the inverse Laplace transform of each individual term in the right-hand side of (4.111):

\[ y(t) = \frac{5+j2}{58} e^{-j2t}u(t) + \frac{5-j2}{58} e^{j2t}u(t) - \frac{5}{29} e^{-5t}u(t) \]  

(4.112)

At this point, we are technically done, however the first two terms in \( y(t) \) are complex and also happen to be complex conjugates of each other. So we can simplify further by noting that

\[ \frac{5+j2}{58} e^{-j2t}u(t) + \frac{5-j2}{58} e^{j2t}u(t) = 2Re\left( \frac{5-j2}{58} e^{j2t}u(t) \right) = 2Re(0.0928e^{-j0.3805} e^{j2t}u(t)) = 0.1857 \cos(2t - 0.3805)u(t) \]  

(4.113)

The simplified answer is given by

\[ y(t) = 0.1857 \cos(2t - 0.3805)u(t) - 0.1724e^{-5t}u(t) \]  

(4.114)
We note that the answer contains a transient term, $-0.1724e^{-10t}u(t)$, and a steady-state term $0.1857\cos(2t - 0.3805)$. The steady-state term corresponds to the sinusoidal steady-state response of the filter (see Section 3.11). It can be readily seen that the frequency response of the filter is

$$H(j\Omega) = \frac{1}{5 + j\Omega}$$

and therefore $|H(j2)| = 0.1857$ and $\angle H(j2) = -0.3805$ (see Example 3.11.1).

4.6 Transfer Functions and Frequency Response

We saw in Section 4.3.6 that the transfer function of a linear time-invariant system is given by

$$H(s) = \frac{Y(s)}{X(s)}$$

If we assume that $H(s)$ is a rational function of $s$ then we can write

$$H(s) = \frac{(s - \beta_1)(s - \beta_2)\cdots(s - \beta_q)}{(s - \alpha_1)(s - \alpha_2)\cdots(s - \alpha_p)}$$

where $\beta_1, \beta_2, \ldots, \beta_q$ are the zeros, and $\alpha_1, \alpha_2, \ldots, \alpha_p$ are the poles of $H(s)$. The poles and zeros are points in the $s$-plane where the transfer function is either non-existent (for a pole) or zero (for a zero). These points are sometimes plotted in the $s$-plane with “×” representing the location of a pole and a “◦” representing the location of a zero. Since

$$|H(j\Omega)| = |H(s)|_{s=j\Omega}$$

using Table 1.1, we have

$$|H(j\Omega)| = \frac{|j\Omega - \beta_1||j\Omega - \beta_2|\cdots|j\Omega - \beta_q|}{|j\Omega - \alpha_1||j\Omega - \alpha_2|\cdots|j\Omega - \alpha_p|}$$

and

$$\angle H(j\Omega) = \sum_{k=1}^{q} \angle(j\Omega_k - \beta_k) - \sum_{l=1}^{p} \angle(j\Omega_l - \alpha_l)$$

Each of the quantities $j\Omega - \beta_k$ represents a vector from the zero $\beta_k$ to the $j\Omega$ axis in the complex $s$-plane. Likewise, the quantities $j\Omega - \alpha_k$ are vectors from the pole $\alpha_k$ to the $j\Omega$ axis.
Example 4.6.1 Consider a first-order lowpass filter with transfer function

\[ H(s) = \frac{1}{s + 2} \]  

(4.121)

Then

\[ |H(j\Omega)| = \frac{1}{|j\Omega + 2|} \]  

(4.122)

The quantity \( j\Omega + 2 \) is a vector from the pole at \( s = -2 \) to the \( j\Omega \) axis in the complex plane. The magnitude of the frequency response is the inverse of the magnitude of this vector. The length of \( j\Omega + 2 \) increases as \( \Omega \) increases thereby making \( |H(j\Omega)| \) decrease as one would expect of a lowpass filter.

Example 4.6.2 Consider a first-order highpass filter with transfer function

\[ H(s) = \frac{s}{s + 2} \]  

(4.123)

Then

\[ |H(j\Omega)| = \frac{|j\Omega|}{|j\Omega + 2|} \]  

(4.124)

When \( \Omega = 0 \), \( |H(j\Omega)| = 0 \), but as \( \Omega \) approaches infinity, the two vectors \( j\Omega \) and \( j\Omega + 2 \) have equal lengths, so the magnitude of the frequency response approaches unity.

Example 4.6.3 Let’s now look at the transfer function corresponding to a second-order filter

\[ H(s) = \frac{s + 1}{(s + 1 + j5)(s + 1 - j5)} \]  

(4.125)

Or

\[ |H(j\Omega)| = \frac{|j\Omega + 1|}{|j\Omega + 1 + j5||j\Omega + 1 - j5|} \]  

(4.126)

The pole-zero plot for this transfer function is shown in Figure 4.3. The corresponding magnitude and phase of the frequency response are shown in Figure 4.4. The two poles are at \( s = -3 \pm j5 \) and the zero is at \( s = -1 \). When the frequency gets close to either one of the poles, the frequency response magnitude increases since the lengths of one of the vectors \( j\Omega \pm j5 \) is small.
In the previous example we saw that as the poles get close to the $j\Omega$ axis in the $s$-plane, the frequency response magnitude increases at frequencies that are close to the poles. In fact, when a pole is located directly on the $j\Omega$ axis, the filter’s frequency response magnitude becomes infinite, and will begin to oscillate. If a zero is located on the $j\Omega$ axis, the filter’s frequency response magnitude will be zero.

**Example 4.6.4** Suppose a filter has transfer function

$$H(s) = \frac{s}{s^2 + 25} \tag{4.127}$$

The two poles are at $s = \pm j5$ and there is a zero at $s = 0$. From Table 4.2, the impulse response of the filter is $h(t) = \cos(5t)u(t)$. This impulse response doesn’t die out like most of the impulse responses we’ve seen. Instead, it oscillates at a fixed frequency. The filter is called an oscillator. Oscillators are useful for generating high-frequency sinusoids used in wireless communications.

### 4.7 Filter Stability

We define a filter as being *stable* if a bounded input produces a bounded output. This is sometimes called *BIBO* stability.

**Example 4.7.1** Consider the filter seen in Example 4.6.4. The impulse response for this filter is $h(t) = \cos(5t)u(t)$. If the input to this filter is
Figure 4.4: Magnitude and phase of frequency response for Example 4.6.3.

\[ x(t) = \cos(5t)u(t), \text{ then the output is given by} \]

\[ y(t) = \int_{0}^{\infty} \cos(5\tau) \cos(5(t-\tau))u(t-\tau)d\tau \]

\[ = \int_{0}^{t} \cos(5\tau) \cos(5(t-\tau))d\tau \]

\[ = \frac{1}{2} \int_{0}^{t} \cos(10\tau - 5t)d\tau + \frac{1}{2} \cos(5t) \int_{0}^{t} d\tau \]

\[ = \frac{1}{2} \int_{0}^{t} \cos(10\tau - 5t)d\tau + \frac{1}{2} t \cos(5t) \quad (4.128) \]

where in the third line, we have used the trigonometric identity

\[ \cos(\theta_1) \cos(\theta_2) = \frac{1}{2} (\cos(\theta_1 - \theta_2) + \cos(\theta_1 + \theta_2)) \]
As $t$ approaches infinity, then it is clear that $y(t)$ also becomes unbounded. Therefore the filter is unstable.

