Chapter 1

Signals and Systems

1.1 Basic Problems

1.1.1 Problem (1.21)

a It is just a shift for the signal $x(t)$ one unit to the right

b You can do it by shifting the signal first to the lift by two to produce $x(t+2)$ then reverse the signal (replacing each t with $-t$) to produce $x(-t+2)$
Another way is to write $x(2-t)$ as $x(-(t-2))$ which is x(t) flipped about the $t = 0$ axis (to produce the function $x(-t)$) and then shifted by two units to the right to produce $x(-(t-2))$

c You can make the shift first by one in the left direction then apply the compression by dividing the x-axis by two
Again, you can also write $x(2t+1)$ as $x(2(t+1/2))$ and this later function is $x(2t)$ shifted by 1/2 to the left. The function $x(2t)$ is a compression of the t axis in the original function $x(t)$ by two

d This can be simply done by first shifting the signal $x(t)$ to the left by four to produce $x(t+4)$ then make the reverse to produce $x(-t+4)$ and finally, stretch the signal along the x-axis by multiply the x-axis by two to produce $x(4-t/2)$

e You can do it in two steps, first get $x(t)u(t)$ and $x(t-)u(t)$ which have values only in the positive direction then add the two figures to each other to find that the values in the positive direction will cancel each other for $x(t)$ 1

f Note that $\delta(t+3/2)$ has value only at $t = -3/2$ and $\delta(t-3/2)$ has value only at $t = 3/2$. So, you will keep the values of $x(t)$ at $t = -3/2, 3/2$ only. Note that the solution in the graph for the signal of the following form: $x(t)[\delta(t-3/2) - \delta(t+3/2)]$

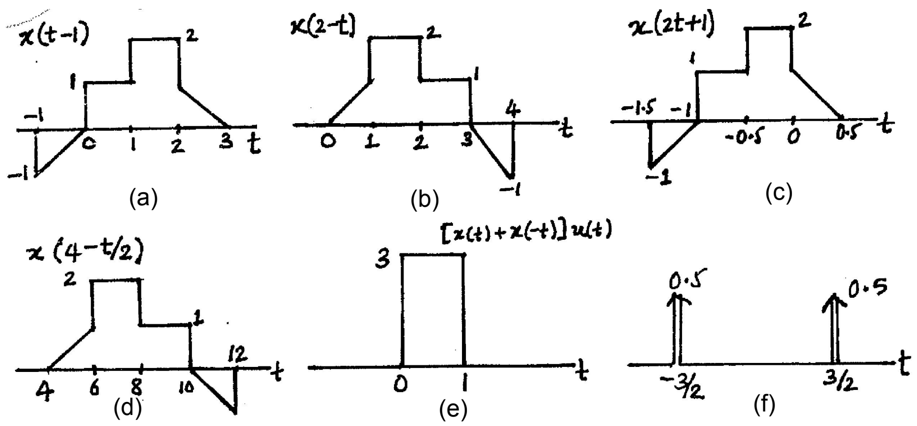


Figure 1.1: Problem Solution 1.21

1.1.2 Problem (1.22)

a Simply, shift to the right direction by four

b Shift to the left direction by three then reverse

c You shall scale the x-axis by 1/3 but note that in discrete you have values only for the integer indexes of x-axis. Then you will have the values for $x[n]$ at n is multiple of three (i.e. -3,0,3 in this case) then put them at $n/3$

d As in (c) but first, make a shift by one in the negative direction

e Note that $u[3-n]$ equals to one in the range of $-\inf < n <= 3$ which covers the range of the entire signal. So, $x[n]u[3-n] = x[n]$

f Get $x[n-2]$ first by shift in the right direction by two. Then, keep only the value of the resulted signal at $n = 2$ as $\delta[n-2]$ has value only at $n = 2$

g You can notice that the output signal equals to $x[n]$ when n is even or zero, and equals to zero when n is odd

