Iterated Filter Banks

Analysis Bank: \( \{y_1[n], y_1^2[n], y_1^3[n], y_0^3[n]\} \) are the Discrete Wavelet Transform (DWT) coefficients
Iterated Filter Banks (cont.)

since each stage is a 2-channel perfect reconstruction filter bank:

\[
\hat{y}_0^2[n] = y_0^2[n] \\
\hat{y}_0^1[n] = y_0^1[n] \quad \Rightarrow \hat{x}[n] = x[n]
\]
DWT Coefficients and Subbands

- Each DWT coefficient corresponds to a different subband.
- If all but one of the DWT coefficients, $y_i[n]$, is zeroed out, can identify the appropriate subband where resulting $\hat{x}[n]$ lives.

![Diagram showing subbands B1, B2, B3, B4 with frequencies $0$, $\pi/8$, $\pi/4$, $\pi/2$, and $\pi$.]
DWT Coefficients and Subbands (cont.)

<table>
<thead>
<tr>
<th>Non-zero DWT coefficient</th>
<th>$y_0^3[n]$</th>
<th>$y_1^3[n]$</th>
<th>$y_1^2[n]$</th>
<th>$y_1^1[n]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Band where $\hat{x}[n]$ lies</td>
<td>B1</td>
<td>B2</td>
<td>B3</td>
<td>B4</td>
</tr>
</tbody>
</table>
Eight equal subbands:
Eight equal subbands (cont.)

\[ \hat{x}[n] \]

\[ 0 \pi/8 \pi/4 \pi/2 3\pi/4 \pi \]

B1 B2 B3 B4 B5 B6 B7 B8
Subband Coding

- Subband coefficients are quantized and encoded.
- Typical quantizer is DPCM.
- Typical encoder is the Huffman (entropy) encoder.
Example of 2-channel Subband Coder

\[
x[n] \xrightarrow{h_1[n]} y_1[n] \xrightarrow{2\downarrow} \text{DPCM} \xrightarrow{\text{entropy encoder}} c_1[n] \\
x[n] \xrightarrow{h_0[n]} y_0[n] \xrightarrow{2\downarrow} \text{DPCM} \xrightarrow{\text{entropy encoder}} c_0[n]
\]
Bit Allocation

\[ x[n] \]

\[ y_0[n] \]

\[ y_1[n] \]

can be encoded using DPCM, 1 or 2 bits per sample

can allocate more bits per sample here
SNR for uniform PCM Quantizer

\[ SNR = \frac{\sigma_x^2}{\sigma_e^2} = \frac{E[x^2[n]]}{E[e^2[n]]} \]

\[ e[n]: \text{quantization error} \]

\[ E[\cdot] : \text{expectation operation} \]

\[ \sigma_e^2 = \frac{\Delta^2}{12} = \frac{X_{\text{max}}^2}{3 \cdot 2^{2N}} \]

\[ SNR = \frac{3 \cdot 2^{2N}}{\left( \frac{X_{\text{max}}}{\sigma_x} \right)^2} \]

\[ 2^N = \text{number of quantizer levels} \]
Number of bits/sample vs. Quantizer SNR for PCM

\[ \sigma_e^2 = \frac{\sigma_x^2}{SNR} \quad \text{SNR}^{-1} = \frac{X_{max}^2}{3 \cdot 2^{2N} \sigma_x^2} \equiv \varepsilon 2^{-2N} \]

\[ \sigma_e^2 = \varepsilon 2^{-2N} \sigma_x^2 \]

typically, \( \varepsilon > 1 \)

solving for \( N \):

\[ N = \frac{1}{2} \log_2 \left( \frac{\sigma_x^2}{\sigma_e^2} \right) + \frac{1}{2} \log_2(\varepsilon) \]

bits/sample needed to achieve a given SNR
SNR Gain in Subband Coding (SBC)

- Assume there are $M$ equal-width subbands in analysis filter bank.
- We use $N_k$ bits per subband-$k$ sample versus $N$ bits per sample for PCM.
- We impose the following constraint:

$$\sum_{k=1}^{M} N_k = MN$$

- This does not mean that we allow more bits for SBC since each subband is sampled at $1/M$ rate of PCM due to downsamplers. Total number of bits remains the same.
SBC SNR Gain (cont.)

- Assume each subband is quantized with uniform PCM.
- Assume quantization errors are white and uncorrelated across subbands.
- Total quantization error:

\[
\sigma_{e,SBC}^2 = \sum_{k=1}^{M} \sigma_{e,k}^2
\]

\[
\sigma_{e,k}^2 : \text{quantization error variance for } k^{th} \text{ subband}
\]
SBC SNR Gain (cont.)

\[ \sigma_{e,SBC}^2 = \sum_{k=1}^{M} \varepsilon_k 2^{-2N_k} \sigma_{x,k}^2 \]

- Assume \( \varepsilon_k = \varepsilon \), all \( k \)
- SBC SNR Gain:

\[ G_{SBC} \equiv \frac{2^{-2N} \sigma_x^2}{\sum_{k=1}^{M} 2^{-2N_k} \sigma_{x,k}^2} \]

- \( G_{SBC} \) can be made > 1 if spectrum of \( x[n] \) is not flat (as is the case for differential quantization).
Optimized Bit Allocation

- Want to minimize $\sigma_{e,SBC}^2 = \sum_{k=1}^{M} \varepsilon_k 2^{-2N_k} \sigma_{x,k}^2$ subject to the constraint:

$$\sum_{k=1}^{M} N_k = MN$$

- This can be solved via the method of Lagrange multipliers. The objective function we want to minimize is:

$$\sigma_{e,SBC}^2 - \lambda \left( MN - \sum_{k=1}^{M} N_k \right)$$

to be minimized in terms of $N_k$ and $\lambda$
Optimized Bit Allocation (cont.)

- Setting the derivative of:

\[ \sigma_{e,SBC}^2 - \lambda \left( MN - \sum_{k=1}^{M} N_k \right) \]

w.r.t. \( N_k \) and \( \lambda \) equal to 0 gives:

\[ N_k = N + \frac{1}{2} \log_2 \frac{\sigma_{x,k}^2}{\left( \prod_{j=1}^{M} \sigma_{x,j}^2 \right)^{1/M}} \]