If a transfer function is rational, it can be expressed as a sum of any direct term that may be present plus a proper rational function ($q < p$).

$$H(s) = c_ms^m + c_{m-1}s^{m-1} + \cdots + c_1s + c_0 + \frac{b_q s^q + b_{q-1}s^{q-1} + \cdots + b_1s + b_0}{a_p s^p + a_{p-1}s^{p-1} + \cdots + a_1s + a_0}$$ \hfill (4.129)

The direct terms can be shown to produce unbounded outputs when the input is a step function (which of course, is bounded). The proper rational function produces output terms that depend on whether poles are distinct or repeated and whether these are real or complex:

- **Distinct real poles**, $s = \sigma_k$ lead to an impulse response with terms:
  $$Ke^{\sigma_k t}u(t)$$

- **Distinct complex conjugate poles** $s = \sigma_k \pm \omega_k$ produce impulse response terms:
  $$Ke^{\sigma_k t} \cos(\omega_k + \theta)u(t)$$

- **Repeated real poles** $s = \sigma_k$ produce impulse response terms having the form:
  $$Kt^n e^{\sigma_k t}u(t)$$

The quantities $K$ and $\theta$ are constants while $n$ is a positive integer. In all of these cases the filter impulse response dies out with time if the poles have negative real parts. If the poles have zero or positive real parts, then the impulse response terms either oscillate or grow with time. When this happens then one can always find a bounded input that produces an unbounded output. Therefore, in order for a filter with a rational transfer function to have BIBO stability, the transfer function should be proper and the poles of the transfer function should have negative real parts.

$$\quad$$

$$\quad$$

(4.130)

(4.131)
4.8 Exercises

1. Find the Laplace Transform of the following signals, for each case indicate the Laplace transform property that was used:

(a) \( x(t) = 4e^{-0.2t}u(t) \)
(b) \( x(t) = 4te^{-0.2t}u(t) \)
(c) \( x(t) = 4e^{-0.2(t-10)}u(t - 10) \)
(d) \( x(t) = \delta(t - 5) \)
(e) \( x(t) = 10tu(t) \)
(f) \( x(t) = \sin(10\pi t)u(t) \)
(g) \( x(t) = e^{-3t}\sin(10\pi t)u(t) \)
(h) \( x(t) = \text{rect}(t - 0.5, 1) \)

2. Suppose that two filters having impulse responses \( h_1(t) \) and \( h_2(t) \) are cascaded (i.e. connected in series). Find the transfer function of the equivalent filter assuming \( h_1(t) = 10e^{-10t}u(t) \) and \( h_2(t) = 5e^{-5t}u(t) \).

3. Find the inverse Laplace transforms of the following:

(a) \( X(s) = \frac{e^{-2s}}{s+5} \)
(b) \( X(s) = \frac{se^{-s}}{s^2+9} \)
(c) \( X(s) = \frac{1}{(s+3)^2} \)
(d) \( X(s) = 10 \)
(e) \( X(s) = \frac{10}{s^2} \)
(f) \( X(s) = \frac{e^{-s}}{s} \)

4. Use partial fraction expansions to find the inverse Laplace transforms of the following:

(a) \( X(s) = \frac{s+2}{(s+5)(s+2)(s+1)} \)
(b) \( X(s) = \frac{s+1}{(s+2)^3(s+3)} \)
(c) \( X(s) = \frac{s}{(s^2+9)(s+2)} \)
(d) \( X(s) = \frac{s^2-3s+1}{(s+1)(s+2)} \)
5. Consider a filter having impulse response \( h(t) = e^{-2t}u(t) \). Use Laplace transforms to find the output of the filter when the input is given by:

(a) \( x(t) = u(t) \)

(b) \( x(t) = tu(t) \)

(c) \( x(t) = e^{-4t}u(t) \)

(d) \( x(t) = \cos(10t)u(t) \)
Chapter 5

Sampling Continuous-Time Signals

5.1 Discrete-Time Signals

A discrete-time signal is defined as:

\[ x[n] \equiv x(nT_s), n = \ldots, -2, -1, 0, 1, \ldots \] (5.1)

In other words, a discrete-time signal \( x[n] \) results by sampling the continuous time signal \( x(t) \) every \( T_s \) seconds where \( T_s \) is called the sampling interval. The sampling frequency is given by \( 1/T_s \) in \( \text{samples/sec} \). Note the difference in notation, discrete-time signals use “square brackets” while continuous-time signals use “parentheses”. Also, discrete-time signals are sequences of numbers, indexed by the integers, whereas continuous-time signals are functions of the continuous real variable \( t \). This is illustrated in Figure 5.1. As we shall see, much of what we have already learned regarding continuous-time signals still applies to discrete-time signals, so much of our work is already done!

5.2 Why Sample?

Now that we know what a discrete-time signal is, let’s try to look at why they are important. The most important reason for sampling signals is that in

\(^1\text{technically, the correct units are sec}^{-1}\)
Figure 5.1: Sampling a continuous-time signal, $x(t)$, every $T_s$ seconds produces the discrete-time signal, $x[n] = x(nT_s)$.

discrete-time, signals are easier to manipulate on a computer. Discrete-time signals can be manipulated in such a way as to enable them to be stored or transmitted efficiently. Examples of discrete-time signals in everyday life include the speech signal inside your cell phone and the data which is stored in a music CD or MP3 file. Of course when you listen to an MP3 file, you are listening to a continuous-time signal. The discrete-time data in the MP3 file must be converted back to a continuous-time signal in order to hear it.