h You can do it simply by substitution over the entire range

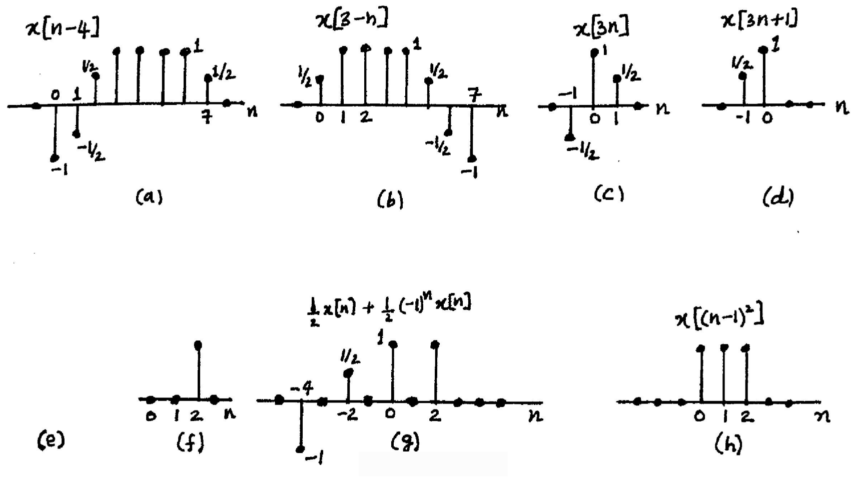


Figure 1.2: Problem Solution 1.22

1.1.3 Problem (1.25)

(a) Periodic, period $= 2\pi/(4) = \pi/2$.

(b) Periodic, period $= 2\pi/(\pi) = 2$.

(c) $x(t) = [1 + \cos(4t - 2\pi/3)]/2$. Periodic, period $= 2\pi/(4) = \pi/2$.

(d) $x(t) = \cos(4\pi t)/2$. Periodic, period $= 2\pi/(4\pi) = 1/2$.

(e) $x(t) = [\sin(4\pi t)u(t) - \sin(4\pi t)u(-t)]/2$. Not periodic.

(f) Not periodic.

Note: Sinusoidal signals ($sin(t)$, $Cos(t)$ and $exp(t)$) are always periodic with period 2π. Shifting the independent variable (t) does not affect periodicity or period. Scaling the independent variable keeps the signal periodic but only changes the period to be $2\pi/scale$. Finally, in point number (e) the signal is not periodic as the sign of sin function in the negative direction is reversed.

1.1.4 Problem (1.31)

(a) Note that $x_2(t) = x_1(t) - x_1(t-2)$. Therefore, using linearity we get $y_2(t) = y_1(t) - y_1(t-2)$. This is as shown in Figure S1.31.

(b) Note that $x_3(t) = x_1(t) + x_1(t+1)$. Therefore, using linearity we get $y_3(t) = y_1(t) + y_1(t+1)$. This is as shown in Figure S1.31.

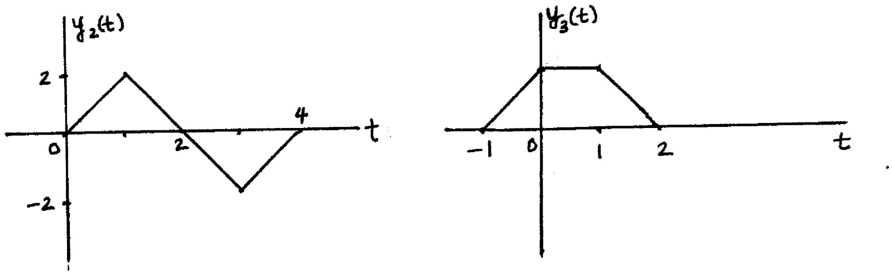


Figure 1.3: Problem Solution 1.31

1.2 Advanced Problems

1.2.1 Problem (1.34)

(a) Consider

$$\sum_{n=-\infty}^{\infty} x[n] = x[0] + \sum_{n=1}^{\infty}\{x[n] + x[-n]\}.$$

If $x[n]$ is odd, $x[n] + x[-n] = 0$. Therefore, the given summation evaluates to zero.

