EE 5345
Biomedical Instrumentation
Lecture 19: slides 371-391

Carlos E. Davila, Electrical Engineering Dept.
Southern Methodist University
slides can be viewed at:
http://www.seas.smu.edu/~cd/ee5345.html
Matrices

Consider the matrix $A$ having $m$ rows and $n$ columns, the size of the matrix is $m$ by $n$.

$$A = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}$$

transpose of $A$: $A^T = \begin{bmatrix}
a_{11} & a_{21} & \cdots & a_{m1} \\
a_{12} & a_{22} & \cdots & a_{m2} \\
\vdots & \vdots & \ddots & \vdots \\
a_{1n} & a_{2n} & \cdots & a_{mn}
\end{bmatrix}$
Matrix Arithmetic

Let \( x = [x_1 \ x_2 \ \cdots \ x_n]^T \in \mathbb{R}^n \) and
\[
y = [y_1 \ y_2 \ \cdots \ y_n]^T \in \mathbb{R}^n
\]
then
\[
x + y = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix} \quad cx = \begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{bmatrix} \quad c, \text{ scalar}
\]
Matrix Arithmetic (cont.)

Let \( x = [x_1 \ x_2 \ \ldots \ x_n]^T \in \mathbb{R}^n \) then

\[
[Ax]_i = \sum_{j=1}^{n} a_{ij}x_j, \quad i = 1, \ldots m
\]

Let \( B \) be \( n \) by \( r \), then

\[
[AB]_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}, \quad i = 1, \ldots m, \quad j = 1, \ldots r
\]

Inner product between two vectors in \( \mathbb{R}^n \):

\[
x^T y = \sum_{k=1}^{n} x_k y_k
\]
Brief Introduction to Linear Algebra

- **Euclidean Vector Space** $\mathbb{R}^n$: set consisting of all $n \times 1$ vectors.

- **Subspace of** $\mathbb{R}^n$, $S$:
  - if $x \in S$ and $y \in S$ then $x + y \in S$
  - if $x \in S$ then $cx \in S$, $c = $ scalar (1x1).

- **Span of Subspace** $S$:
  - $\{w_1, w_2, ..., w_r\}$ is a span of $S$ if every vector in $S$ can be expressed as the linear combination:

$$
\sum_{k=1}^{r} \beta_k w_k
$$
Orthogonality Relationships

- Two vectors $x$ and $y$ are orthogonal $(x \perp y)$ if:
  \[ x^T y = 0 \]

- Two subspaces $P \subseteq \mathbb{R}^n$ and $Q \subseteq \mathbb{R}^n$ are orthogonal if
  \[ x \perp y, \quad x \in P, \quad y \in Q \]
  we say $P \perp Q$
Brief Introduction to Linear Algebra (cont.)

- Basis of $S$ is a set of vectors $\{v_1, v_2, \ldots, v_l\}$ which:
  - spans $S$
  - is linearly independent
- $\{v_1, v_2, \ldots, v_l\}$ are linearly independent if:
  \[ \sum_{k=1}^{l} \beta_k v_k = 0, \text{ iff } \beta_k = 0, \quad k = 1, \ldots, l \]
- dimension of $S = \text{number of basis vectors.}$
- $\theta$: zero vector belongs to any subspace
Subspaces Associated with Matrices

- Column space (range) of $A$: subspace consisting of all possible linear combinations of the columns of $A$, denoted as $R(A)$.  

- Row space of $A$ is the subspace consisting of all possible linear combinations of the rows of $A$, denoted $R(A^T)$.  

- Null space of $A$ is the subspace consisting of all $n \times 1$ vectors $x$ satisfying: $Ax = 0$, denoted $N(A)$.  

- Left null space of $A$ is the subspace consisting of all $m \times 1$ vectors $x$ satisfying: $x^T A = 0$, denoted $N(A^T)$.
Some Important Theorems

- \( N(A) \perp R(A^T) \), (\( n \) by 1 vectors)
- \( R(A) \perp N(A^T) \), (\( m \) by 1 vectors)
- def’n: rank of \( A \) = number of linearly independent columns of \( A = \text{rank}(A) \).
- rank of \( A \) = dimension of \( R(A) \).
- def’n: Let \( C \) be an \( n \) by \( n \) square matrix, \( C^{-1} \) is the inverse of \( C \) if:
  
  \[
  C^{-1}C = I
  \]
- \( \text{rank}(A) = n \) if and only if \( A^TA \) has an inverse.
Identity Matrix

\[ I = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
0 & 0 & \ddots & 0 \\
0 & 0 & \cdots & 1
\end{bmatrix} \]

\[ AI = A \quad IA = A \]