5.3 The Sampling Theorem

Our goal in this section is to determine conditions which are necessary to recover $x(t)$ from its samples ($x[n] = x(nT_s)$). To this end, lets begin by forming the signal

$$x_s(t) = \sum_{k=-\infty}^{\infty} x[k] \delta(t - kT_s)$$ (5.2)
5.3. THE SAMPLING THEOREM

A picture of $x_s(t)$ is shown in Figure 5.2. Notice that it consists of a series of impulse functions, occurring every $T_s$ seconds, where each impulse function has an area given by successive elements of the discrete-time signal $x[n]$. Alternately, we can write

$$x_s(t) = x(t)s(t) \quad (5.3)$$

where $s(t)$ is give by

$$s(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s) \quad (5.4)$$

This is shown in Figure 5.3 which shows $s(t)$ to be a sequence of unit-area impulse functions. While $x_s(t)$ is certainly not equal to $x(t)$ it is a continuous-time signal, and so we hold out hope that some processing, perhaps a filtering operation of $x_s(t)$, will give us $x(t)$. We will now look at the relationship between $X_s(j\Omega)$ and $X(j\Omega)$. Since $x_s(t)$ is the product of two signals, $x(t)$ and $s(t)$, we can apply the modulation property of the Fourier Transform:

$$X_s(j\Omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Theta)S(j(\Omega - \Theta))d\Theta \quad (5.5)$$

where $X(j\Omega)$ and $S(j\Omega)$ are the Fourier Transforms of $x(t)$ and $s(t)$, respectively. To find $S(j\Omega)$, we note that it is a periodic signal having period $T_s$. Therefore, we can apply the formula for the Fourier Transform of a periodic signal derived in Section 3.6. In the exercises, we you will be asked to show that complex form of the Fourier Series of $s(t)$ is

$$s(t) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} e^{j\Omega_s n} \quad (5.6)$$
where $\Omega_s = 2\pi/T_s$ is the sampling frequency in rad/sec. The Fourier Transform of $s(t)$ is therefore given by

$$S(j\Omega) = \frac{2\pi}{T_s} \sum_{n=-\infty}^{\infty} \delta(\Omega - n\Omega_s) \quad (5.7)$$

Inserting (5.7) into (5.5), using the fact that an integral of a sum of functions is the same as the sum of the integral of the individual functions, and applying the sifting property of the unit impulse leads to the following steps:

$$X_s(j\Omega) = \frac{1}{T_s} \int_{-\infty}^{\infty} X(j\Theta) \left[ \sum_{n=-\infty}^{\infty} \delta(\Omega - \Theta - n\Omega_s) \right] d\Theta \quad (5.8)$$

$$= \frac{1}{T_s} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} X(j\Theta) \delta(\Omega - \Theta - n\Omega_s) d\Theta \quad (5.9)$$

$$= \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X(j(\Omega - n\Omega_s)) \quad (5.10)$$

It appears as though $X_s(j\Omega)$ consists of a sum of frequency-shifted versions of $X(j\Omega)$, where the amount of the frequency shifts, $n\Omega_s$, is an integer multiple of the sampling frequency. There is also a scaling by $1/T_s$. Figure 5.4 shows the relationship between $X(j\Omega)$ and $X_s(j\Omega)$. Note that in Figure 5.4 we have assumed that adjacent spectra in $X_s(j\Omega)$ do not overlap. In this case, we can recover $X(j\Omega)$ from $X_s(j\Omega)$ by simply lowpass filtering $x_s(t)$! This is illustrated in Figure 5.4.

$$X_s(j\Omega) = \frac{1}{T_s} \int_{-\infty}^{\infty} X(j\Theta) \left[ \sum_{n=-\infty}^{\infty} \delta(\Omega - \Theta - n\Omega_s) \right] d\Theta \quad (5.8)$$

$$= \frac{1}{T_s} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} X(j\Theta) \delta(\Omega - \Theta - n\Omega_s) d\Theta \quad (5.9)$$

$$= \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X(j(\Omega - n\Omega_s)) \quad (5.10)$$

From the plot of $X_s(j\Omega)$ in Figure 5.5, it can be seen that adjacent frequency-shifted versions of $X(j\Omega)$ will not overlap as long as $\Omega_N < \Omega_s - \Omega_N$ or

$$\Omega_s > 2\Omega_N \quad (5.12)$$

If this condition is not met, then $x(t)$ cannot be recovered from $x_s(t)$ by lowpass filtering. Hence we have the famous Sampling Theorem:

*If $x(t)$ is band-limited to $\Omega_N$, and is sampled a frequency of $1/T_s$ samples per second to give the discrete-time signal $x[n]$, then $x(t)$ may be recovered
5.4. Converting from Discrete Time to Continuous Time

from \( x[n] \) as long as \( \Omega_s = 2\pi/T_s > 2\Omega_N \). If the requirements in the sampling theorem are not met, then a condition known as aliasing results. Aliasing occurs when adjacent frequency-shifted versions of \( X(j\Omega) \) overlap. Since the copies are being added (see equation (5.10)), the information contained in the aliased frequency bands is lost. This is illustrated in Figure 5.6.

5.4 Converting from Discrete Time to Continuous Time

In the previous section it was mentioned that, if the conditions in the Sampling Theorem are met, it is possible to recover \( x(t) \) from \( x[n] \) by lowpass filtering. In this section, we will take a closer look at this. As indicated previously the steps for going from discrete-time to continuous-time are:

1. Form the signal:
   \[
x_s(t) = \sum_{k=-\infty}^{\infty} x[k]\delta (t - kT_s) \tag{5.13}
   \]

2. Filter \( x_s(t) \) with an ideal lowpass filter \( h_{LP}(t) \) to obtain \( \hat{x}(t) \):
   \[
x(t) = x_s(t) * h_{LP}(t) \tag{5.14}
   \]
   \[
   = \int_{-\infty}^{\infty} x_s(\tau)h_{LP}(t - \tau)d\tau
   \]
   The filter \( h_{LP}(t) \) is sometimes called a reconstruction filter. As mentioned in the previous section, the frequency response of the reconstruction filter should be that of an ideal lowpass filter:

   \[
   H_{LP}(j\Omega) = \begin{cases} 
   T_s, & |\Omega| < \frac{\Omega_s}{2} \\
   0, & \text{otherwise}
   \end{cases} \tag{5.15}
   \]

Under these conditions, \( \hat{x}(t) = x(t) \).

The above steps raise a disturbing question. When generating the signal \( x_s(t) \), how practical is it to generate impulse functions? Since these are infinite in amplitude and infinitesimally narrow there are likely to be few signal generators on the planet which are capable of generating unit impulse
functions. To avoid having to generate an impulse function, we can instead substitute the expression for \( x_s(t) \) in (5.13) into (5.14) to get:

\[
\hat{x}(t) = \int_{-\infty}^{\infty} \left[ \sum_{k=-\infty}^{\infty} x[k] \delta(\tau - kT_s) \right] h_{LP}(t - \tau) d\tau \tag{5.16}
\]

Rearranging the order of summation and integration and invoking the sifting property of the impulse function gives:

\[
\hat{x}(t) = \sum_{k=-\infty}^{\infty} x[k] \int_{-\infty}^{\infty} \delta(\tau - kT_s) h_{LP}(t - \tau) d\tau \tag{5.17}
\]

\[
= \sum_{k=-\infty}^{\infty} x[k] h_{LP}(t - kT_s) \tag{5.18}
\]

Equation (5.18) has no impulse function! On the other hand, it does require the impulse response of an ideal lowpass filter. Using a result from one of the exercises in Chapter 3, we get

\[
h_{LP}(t) = \frac{\sin \left( \frac{\Omega_s}{2} t \right)}{\frac{\Omega_s}{2} t} \tag{5.19}
\]

substituting this result into (5.18) gives

\[
\hat{x}(t) = \sum_{k=-\infty}^{\infty} x[k] \frac{\sin \left( \frac{\Omega_s}{2} (t - kT_s) \right)}{\frac{\Omega_s}{2} (t - kT_s)} \tag{5.20}
\]

This expression can be interpreted to mean that \( x(t) \) can be obtained by interpolating its samples with the sinc function \( h_{LP}(t) \).