(b) Let $y[n] = x_1[n]x_2[n]$. Then

$$y[-n] = x_1[-n]x_2[-n] = -x_1[n]x_2[n] = -y[n].$$

This implies that $y[n]$ is odd.

(c) Consider

$$\begin{aligned}\sum_{n=-\infty}^{\infty} x^2[n] &= \sum_{n=-\infty}^{\infty} \{x_e[n] + x_o[n]\}^2 \\ &= \sum_{n=-\infty}^{\infty} x_e^2[n] + \sum_{n=-\infty}^{\infty} x_o^2[n] + 2\sum_{n=-\infty}^{\infty} x_e[n]x_o[n].\end{aligned}$$

Using the result of part (b), we know that $x_e[n]x_o[n]$ is an odd signal. Therefore, using the result of part (a) we may conclude that

$$2\sum_{n=-\infty}^{\infty} x_e[n]x_o[n] = 0.$$

Therefore,

$$\sum_{n=-\infty}^{\infty} x^2[n] == \sum_{n=-\infty}^{\infty} x_e^2[n] + \sum_{n=-\infty}^{\infty} x_o^2[n].$$

(d) Consider

$$\int_{-\infty}^{\infty} x^2(t)dt = \int_{-\infty}^{\infty} x_e^2(t)dt + \int_{-\infty}^{\infty} x_o^2(t)dt + 2\int_{-\infty}^{\infty} x_e(t)x_o(t)dt.$$

Again, since $x_e(t)x_o(t)$ is odd,

$$\int_{-\infty}^{\infty} x_e^2(t)x_o(t)dt = 0.$$

Therefore,

$$\int_{-\infty}^{\infty} x^2(t)dt = \int_{-\infty}^{\infty} x_e^2(t)dt + \int_{-\infty}^{\infty} x_o^2(t)dt.$$

1.2.2 Problem (1.36)

(a) If $x[n]$ is periodic $e^{j\omega_0(n+N)T} = e^{j\omega_0 nT}$, where $\omega_0 = 2\pi/T_0$. This implies that

$$\frac{2\pi}{T_0}NT = 2\pi k \qquad \Rightarrow \qquad \frac{T}{T_0} = \frac{k}{N} = \text{a rational number.}$$

(b) If $T/T_0 = p/q$ then $x[n] = e^{j2\pi n(p/q)}$. The fundamental period is $q/\gcd(p,q)$ and the fundamental frequency is

$$\frac{2\pi}{q}\gcd(p,q) = \frac{2\pi}{p}\frac{p}{q}\gcd(p,q) = \frac{\omega_0}{p}\gcd(p,q) = \frac{\omega_0 T}{p}\gcd(p,q).$$

(c) $p/\gcd(p,q)$ periods of $x(t)$ are needed.

1.3 Mathematical Review

1.3.1 Problem (1.51)

(a) We have

$$e^{j\theta} = \cos\theta + j\sin\theta. \tag{S1.51-1}$$

and

$$e^{-j\theta} = \cos\theta - j\sin\theta. \tag{S1.51-2}$$

Summing eqs. (S1.51-1) and (S1.51-2) we get

$$\cos\theta = \frac{1}{2}(e^{j\theta} + e^{-j\theta}).$$

(b) Subtracting eq. (S1.51-2) from (S1.51-1) we get

$$\sin\theta = \frac{1}{2j}(e^{j\theta} - e^{-j\theta}).$$

(c) We now have $e^{j(\theta+\phi)} = e^{j\theta}e^{j\phi}$. Therefore,

$$\begin{aligned} \cos(\theta+\phi) + j\sin(\theta+\phi) &= (\cos\theta\cos\phi - \sin\theta\sin\phi) \\ &+ j(\sin\theta\cos\phi + \cos\theta\sin\phi) \end{aligned} \tag{S1.51-3}$$