Euclidean norm of a vector:

\[ \|x\| = \sum_{k=1}^{n} x_k^2 \]
Projection onto a Subspace $S$

Suppose $y \not\in S$, the projection of $y$ onto $S$ is denoted $y_P$. Then

$$P_s y \in S \quad y - P_s y \perp x \in S$$

$P_s$ : projection matrix

$\theta = \text{zero vector}$
Projection Theorem

- We seek a vector $x$ in $S$ that is “closest” to $y$ in the sense of minimizing

$$\min_{x \in S} \|x - y\|$$

- The vector minimizing the above norm is given by

$$y^* \equiv P_S y$$

and $y^*$ is unique.
Computation of Projection Matrix

If $S$ has the basis:

\[ \{v_1, v_2, \ldots, v_l\} \]

Let $V = [v_1, v_2, \ldots, v_l]$, then:

\[ P_S = V(V^TV)^{-1}V^T \]

$P_S$ is idempotent: \[ P_S P_S = P_S \]

$P_S$ is symmetric: \[ P_S = P_S^T \]
Least Squares Linear Prediction

- Available Data: $x[0], x[1], x[2], \ldots, x[M - 1]$

- LS Equations: $X\alpha \approx d$

$$X = \begin{bmatrix}
x[p-1] & x[p-2] & \cdots & x[0] \\
x[p] & x[p-1] & \cdots & x[1] \\
\vdots & \vdots & \ddots & \vdots \\
\end{bmatrix}$$
Least Squares Linear Prediction (cont.)

\[ d = \begin{bmatrix} x[p] & x[p+1] & \cdots & x[M-1] \end{bmatrix}^T \]

\[ \alpha = \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_p \end{bmatrix}^T \]

• usually, \( X \) has more rows than columns (overdetermined system of equations).
• note that in general, the left-hand side of the LS equations \( X\alpha \approx d \) do not equal the right-hand side.
LS Equations (Covariance Method)

\[
\begin{bmatrix}
  x[p-1] & x[p-2] & \cdots & x[0] \\
  x[p] & x[p-1] & \cdots & x[1] \\
  \vdots & \vdots & \ddots & \vdots \\
\end{bmatrix}
\begin{bmatrix}
  \alpha_1 \\
  \alpha_2 \\
  \vdots \\
  \alpha_p
\end{bmatrix}
\approx
\begin{bmatrix}
  x[p] \\
  x[p-1] \\
  \vdots \\
  x[M-1]
\end{bmatrix}
\]

\(X\begin{bmatrix}\alpha \\ d\end{bmatrix}\)

\(M - p\) equations

\(p\) unknowns: \(\alpha_k, \ k = 1, \ldots, p\)
Solution of LS Problem

- We seek $\alpha$ so that prediction error vector $\|X\alpha - d\|$ is minimized.
- $X\alpha$ lies in the range of $X$.
- So we seek the vector in $R(A)$ that is closest to $d$ in Euclidean norm.
- From Projection Theorem:

$$d - X\alpha \perp R(A)$$

equivalently:

$$X^T(d - X\alpha) = 0$$
Solution of LS Problem (cont.)

- **LS normal equations:**

  \[ X^T X \alpha = X^T d \]

- **LS Solution:**

  \[ \alpha = (X^T X)^{-1} X^T d \]

  provided dimension of \( R(X) = p \) (columns of \( X \) are linearly independent).
Autocorrelation Method: \( X_a \alpha \approx d_a \)

\[
\begin{bmatrix}
  x[0] & 0 & \ldots & 0 \\
  x[1] & x[0] & \ldots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  x[p-1] & x[p-2] & \ldots & x[0] \\
  \vdots & \vdots & \ddots & \vdots \\
  x[N-2] & x[N-3] & \ldots & x[N-p-1] \\
  x[N-1] & x[N-2] & \ldots & x[N-p] \\
  0 & x[N-1] & \ldots & x[N-p+1] \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \ldots & x[N-1]
\end{bmatrix}
\begin{bmatrix}
  \alpha_1 \\
  \alpha_2 \\
  \vdots \\
  \alpha_p
\end{bmatrix}
\approx
\begin{bmatrix}
  x[1] \\
  x[2] \\
  \vdots \\
  x[p] \\
  \vdots \\
  x[N-1]
\end{bmatrix}
\]
Autocorrelation Method (cont.)

Solution:

\[ \alpha = \left( X_a^T X_a \right)^{-1} X_a^T d_a \]

The matrix \( X_a^T X_a \) is Toeplitz, meaning elements along any diagonal are equal:
Computational Considerations

- Covariance method requires $O(p^3)$ operations to invert $X^T X$ (this can be expensive).
- Autocorrelation method requires $O(p^2)$ operations to invert $X_a^T X_a$
- In differential quantization, must also encode prediction filter $\alpha$.
- If prediction filter is recomputed every $N$ samples, get adaptive DPCM (ADPCM).