### 5.5 An Alternative Expression for \( X_s(j\Omega) \)

Finally, we will examine a different formulation for \( S_s(j\Omega) \). Recall that \( x_s(t) \) is formed as follows:

\[
x_s(t) = \sum_{k=-\infty}^{\infty} x[k] \delta(t - kT_s) \tag{5.21}
\]
Applying the notation, \( X(k\Omega) = F\{x(t)\} \) to (5.23) gives

\[
X_s(j\Omega) = \sum_{k=-\infty}^{\infty} x[k]F\{\delta(t-kT_s)\} = \sum_{k=-\infty}^{\infty} x[k]e^{-j\Omega T_s k}
\]

\[(5.22)\]

\[(5.23)\]

5.6 Exercises

1. A continuous-time signal has the following Fourier transform:

\[
\begin{array}{c}
\text{1} \\
\text{\(-\Omega_N\)} \\
\text{\(\Omega_N\)} \\
\text{\(\Omega\)}
\end{array}
\]

Assume that \( \Omega_N = 10\pi \).

(a) What is the maximum sampling interval, \( T \) which will enable \( x(t) \) to be recovered from its samples?

(b) Sketch the Fourier transform of \( x_s(t) \) for the value of \( T \) selected in part a.

(c) Sketch the discrete-time Fourier transform, \( X(e^{j\omega}) \) of \( x[n] \) for the value of \( T \) selected in part a.

2. Repeat part 1 for \( T = 0.2 \).

3. Suppose \( x(t) = \cos(10\pi t) \) is sampled using \( T = 0.05 \).

(a) Sketch the Fourier transform of \( x(t) \).

(b) Sketch the Fourier transform of \( x_s(t) \).

(c) Sketch the discrete-time Fourier transform, \( X(e^{j\omega}) \) of \( x[n] \).
Figure 5.3: The signal $x_s(t)$ can also be obtained by multiplying $x(t)$ and $s(t)$ since $x[n] = x(nT_s)$. 

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Chapter 5. Sampling Continuous-Time Signals
Figure 5.4: Relationship between $X(j\Omega)$ and $X_s(j\Omega)$, assuming no aliasing has occurred.
Figure 5.5: Lowpass filtering $x_s(t)$ with an ideal lowpass filter allows us to recover $x(t)$. 
Figure 5.6: Aliasing results when $\Omega_s < 2\Omega_N$. 
Chapter 6

The Discrete-Time Fourier Transform

6.1 Definition of \( X(e^{j\omega}) \)

The discrete-time Fourier Transform is defined as

\[
X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \quad (6.1)
\]

The variable \( \omega \) corresponds to discrete-time frequency and has units of rad. If we compare \( X(e^{j\omega}) \) with the alternative expression for \( X_s(j\Omega) \) in (5.23) we find that the two expressions differ only in the frequency variable used. In fact, if we set

\[
\omega = \Omega T_s \quad (6.2)
\]

in (6.1), we obtain (5.23)! This relationship between \( X(e^{j\omega}) \) and \( X_s(j\Omega) \) is important because it tells us how the discrete-time Fourier Transform is related to the continuous-time Fourier Transform. In fact, using (5.10), the exact relationship is given by

\[
X(e^{j\omega}) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X\left(j \left(\frac{\omega}{T_s} - 2\pi n\right)\right) \quad (6.3)
\]

We saw in Chapter 5.1 that \( X_s(j\Omega) \) is periodic with period \( \Omega_s = 2\pi/T_s \). Using (6.2), it follows that \( X(e^{j\omega}) \) is periodic with period \( 2\pi \), i.e. \( \Omega_s T_s = 2\pi \). Figure 6.1 illustrates this relationship. Note also that the periodicity of
$X(e^{j\omega})$ is implicit from its definition in (6.1) since $e^{j\omega} = e^{j(\omega + 2\pi k)}$ for any integer $k$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure6_1.png}
\caption{Relationship between $X(j\Omega)$ and $X(e^{j\omega})$.}
\end{figure}

### 6.2 Properties of the Discrete-Time Fourier Transform

The properties associated with $X(e^{j\omega})$ are very similar to those of the continuous-time FT. Therefore, we will leave their proof as exercise problems. The properties are summarized in Table 6.2.

### 6.3 Some Common Discrete-Time Signals

Table 6.3 lists some common discrete-time Fourier Transform pairs.
### 6.4 Exercises

1. Consider the following discrete-time signal:

\[ x[n] = (0.5)^n u[n] \]

Find the discrete-time Fourier transform \( Y(e^{j\omega}) \) and then sketch \( |Y(e^{j\omega})| \) for:

(a) \( y[n] = x[n] \)
(b) \( y[n] = x[n] - 10 \)
(c) \( y[n] = x[n] e^{j0.3\pi n} \)

---

<table>
<thead>
<tr>
<th>Property</th>
<th>( y[n] )</th>
<th>( Y(e^{j\omega}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linearity</td>
<td>( \alpha x_1[n] + \beta x_2[n] )</td>
<td>( \alpha X_1(e^{j\omega}) + \beta X_2(e^{j\omega}) )</td>
</tr>
<tr>
<td>Time Shift</td>
<td>( x[n - n_0] )</td>
<td>( X(e^{j\omega}) e^{-j\omega n_0} )</td>
</tr>
<tr>
<td>Frequency Shift</td>
<td>( x[n] e^{j\omega_0 n} )</td>
<td>( X(e^{j(\omega - \omega_0)}) )</td>
</tr>
<tr>
<td>Convolution</td>
<td>( x[n] * h[n] )</td>
<td>( X(e^{j\omega}) H(e^{j\omega}) )</td>
</tr>
<tr>
<td>Modulation</td>
<td>( x[n] w[n] )</td>
<td>( \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j(\omega - \theta)}) W(e^{j\theta}) d\theta )</td>
</tr>
</tbody>
</table>

Table 6.1: Discrete-Time Fourier Transform properties.