Putting $\theta = \phi$ in eq. (S1.51-3), we get

$$\cos 2\theta = \cos^2\theta - \sin^2\theta.$$

Putting $\theta = -\phi$ in eq. (S1.51-3), we get

$$1 = \cos^2\theta + \sin^2\theta.$$

Adding the two above equations and simplifying

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta).$$

(d) Equating the real parts in eq. (S1.51-3) with arguments $(\theta+\phi)$ and $(\theta-\phi)$ we get

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

and

$$\cos(\theta-\phi) = \cos\theta\cos\phi + \sin\theta\sin\phi.$$

Subtracting the two above equations, we obtain

$$\sin\theta\sin\phi = \frac{1}{2}[\cos(\theta-\phi) - \cos(\theta+\phi)].$$

(e) Equating imaginary parts in in eq. (S1.51-3), we get

$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi.$$

1.3.2 Problem (1.52)

(a) $zz^* = re^{j\theta}re^{-j\theta} = r^2$

(b) $z/z^* = re^{j\theta}r^{-1}e^{j\theta} = e^{j2\theta}$

(c) $z + z^* = x + jy + x - jy = 2x = 2\mathcal{R}e\{z\}$

(d) $z - z^* = x + jy - x + jy = 2jy = 2\mathcal{I}m\{z\}$

(e) $(z_1 + z_2)^* = ((x_1 + x_2) + j(y_1 + y_2))^* = x_1 - jy_1 + x_2 - jy_2 = z_1^* + z_2^*$

(f) Consider $(az_1z_2)^*$ for $a > 0$.

$$(az_1z_2)^* = (ar_1r_2e^{j(\theta_1+\theta_2)})^* = ar_1e^{-j\theta_1}r_2e^{-j\theta_2} = az_1^*z_2^*.$$

For $a < 0$, $a = |a|e^{j\pi}$. Therefore,

$$(az_1z_2)^* = (|a|r_1r_2e^{j(\theta_1+\theta_2+\pi)})^* = |a|e^{-j\pi}r_1e^{-j\theta_1}r_2e^{-j\theta_2} = az_1^*z_2^*.$$

(g) For $|z_2| \neq 0$,

$$\left(\frac{z_1}{z_2}\right)^* = \frac{r_1}{r_2}e^{-j\theta_1}e^{j\theta_2} = \frac{r_1e^{-j\theta_1}}{r_2e^{-j\theta_2}} = \frac{z_1^*}{z_2^*}.$$

(h) From (c), we get

$$\mathcal{R}e\{\frac{z_1}{z_2}\} = \frac{1}{2}\left[\left(\frac{z_1}{z_2}\right) + \left(\frac{z_1}{z_2}\right)^*\right].$$

Using (g) on this, we get

$$\mathcal{R}e\{\frac{z_1}{z_2}\} = \frac{1}{2}\left[\left(\frac{z_1}{z_2}\right) + \left(\frac{z_1^*}{z_2^*}\right)\right] = \frac{1}{2}\left[\frac{z_1z_2^* + z_1^*z_2}{z_2z_2^*}\right].$$

1.3.3 Problem (1.54)

(a) For $\alpha = 1$, it is fairly obvious that

$$\sum_{n=0}^{N-1} \alpha^n = N.$$

For $\alpha \neq 1$, we may write

$$(1-\alpha)\sum_{n=0}^{N-1} \alpha^n = \sum_{n=0}^{N-1} \alpha^n - \sum_{n=0}^{N-1} \alpha^{n+1} = 1 - \alpha^N.$$

Therefore,

$$\sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}.$$

(b) For $|\alpha| < 1$,

$$\lim_{N\to\infty} \alpha^N = 0.$$

Therefore, from the result of the previous part,

$$\lim_{N\to\infty} \sum_{n=0}^{N-1} \alpha^n = \sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}.$$

(c) Differentiating both sides of the result of part (b) wrt α, we get

$$\begin{aligned} \frac{d}{d\alpha}\left(\sum_{n=0}^{\infty}\alpha^N\right) &= \frac{d}{d\alpha}\left(\frac{1}{1-\alpha}\right) \\ \sum_{n=0}^{\infty} n\alpha^{n-1} &= \frac{1}{(1-\alpha)^2} \end{aligned}$$

(d) We may write

$$\sum_{n=k}^{\infty} \alpha^n = \alpha^k \sum_{n=0}^{\infty} \alpha^n = \frac{\alpha^k}{1-\alpha} \text{ for } |\alpha| < 1.$$

Chapter 2

Linear Time Invariant

2.1 Basic Problems

2.1.1 Problem (2.1)

(a) We know that

$$y_1[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] \tag{S2.1−1}$$

The signals $x[n]$ and $h[n]$ are as shown in Figure S2.1.

