Assumptions for Quantizer SNR: $SNR = \frac{3 \cdot 2^{2N}}{(X_{\text{max}})^2}$

- Quantization error is stationary, white noise:
  $$E[e[n]e[n+m]] = \begin{cases} \sigma_e^2 & , m = 0 \\ 0, & m \neq 0 \end{cases}$$

- Quantization error is uncorrelated with the input signal:
  $$E[x[n]e[n+m]] = 0, \forall m$$

- Quantization error is uniformly distributed on $\left[-\frac{\Delta}{2}, \frac{\Delta}{2}\right]$
Derivation of Quantizer SNR

- Quantizer input range is $2X_{\text{max}}$, step height is:

$$\Delta = \frac{2X_{\text{max}}}{2^N}$$

- Since $e[n]$ is uniformly distributed:

$$\sigma_e^2 = \frac{\Delta^2}{12} = \frac{X_{\text{max}}^2}{3 \cdot 2^{2N}}$$
Derivation of Quantizer SNR (cont.)

- substituting $\sigma_e^2$ into $SNR = \frac{\sigma_x^2}{\sigma_e^2}$ gives:

$$SNR = \frac{3 \cdot 2^{2N}}{\left( \frac{X_{\text{max}}}{\sigma_x} \right)^2}$$

- If $X_{\text{max}} = 4\sigma_x$ then:

$$SNR(\text{dB}) = 6N - 7.2$$
Differential Quantization Coder

\[ x[n] + d[n] \xrightarrow{Q[\ ]} \hat{d}[n] \xrightarrow{\text{encoder}} c[n] \]

\[ \tilde{x}[n] \xrightarrow{P(z)} \phi[n] \xrightarrow{+} \tilde{x}[n] \]
Analysis of Differential Quantization

- $P(z)$ is a prediction filter, tries to predict the present value of $x[n]$ based on previous values of $\phi[n]$
- prediction error:
  \[ d[n] = x[n] - \tilde{x}[n] \]  
  (1)
- quantizer output:
  \[ \hat{d}[n] = d[n] + e[n] \]  
  (2)
- predictor input: quantization error
  \[ \phi[n] = \tilde{x}[n] + \hat{d}[n] \]  
  (3)
Analysis of Differential Quantization (cont.)

- subst. (1) and (2) in (3) gives:

\[
\phi[n] = x[n] + e[n] \equiv \hat{x}[n]
\]

- \( \phi[n] \) is a quantized version of \( x[n] \) but the quantization error \( e[n] \) corresponds to quantizing the prediction error, \( d[n] \), which is typically small if the prediction is good, so \( e[n] \) is small!
Differential Quantization SNR

\[ SNR = \frac{\sigma_x^2}{\sigma_e^2} \]

\[ SNR = \frac{\sigma_x^2}{\sigma_d^2} \cdot \frac{\sigma_d^2}{\sigma_e^2} = G_P SNR_Q \]

SNR of quantizer: \[ SNR_Q = \frac{\sigma_d^2}{\sigma_e^2} \]

SNR gain due to DQ: \[ G_P = \frac{\sigma_x^2}{\sigma_d^2} \] (typically 4 - 11 dB)
Prediction Filter

\[ P(z) = \sum_{k=1}^{p} \alpha_k z^{-k} \] (length \( p \) FIR filter)

- prediction filter difference equation:

\[ \tilde{x}[n] = \sum_{k=1}^{p} \alpha_k \hat{x}[n-k] \]

- If the \( x[n] \) are highly correlated, then the prediction can be very good, keeping the prediction error and quantization error small.
Differential Quantization Decoder

\[ c[n] \rightarrow \text{decoder} \rightarrow \hat{d}[n] \rightarrow + \rightarrow \hat{x}[n] \]

\[ P(z) \]

\[ \tilde{x}[n] \rightarrow \]
Delta Modulation (DM) Coder

\[ x[n] + d[n] \rightarrow Q[ ] \rightarrow \hat{d}[n] \rightarrow \text{encoder} \rightarrow c[n] \]

\[ \tilde{x}[n] \rightarrow \alpha z^{-1} \rightarrow \hat{x}[n] \rightarrow + \]

\[ \alpha z^{-1} \]
Delta Modulation Decoder

\[ c[n] \rightarrow \text{decoder} \rightarrow \hat{d}[n] \rightarrow + \rightarrow \hat{x}[n] \]

\[ \hat{x}[n] \rightarrow c[n] \rightarrow \text{decoder} \rightarrow \hat{d}[n] \rightarrow + \rightarrow \hat{x}[n] \]

\[ \alpha z^{-1} \]
Delta Modulation Uses a 1-Bit Quantizer

\[ \hat{d}[n] \]

\[ \Delta \]

\[ c[n] = 1 \]

\[ d[n] \]

\[ c[n] = 0 \]

\[ -\Delta \]
DM Encoder and Decoder

<table>
<thead>
<tr>
<th>$\hat{d}[n]$</th>
<th>$c[n]$</th>
<th>$c[n]$</th>
<th>$\hat{d}[n]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta$</td>
<td>1</td>
<td>1</td>
<td>$\Delta$</td>
</tr>
<tr>
<td>$-\Delta$</td>
<td>0</td>
<td>0</td>
<td>$-\Delta$</td>
</tr>
</tbody>
</table>

encoder decoder
Delta Modulation (cont.)

- Need only store 1 bit per sample.
- If a 1-bit quantizer is used, to get good performance, prediction error must be very small.
- To get small prediction error, must sample ECG at a high rate. Typically many times the Nyquist rate.
Selection of $\Delta$

- Decoder difference equation:

$$\hat{x}[n] = \alpha \hat{x}[n-1] + \hat{d}[n]$$

- If $\alpha \approx 1$ then decoder approximates an integrator that accumulates positive or negative increments of magnitude $\Delta$.

- Quantizer input:

$$d[n] = x[n] - \hat{x}[n-1] = x[n] - x[n-1] - e[n-1]$$

resembles a discrete-time differentiation.
Selection of $\Delta$ (cont.)

Therefore in order for the integrator to be able to track $x[n]$: 

$$\frac{\Delta}{T} \geq \max \left| \frac{dx(t)}{dt} \right|$$

$T =$ sampling interval 

otherwise, get slope overload distortion.
Slope Overload Distortion

$x(t)$

$\hat{x}[n]$

$\Delta$

$c[n]$: 1 1 1 1 1 1 1 1 0 1 0 0 1
Differential Pulse Code Modulation (DPCM)

- Predictor filter \((p = 4 \text{ or } 5)\):
  \[
P(z) = \sum_{k=1}^{p} \alpha_k z^{-k}
  \]

- Quantizer generally has more than one level.

- Prediction filter coefficients can be chosen to minimize the mean squared prediction error:
  \[
  E[d^2[n]] = E[(x[n] - \bar{x}[n])^2]
  \]
Linear Prediction

- Prediction error:

\[ d[n] = x[n] - \sum_{k=1}^{p} \alpha_k \hat{x}[n - k] \]

- We can ignore the quantization error if it is small, prediction error becomes:

\[ d[n] = x[n] - \sum_{k=1}^{p} \alpha_k x[n - k] \]
Least Squares Linear Prediction

- Assume the following $M$ data samples are available:

$$x[0], x[1], x[2], \ldots, x[M - 1]$$

- Least squares normal equations:

$$X\overline{\alpha} = \bar{x}$$

$$\overline{\alpha} = [\alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_p]^T$$