<table>
<thead>
<tr>
<th>( x[n] )</th>
<th>( X(e^{j\omega}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta[n - n_0] )</td>
<td>( e^{-j\omega n_0} )</td>
</tr>
<tr>
<td>( e^{j\omega_0 n} )</td>
<td>( 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi k) )</td>
</tr>
<tr>
<td>( \cos(\omega_0 n) )</td>
<td>( \pi \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi k) + \pi \sum_{k=-\infty}^{\infty} \delta(\omega + \omega_0 - 2\pi k) )</td>
</tr>
<tr>
<td>( Ka^n u[n] )</td>
<td>( \frac{K}{1 - ae^{-j\omega}} ), (</td>
</tr>
</tbody>
</table>

Table 6.2: Some common Fourier Transform pairs.
(d) \( y[n] = x[n] \sin[0.1\pi n] \)
(e) \( y[n] = x[n] \ast \delta[n - 10] \)
(f) \( y[n] = x^2[n] \)

2. Find the discrete-time Fourier Transforms of:

(a) \( x[n] = (-0.1)^{n-5}u[n - 5] \)
(b) \( x[n] = \delta[n - 10] \)
(c) \( x[n] = e^{j0.3\pi n} \)
(d) \( x[n] = \cos(0.3\pi n) \)
(e) \( x[n] = e^{j0.3\pi n} \cos(0.3\pi n) \)
Chapter 7

Discrete-Time Filters

7.1 Linear, Time Invariant Filters

A discrete-time filter can be thought of as a black box with a single input signal, \( x[n] \) and a single output signal \( y[n] \) (see Figure 7.1). We can represent the operation performed by the filter using the operator \( L[\cdot] \), i.e.

\[
y[n] = L[x[n]]
\]

(7.1)

A filter is time invariant if \( y[n] = L[x[n]] \) implies that \( y[n-n_0] = L[x[n-n_0]] \). In other words, the output of a time-invariant filter is always the same (up to a time shift), regardless of when the input signal is applied. If we now assume that \( y_1[n] = L[x_1[n]] \) and \( y_2[n] = L[x_2[n]] \), then if

\[
\alpha y_1[n] + \beta y_2[n] = L[\alpha x_1[n] + \beta x_2[n]]
\]

(7.2)

then the filter is said to be linear and it satisfies the superposition principle.

![Figure 7.1: A discrete-time filter.](image-url)
As was seen in the previous chapter, any discrete-time signal can be composed of sums of weighted, time delayed impulses

\[ x[n] = \sum_{m=-\infty}^{\infty} x[m] \delta[n-m] \] (7.3)

Now suppose that \( h[n] = L[\delta[n]] \). The signal \( h[n] \) is called the impulse response of the filter. It follows by linearity and time invariance that

\[ y[n] = L[x[n]] \]
\[ = L \left[ \sum_{m=-\infty}^{\infty} x[m] \delta[n-m] \right] \]
\[ = \sum_{m=-\infty}^{\infty} x[m] L[\delta[n-m]] \]
\[ = \sum_{m=-\infty}^{\infty} x[m] h[n-m] \equiv x[n] * h[n] \] (7.4)

The right hand side of (7.4) is called a convolution sum. A simple change of variables can also be used to show that (see Exercises)

\[ y[n] = \sum_{m=-\infty}^{\infty} h[m] x[n-m] \] (7.5)

### 7.2 Frequency Response

The frequency response of a discrete-time filter is the Fourier Transform of the impulse response, \( H(e^{j\omega}) \). We observe that since \( y[n] = x[n] * h[n] \), using the Fourier Transform of a convolution gives \( Y(e^{j\omega}) = X(e^{j\omega}) H(e^{j\omega}) \). The frequency response of a discrete-time filter is therefore given by

\[ H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} \] (7.6)

The magnitude of the frequency response gives us the filter gain at different frequencies. A lowpass filter, allows low frequencies to pass but attenuates (removes) high frequencies. The magnitude of a lowpass filter is shown in Figure 7.2, along with the frequency response magnitude for a highpass filter.

In the next section, we will look at some special cases of discrete-time filters.
7.3 Finite Impulse Response (FIR) Filters

The first type of discrete-time filter we will look at is called a finite impulse response (FIR) filter. The relationship between the output and input of this filter is given by

\[ y[n] = \sum_{k=0}^{q} b_k x[n - k] \]  

(7.7)

This equation is called a difference equation. The quantities \( b_k, k = 0, \ldots, q \) in (FIR) are called the coefficients for the FIR filter. Comparing (7.7) with (7.5), it follows that

\[ h[n] = \begin{cases} 
  b_n, & 0 \leq k \leq q \\
  0, & \text{otherwise}
\end{cases} \]  

(7.8)

Figure 7.2: Frequency response magnitudes for (a) lowpass and (b) highpass filters.
CHAPTER 7. DISCRETE-TIME FILTERS

The name “FIR” results from the fact that \( h[n] \) has only a finite number of nonzero elements. Given the impulse response of the filter, and an input signal, it is possible to compute the output of the filter. This can be done in a manner very similar to continuous-time convolution. The procedure is as follows:

1. Set \( n = 0 \).

2. Sketch \( h[m] \) and \( x[n-m] \) as discrete-time signals having \( m \) as the time variable.

3. Compute \( h[m]x[n-m] \) and sketch the result as a function of the discrete time variable, \( m \).

4. Set \( y[n] \) to the sum of \( h[m]x[n-m] \) over all values of \( m \).

5. Change \( n \) to \( n + 1 \) and go to step 2.

Click on the following link after you have familiarized yourself with it, consider the following questions:

- What is the impulse response \( h[n] \)?
- Why is \( y[n] = 0 \) for \( n < 0 \)?
- What is \( q \) for this example?
- If in general, \( x[n] \) has a length of \( N \), and \( h[n] \) has a length of \( q + 1 \), what is the length of \( y[n] \)?

The frequency response of an FIR filter can be found by computing the discrete-time Fourier Transform of both sides of (7.7). If we let \( F \{ \cdot \} \) be the Fourier Transform operation, then we have

\[
Y \left( e^{j\omega} \right) = F \left\{ \sum_{k=0}^{q} b_k x[n-k] \right\} \quad (7.9)
\]

If we now use the linearity property of the Fourier Transform, we have

\[
Y \left( e^{j\omega} \right) = \sum_{k=0}^{q} b_k F \{ x[n-k] \} \quad (7.10)
\]
Next, using the time shift property of the Fourier Transform gives

\[ Y(e^{j\omega}) = \sum_{k=0}^{q} b_k X(e^{j\omega}) e^{-j\omega k} \] (7.11)

Finally, solving for \( H(e^{j\omega}) = Y(e^{j\omega}) / X(e^{j\omega}) \) gives

\[ H(e^{j\omega}) = \sum_{k=0}^{q} b_k e^{-j\omega k} \] (7.12)

**Example 7.3.1** Suppose that we have a filter with difference equation

\[ y[n] = x[n] + 2x[n - 1] + x[n - 2] \]

Comparing this difference equation with (7.7), we see that for this filter, \( q = 2, b_0 = 1, b_1 = 2, \) and \( b_2 = 1 \). Using (7.12), we find that

\[ H(e^{j\omega}) = 1 + 2e^{-j\omega} + e^{-2j\omega} \]

To plot the magnitude of the \( H(e^{j\omega}) \), we can first factor out an \( e^{-j\omega} \):

\[ H(e^{j\omega}) = e^{-j\omega} (e^{j\omega} + 2 + e^{-j\omega}) = e^{-j\omega} (2 + 2 \cos \omega) \] (7.13)