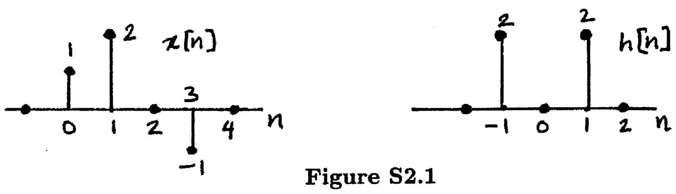


Figure S2.1

From this figure, we can easily see that the above convolution sum reduces to

$$\begin{aligned} y_1[n] &= h[-1]x[n+1] + h[1]x[n-1] \\ &= 2x[n+1] + 2x[n-1] \end{aligned}$$

This gives

$$y_1[n] = 2\delta[n+1] + 4\delta[n] + 2\delta[n-1] + 2\delta[n-2] - 2\delta[n-4]$$

(b) We know that

$$y_2[n] = x[n+2] * h[n] = \sum_{k=-\infty}^{\infty} h[k]x[n+2-k]$$

Comparing with eq. (S2.1-1), we see that

$$y_2[n] = y_1[n+2]$$

(c) We may rewrite eq. (S2.1-1) as

$$y_1[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Similarly, we may write

$$y_3[n] = x[n] * h[n+2] = \sum_{k=-\infty}^{\infty} x[k]h[n+2-k]$$

Comparing this with eq. (S2.1), we see that

$$y_3[n] = y_1[n+2]$$

2.1.2 Problem (2.3)

Let us define the signals

$$x_1[n] = \left(\frac{1}{2}\right)^n u[n]$$

and

$$h_1[n] = u[n].$$

We note that

$$x[n] = x_1[n-2] \qquad \text{and} \qquad h[n] = h_1[n+2]$$

Now,

$$\begin{aligned} y[n] &= x[n] * h[n] = x_1[n-2] * h_1[n+2] \\ &= \sum_{k=-\infty}^{\infty} x_1[k-2] h_1[n-k+2] \end{aligned}$$

By replacing k with $m+2$ in the abovr summation, we obtain

$$y[n] = \sum_{m=-\infty}^{\infty} x_1[m] h_1[n-m] = x_1[n] * h_1[n]$$

Using the results of Example 2.1 in the text book, we may write

$$y[n] = 2\left[1 - \left(\frac{1}{2}\right)^{n+1}\right] u[n]$$

2.1.3 Problem (2.8)

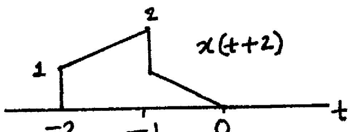
Using the convolution integral,

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau.$$

Given that $h(t) = \delta(t+2) + 2\delta(t+1)$, the above integral reduces to

$$x(t) * y(t) = x(t+2) + 2x(t+1)$$

The signals $x(t+2)$ and $2x(t+1)$ are plotted in Figure S2.8.



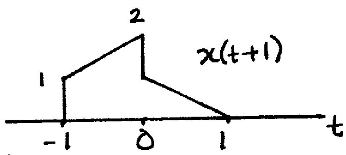


Figure S2.8

Using these plots, we can easily show that

$$y(t) = \begin{cases} t+3, & -2 < t \leq -1 \\ t+4, & -1 < t \leq 0 \\ 2-2t, & 0 < t \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

2.1.4 Problem (2.21)

(a) The desired convolution is

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\ &= \beta^n \sum_{k=0}^{n} (\alpha/\beta)^k \text{ for } n \geq 0 \\ &= [\frac{\beta^{n+1} - \alpha^{n+1}}{\beta - \alpha}]u[n] \text{ for } \alpha \neq \beta. \end{aligned}$$

(b) From (a),

$$y[n] = \alpha^n \left[\sum_{k=0}^{n} 1\right] u[n] = (n+1)\alpha^n u[n].$$

(c) For $n \leq 6$,

$$y[n] = 4^n \left\{\sum_{k=0}^{\infty} (-\frac{1}{8})^k - \sum_{k=0}^{3} (-\frac{1}{8})^k\right\}.$$

For $n > 6$,

$$y[n] = 4^n \left\{\sum_{k=0}^{\infty} (-\frac{1}{8})^k - \sum_{k=0}^{n-1} (-\frac{1}{8})^k\right\}.$$

Therefore,

$$y[n] = \begin{cases} (8/9)(-1/8)^4 4^n, & n \leq 6 \\ (8/9)(-1/2)^n, & n > 6 \end{cases}$$

(d) The desired convolution is

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\ &= x[0]h[n] + x[1]h[n-1] + x[2]h[n-2] + x[3]h[n-3] + x[4]h[n-4] \\ &= h[n] + h[n-1] + h[n-2] + h[n-3] + h[n-4]. \end{aligned}$$

This is as shown in Figure S2.21.

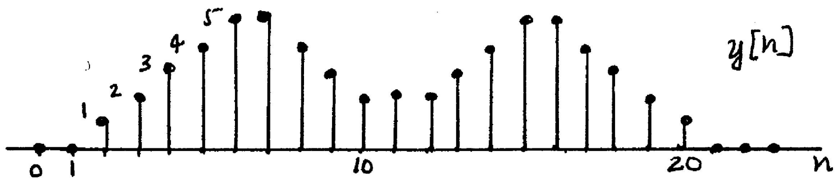


Figure S2.21

2.1.5 Problem (2.22)

(a) The desired convolution is

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ &= \int_0^t e^{-\alpha t}e^{-\beta(t-\tau)}d\tau, \ t \geq 0 \end{aligned}$$

Then

$$y(t) = \begin{cases} \frac{e^{-\beta t}\{e^{-(\alpha-\beta)t}-1\}}{\beta-\alpha}u(t) & \alpha \neq \beta \\ te^{-\beta t}u(t) & \alpha = \beta \end{cases}.$$

(b) The desired convolution is

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ &= \int_0^2 h(t-\tau)d\tau - \int_2^5 h(t-\tau)d\tau. \end{aligned}$$

This may be written as

$$y(t) = \begin{cases} \int_0^2 e^{2(t-\tau)}d\tau - \int_2^5 e^{2(t-\tau)}d\tau, & t \leq 1 \\ \int_{t-1}^2 e^{2(t-\tau)}d\tau - \int_2^5 e^{2(t-\tau)}d\tau, & 1 \leq t \leq 3 \\ -\int_{t-1}^5 e^{2(t-\tau)}d\tau, & 3 \leq t \leq 6 \\ 0, & 6 < t \end{cases}$$

Therefore,

$$y(t) = \begin{cases} (1/2)[e^{2t} - 2e^{2(t-2)} + e^{2(t-5)}], & t \leq 1 \\ (1/2)[e^{2} + e^{2(t-5)} - 2e^{2(t-2)}], & 1 \leq t \leq 3 \\ (1/2)[e^{2(t-5)} - e^{2}], & 3 \leq t \leq 6 \\ 0, & 6 < t \end{cases}$$

(c) The desired convolution is

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ &= \int_{0}^{2} \sin(\pi\tau)h(t-\tau)d\tau. \end{aligned}$$

This gives us

$$y(t) = \begin{cases} 0, & t<1 \\ (2/\pi)[1-\cos\{\pi(t-1)\}], & 1<t<3 \\ (2/\pi)[\cos\{\pi(t-3)\}-1], & 3<t<5 \\ 0, & 5<t \end{cases}$$