Taking the magnitude of this expression gives

\[ H(e^{j\omega}) = |(2 + 2 \cos \omega)| \]

Which is plotted in Figure 7.3 along with the phase of \( H(e^{j\omega}) \). The phase is found by noting that

\[ \angle H(e^{j\omega}) = \angle e^{-j\omega} = -\omega \]

### 7.4 Infinite Impulse Response (IIR) Filters

The difference equation relating the output and input of an infinite impulse response (IIR) filter is given by

\[ y[n] = \sum_{k=0}^{q} b_k x[n - k] - \sum_{l=1}^{p} a_l y[n - l] \] (7.14)
CHAPTER 7. DISCRETE-TIME FILTERS

The filter coefficients of an IIR filter are the $b_k, k = 0, \ldots, q$ and $a_l, l = 1, \ldots, p$. The relationship between the filter coefficients of an IIR filter and its impulse response is not as obvious as it was for the FIR filter. In fact, it is generally difficult to find neat expressions for $h[n]$ in terms of the filter coefficients. Instead we will have to satisfy ourselves with looking at several special cases.

Example 7.4.1 Suppose that

$$y[n] = x[n] + 0.5y[n - 1]$$  \hspace{1cm} (7.15)

Comparing (7.15) with (7.14) leads to $q = 0, p = 1, b_0 = 1$, and $a_1 = -0.5$.

Setting $x[n] = \delta[n]$ leads to

$$h[n] = \delta[n] + 0.5h[n - 1]$$  \hspace{1cm} (7.16)

Setting $n = 0$ in (7.16), and assuming that $h[n] = 0, n < 0$, we find that $h[0] = 1$. Setting $n = 1$ in (7.16) and using the result from $n = 0$ then leads to $h[1] = 0.5$. Continuing in this manner leads to

$$h[n] = \begin{cases} 0.5^n, & n \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Figure 7.4 shows a plot of this impulse response.

From this example we find that IIR filters generally have impulse responses which have infinite duration. The reason for this is that the difference equation for an IIR filter has feedback, in other words, the output at the present time, $y[n]$, depends on the outputs at a previous time, $y[n - 1], \ldots, y[n - p]$. For this reason IIR filters are sometimes called recursive filters while FIR filters are called non-recursive filters. If, in the previous example, the filter coefficient $a_1$ had a magnitude greater than one, then the magnitude of $h[n]$ would have gotten bigger and bigger with increasing $n$. This results in a filter which is unstable. We will discuss IIR filter stability in greater detail in Chapter ??.

Proceeding as we did for the FIR filter, the frequency response of an IIR filter can be seen to be

$$H(e^{j\omega}) = \frac{\sum_{k=0}^{q} b_k e^{-j\omega k}}{1 + \sum_{l=1}^{p} a_l e^{-j\omega l}}$$  \hspace{1cm} (7.17)
Example 7.4.2 The frequency response of the filter in Example 7.4.1 is given by

\[ H(e^{j\omega}) = \frac{1}{1 - 0.5e^{-j\omega}} \]

The magnitude and phase of the frequency response is shown in Figure 7.5.

7.5 The Sinusoidal Steady State Response

In order to gain some insight into the behavior of different types of filters, it is useful to look at the response of a filter to a sinusoidal signal, say, \( x[n] = \cos \omega_0 n \). Using the formula for the convolution sum (7.5), the output of the filter is given by

\[ y[n] = \sum_{m=-\infty}^{\infty} h[m] \cos \omega_0 (n - m) \] (7.18)

Using Euler’s Identity for the cosine and doing some basic algebra gives

\[
\begin{align*}
    y[n] &= \frac{1}{2} \sum_{m=-\infty}^{\infty} h[m] \left( e^{j\omega_0(n-m)} + e^{-j\omega_0(n-m)} \right) \\
    &= \frac{1}{2} e^{j\omega_0 n} \sum_{m=-\infty}^{\infty} h[m] e^{-j\omega_0 m} + \frac{1}{2} e^{-j\omega_0 n} \sum_{m=-\infty}^{\infty} h[m] e^{j\omega_0 m} \\
    &= \frac{1}{2} e^{j\omega_0 n} H(e^{j\omega_0}) + \frac{1}{2} e^{-j\omega_0 n} H^*(e^{j\omega_0})
\end{align*}
\] (7.19)

The two terms in (7.19) are complex conjugates of each other. Therefore

\[ y[n] = 2\text{Re}\left\{\frac{1}{2} e^{j\omega_0 n} H(e^{j\omega_0})\right\} \] (7.20)

Expressing \( H(e^{j\omega_0}) \) in polar coordinates as

\[ H(e^{j\omega_0}) = |H(e^{j\omega_0})| e^{j\angle H(e^{j\omega_0})} \] (7.21)

and substituting (7.21) in (7.20) gives

\[
\begin{align*}
    y[n] &= \text{Re}\left\{|H(e^{j\omega_0})| e^{j\omega_0 n + \angle H(e^{j\omega_0})}\right\} \\
    &= |H(e^{j\omega_0})| \cos (\omega_0 n + \angle H(e^{j\omega_0}))
\end{align*}
\] (7.22)
Equation (7.22) is called the sinusoidal steady-state response. It tells us that if the input to a discrete-time filter is a cosine signal with frequency $\omega_0$, then the output is also a cosine having the same frequency, with an amplitude of $|H(e^{j\omega_0})|$ and a phase shift of $-\angle H(e^{j\omega_0})$.

**Example 7.5.1** Let’s assume the filter is the same as in Example 7.3.1:

$$y[n] = x[n] + 2x[n-1] + x[n-2]$$

Find the output of the filter when the input is $x[n] = \cos(0.5\pi n)$.

**Answer:** The frequency response is given by:

$$H(e^{j\omega}) = 1 + 2e^{-j\omega} + e^{-j2\omega}$$

Evaluating the frequency response at $\omega = 0.5\pi$ gives

$$H(e^{j0.5\pi}) = 2e^{-j\pi/2}$$

Therefore

$$y[n] = 2\cos(0.5\pi n - \pi/2) = 2\sin(0.5\pi n)$$

**Example 7.5.2** Here we will use the filter in Example 7.4.2:

$$y[n] = x[n] + 0.5y[n-1]$$

Find the output of the filter when the input is $x[n] = \cos(0.5\pi n)$.

**Answer:** Since the filter is IIR, the frequency response is given by:

$$H(e^{j\omega}) = \frac{1}{1 - 0.5e^{-j\omega}}$$

Evaluating the frequency response at $\omega = 0.5\pi$ gives

$$H(e^{j0.5\pi}) = 0.8 + j0.4 = 0.8944e^{-j0.4636}$$

Therefore

$$y[n] = 0.8944\cos(0.5\pi n - 0.4636)$$

The reason for the term “steady state” is that the input $x[n] = \cos(\omega_0 n)$ was assumed to first be applied to the filter at $n = -\infty$. In practice all input signals have to be applied to a filter at some finite point in time, say $n = 0$. 
Therefore a more realistic input signal might be $x[n] = \cos(\omega_0 n) u[n]$ where $u[n]$ is the discrete-time unit step function defined by

$$u[n] = \begin{cases} 
1, & n \geq 0 \\
0, & n < 0 
\end{cases}$$

Finding the output of a discrete-time filter to $x[n] = \cos(\omega_0 n) u[n]$ is somewhat more complicated. Let’s express the filter input as $x[n] = x_1[n] + x_2[n]$ where

$$x_1[n] = \cos(\omega_0 n) \quad (7.23)$$

and

$$x_2[n] = \begin{cases} 
-cos(\omega_0 n), & n < 0 \\
0, & n \geq 0 
\end{cases} \quad (7.24)$$

We can now apply the principle of superposition to determine the output of the filter. Let $y_1[n]$ and $y_2[n]$ be the filter responses to $x_1[n]$ and $x_2[n]$, respectively. Then it follows that if the input is $x[n] = \cos(\omega_0 n) u[n]$ the filter output will be given by $y_1[n] + y_2[n]$. So our task is to find $y_1[n]$ and $y_2[n]$ and add them up. Since $x_1[n] = \cos(\omega_0 n)$, $y_1[n]$ is just the sinusoidal steady-state response

$$y_1[n] = \left| H(e^{j \omega_0}) \right| \cos \left( \omega_0 n + \angle H(e^{j \omega_0}) \right)$$

Finding $y_2[n]$ is a bit more challenging. Since $x_2[n]$ was applied at $n = -\infty$, then it follows that for all $n < 0$, $y_2[n] = -y_1[n]$ (by linearity). However since $x_2[n]$ is “turned off” at $n = 0$, what happens to $y_2[n]$ for $n \geq 0$? To answer this we need to look at the difference equation for the particular type of filter. For the FIR filter we have

$$y_2[n] = \begin{cases} 
-y_1[n], & n < 0 \\
\sum_{k=0}^{q} b_k x_2[n-k], & n \geq 0 
\end{cases} \quad (7.25)$$

and if the filter is IIR we have

$$y_2[n] = \begin{cases} 
-y_1[n], & n < 0 \\
\sum_{k=0}^{q} b_k x_2[n-k] - \sum_{l=1}^{p} a_l y_2[n-l], & n \geq 0 
\end{cases} \quad (7.26)$$
Therefore, the response to the input signal \( x[n] = \cos(\omega_0 n)u[n] \) for an FIR filter is:

\[
y[n] = \begin{cases} 
0, & n < 0 \\
\sum_{k=0}^q b_k x_2[n-k] + |H(e^{j\omega_0})| \cos(\omega_0 n + \angle H(e^{j\omega_0})), & n \geq 0
\end{cases}
\]

(7.27)

Notice that for \( n \geq 0 \), the output consists of the sum of two terms, the sinusoidal steady state response and the quantity \( \sum_{k=0}^q b_k x_2[n-k] \). This second term can be considered to be a startup transient, which goes to zero after a finite number of samples. For IIR filter we have

\[
y[n] = \begin{cases} 
0, & n < 0 \\
\sum_{k=0}^q b_k x_2[n-k] - \sum_{l=1}^p a_l y_2[n-l] + |H(e^{j\omega_0})| \cos(\omega_0 n + \angle H(e^{j\omega_0})), & n \geq 0
\end{cases}
\]

(7.28)

Here the startup transient is given by \( \sum_{k=0}^q b_k x_2[n-k] - \sum_{l=1}^p a_l y_2[n-l] \).

The transient for the IIR filter will eventually die out, but only as \( n \) gets very large.

**Example 7.5.3** Find the output of the filter in Example 7.5.1 when the input is \( x[n] = \cos(0.5\pi n)u[n] \).

**Answer:** We begin by setting \( x[n] = x_1[n] + x_2[n] \) where \( x_1[n] = \cos(0.5\pi n) \) and

\[
x_2[n] = \begin{cases} 
-\cos(0.5\pi n), & n < 0 \\
0, & n \geq 0
\end{cases}
\]

As seen in the previous example, \( y_1[n] = 2\cos(0.5\pi n - \pi/2) \). And since the filter is FIR

\[
y_2[n] = \begin{cases} 
-y_1[n], & n < 0 \\
x_2[n] + 2x_2[n-1] + x_2[n-2], & n \geq 0
\end{cases}
\]

(7.29)

To determine \( y_2[n], n \geq 0 \), we can evaluate (7.29) for different values of \( n \), starting with \( n = 0 \)

\[
y_2[0] = x_2[0] + 2x_2[-1] + x_2[-2]
\]
\[
= 0 + 2\cos(-0.5\pi) + \cos(-\pi)
\]
\[
= -1
\]
7.5. THE SINUSOIDAL STEADY STATE RESPONSE

\[ y_2[1] = x_2[1] + 2x_2[0] + x_2[-1] \]
\[ = 0 + 0 + \cos(-0.5\pi) \]
\[ = 0 \]

The remaining \( y_2[n] \), \( n \geq 0 \) are zero. Therefore, we have

\[ y[n] = \begin{cases} 
0, & n < 0 \\
-\delta[n] + 2\cos(0.5\pi n - \pi/2), & n \geq 0 
\end{cases} \quad (7.30) \]

**Example 7.5.4** Find the output of the filter in Example 7.5.2 when the input is \( x[n] = \cos(0.5\pi n)u[n] \).

**Answer:** Again, we set \( x[n] = x_1[n] + x_2[n] \) where \( x_1[n] = \cos(0.5\pi n) \) and

\[ x_2[n] = \begin{cases} 
-\cos(0.5\pi nn), & n < 0 \\
0, & n \geq 0 
\end{cases} \]

As seen above, \( y_1[n] = 0.8944 \cos(0.5\pi n - 0.4636) \). And since the filter is IIR

\[ y_2[n] = \begin{cases} 
-y_1[n], & n < 0 \\
x_2[n] + 0.5y_2[n - 1], & n \geq 0 
\end{cases} \quad (7.31) \]

To determine \( y_2[n] \), \( n \geq 0 \), we will have to plug different values of \( n \) in (7.31), starting with \( n = 0 \)

\[ y_2[0] = x_2[0] + 0.5y_2[-1] \]
\[ = 0 + 0.5y_2[-1] \]

\[ y_2[1] = x_2[1] + 0.5y_2[0] \]
\[ = 0 + 0.5^2y_2[-1] \]

Proceeding in this manner, we find that \( y_2[n] = y_2[-1]0.5^{n+1}u[n] \). Evaluating (7.31) at \( n = -1 \) gives \( y_2[-1] = 0.4 \) And the final answer is

\[ y[n] = \begin{cases} 
0, & n < 0 \\
(0.4)0.5^{n+1} + 0.8944 \cos(0.5\pi n - 0.4636), & n \geq 0 
\end{cases} \quad (7.32) \]
(7.33)

(7.34)
Figure 7.3: Frequency response magnitude and phase for Example 7.3.1.
Figure 7.4: Impulse response of IIR filter in Example 7.4.1.
Figure 7.5: Frequency response magnitude and phase for Example 7.4.2.
Chapter 8
The Z Transform

Z-Transform Pairs

\[ a^n u[n] \leftrightarrow \frac{1}{1 - a z^{-1}}, |z| > |a| \]
\[ u[n] \leftrightarrow \frac{1}{1 - z^{-1}}, |z| > 1 \]
\[ \delta[n - n_o] \leftrightarrow z^{-n_o}, z \neq 0 \text{ or } z \neq \infty \]
\[ \cos(\omega_o n) u[n] \leftrightarrow \frac{1 - \cos(\omega_o) z^{-1}}{1 - 2 \cos(\omega_o) z^{-1} + z^{-2}}, |z| > 1 \]
\[ x[n] = \begin{cases} 
1, 0 \leq n \leq N - 1 & \leftrightarrow \frac{1 - z^{-N}}{1 - z^{-1}} \\
0, \text{otherwise} & \end{cases} \]

Z-Transform Properties

\[ \alpha x_1[n] + \beta x_2[n] \leftrightarrow \alpha X_1(z) + \beta X_2(z), \text{ ROC } = R_X \cap R_{X_2} \]
\[ x[n - n_0] \leftrightarrow X(z) z^{-n_0}, \text{ ROC } = R_X \text{ excluding } z = \infty \text{ or } z = 0 \]
\[ x[n] a^n \leftrightarrow X \left( \frac{z}{a} \right) , \text{ ROC } = |a| R_X \]
\[ x[n] * h[n] \leftrightarrow X(z) H(z), \text{ ROC } = R_X \cap R_H \]

(8.1)
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(8.2)
## Appendix A

### Fourier Transform Tables

<table>
<thead>
<tr>
<th>$x(t)$</th>
<th>$X(j\Omega)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{rect}(t, \tau)$</td>
<td>$\tau \frac{\sin(\Omega \tau/2)}{\Omega \tau/2}$</td>
</tr>
<tr>
<td>$\delta(t - \tau)$</td>
<td>$e^{-j\Omega \tau}$</td>
</tr>
<tr>
<td>$e^{j\Omega_0 t}$</td>
<td>$2\pi \delta(\Omega - \Omega_0)$</td>
</tr>
<tr>
<td>$\cos(\Omega_0 t)$</td>
<td>$\pi \delta(\Omega - \Omega_0) + \pi \delta(\Omega + \Omega_0)$</td>
</tr>
<tr>
<td>$e^{-\alpha t} u(t)$</td>
<td>$\frac{1}{\alpha + j\Omega}$</td>
</tr>
</tbody>
</table>

Table A.1: Some common Fourier Transform pairs.
### APPENDIX A. FOURIER TRANSFORM TABLES

<table>
<thead>
<tr>
<th>Property</th>
<th>$y(t)$</th>
<th>$Y(j\Omega)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linearity</td>
<td>$\alpha x_1(t) + \beta x_2(t)$</td>
<td>$\alpha X_1(j\Omega) + \beta X_2(j\Omega)$</td>
</tr>
<tr>
<td>Time Shift</td>
<td>$x(t - \tau)$</td>
<td>$X(j\Omega)e^{-j\Omega\tau}$</td>
</tr>
<tr>
<td>Time Reversal</td>
<td>$x(-t)$</td>
<td>$X(j\Omega)^*$</td>
</tr>
<tr>
<td>Frequency Shift</td>
<td>$x(t)e^{j\Omega_0 t}$</td>
<td>$X(j(\Omega - \Omega_0))$</td>
</tr>
<tr>
<td>Convolution</td>
<td>$x(t) * h(t)$</td>
<td>$X(j\Omega)H(j\Omega)$</td>
</tr>
<tr>
<td>Modulation</td>
<td>$x(t)w(t)$</td>
<td>$\frac{1}{2\pi} \int_{-\infty}^{\infty} X(j(\Omega - \Theta))W(j\Theta)d\Theta$</td>
</tr>
</tbody>
</table>

Table A.2: Fourier Transform properties.
## Appendix B

**Laplace Transform Tables**

<table>
<thead>
<tr>
<th>$x(t)$</th>
<th>$X(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta(t)$</td>
<td>1</td>
</tr>
<tr>
<td>$u(t)$</td>
<td>$\frac{1}{s}$</td>
</tr>
<tr>
<td>$tu(t)$</td>
<td>$\frac{1}{s^2}$</td>
</tr>
<tr>
<td>$e^{-\alpha t}u(t)$</td>
<td>$\frac{1}{s+\alpha}$</td>
</tr>
<tr>
<td>$\cos(\Omega_0 t)u(t)$</td>
<td>$\frac{s}{s^2+\Omega_0^2}$</td>
</tr>
<tr>
<td>$\sin(\Omega_0 t)u(t)$</td>
<td>$\frac{\Omega_0}{s^2+\Omega_0^2}$</td>
</tr>
<tr>
<td>$te^{-\alpha t}u(t)$</td>
<td>$\frac{1}{(s+\alpha)^2}$</td>
</tr>
</tbody>
</table>

Table B.1: Some common Laplace Transform pairs.
### APPENDIX B. LAPLACE TRANSFORM TABLES

<table>
<thead>
<tr>
<th>Property</th>
<th>$y(t)$</th>
<th>$Y(j\Omega)$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Linearity</strong></td>
<td>$\alpha x_1(t) + \beta x_2(t)$</td>
<td>$\alpha X_1(s) + \beta X_2(s)$</td>
</tr>
<tr>
<td><strong>Time Delay</strong></td>
<td>$x(t - \tau)$</td>
<td>$X(s)e^{-s\tau}$</td>
</tr>
<tr>
<td><strong>Multiplication by $t$</strong></td>
<td>$tx(t)$</td>
<td>$-\frac{d}{ds}X(s)$</td>
</tr>
<tr>
<td><strong>Multiplication by $t^2$</strong></td>
<td>$t^2x(t)$</td>
<td>$\frac{d^2}{ds^2}X(s)$</td>
</tr>
<tr>
<td><strong>Time Scaling</strong></td>
<td>$x(\alpha t)$</td>
<td>$\frac{1}{\alpha}X\left(\frac{s}{\alpha}\right)$</td>
</tr>
<tr>
<td><strong>$s$-Shift</strong></td>
<td>$x(t)e^{-at}$</td>
<td>$X(s + a)$</td>
</tr>
<tr>
<td><strong>Convolution</strong></td>
<td>$x(t) \ast h(t)$</td>
<td>$X(s)H(s)$</td>
</tr>
</tbody>
</table>

Table B.2: Laplace Transform properties.