(d) Let

$$h(t) = h_1(t) - \frac{1}{3}\delta(t-2),$$

where

$$h_1(t) = \begin{cases} 4/3, & 0 \le t \le 1 \\ 0, & \text{otherwise} \end{cases}.$$

Now,

$$y(t) = h(t) * x(t) = [h_1(t) * x(t)] - \frac{1}{3}x(t-2).$$

We have

$$h_1(t) * x(t) = \int_{t-1}^{t} \frac{4}{3}(a\tau + b)d\tau = \frac{4}{3}[\frac{1}{2}at^2 - \frac{1}{2}a(t-1)^2 + bt - b(t-1)].$$

Therefore,

$$y(t) = \frac{4}{3}[\frac{1}{2}at^2 - \frac{1}{2}a(t-1)^2 + bt - b(t-1)] - \frac{1}{3}[a(t-2)+b] = at + b = x(t).$$

(e) $x(t)$ periodic implies $y(t)$ periodic. $\therefore$ determine 1 period only. We have

$$y(t) = \begin{cases} \int_{t-1}^{-\frac{1}{2}} (t-\tau-1)d\tau + \int_{-\frac{1}{2}}^{t} (1-t+\tau)d\tau = \frac{1}{4} + t - t^2, & -\frac{1}{2} < t < \frac{1}{2} \\ \int_{t-1}^{\frac{1}{2}} (1-t+\tau)d\tau + \int_{\frac{1}{2}}^{t} (t-1-\tau)d\tau = t^2 - 3t + 7/4, & \frac{1}{2} < t < \frac{3}{2} \end{cases}$$

The period of $y(t)$ is 2.

2.1.6 Problem (2.25)

(a) We may write $x[n]$ as

$$x[n] = \left(\frac{1}{3}\right)^{|n|}.$$

Now, the desired convolution is

$$\begin{aligned}
y[n] &= h[n] * x[n] \\
&= \sum_{k=-\infty}^{-1} (1/3)^{-k}(1/4)^{n-k}u[n-k+3] + \sum_{k=0}^{\infty}(1/3)^{k}(1/4)^{n-k}u[n-k+3] \\
&= (1/12)\sum_{k=0}^{\infty}(1/3)^{k}(1/4)^{n+k}u[n+k+4] + \sum_{k=0}^{\infty}(1/3)^{k}(1/4)^{n-k}u[n-k+3]
\end{aligned}$$

By consider each summation in the above equation separately, we may show that

$$y[n] = \begin{cases} (12^4/11)3^n, & n < -4 \\ (1/11)4^4, & n = -4 \\ (1/4)^n(1/11) + -3(1/4)^n + 3(256)(1/3)^n, & n \geq -3 \end{cases}.$$

(b) Now consider the convolution

$$y_1[n] = [(1/3)^n u[n]] * [(1/4)^n u[n+3]].$$

We may show that

$$y_1[n] = \begin{cases} 0, & n < -3 \\ -3(1/4)^n + 3(256)(1/3)^n, & n \geq -3 \end{cases}.$$

Also, consider the convolution

$$y_2[n] = [(3)^n u[-n-1]] * [(1/4)^n u[n+3]].$$

We may show that

$$y_2[n] = \begin{cases} (12^4/11)3^n, & n < -4 \\ (1/4)^n(1/11), & n \geq -3 \end{cases}.$$

Clearly, $y_1[n] + y_2[n] = y[n]$ obtained in the previous part.

2.1.7 Problem (2.29)

(a) Causal because $h(t) = 0$ for $t < 0$. Stable because $\int_{-\infty}^{\infty} |h(t)|dt = e^{-8}/4 < \infty$.

(b) Not causal because $h(t) \neq 0$ for $t < 0$. Unstable because $\int_{-\infty}^{\infty} |h(t)| = \infty$.

(c) Not causal because $h(t) \neq 0$ for $t < 0$. a Stable because $\int_{-\infty}^{\infty} |h(t)|dt = e^{100}/2 < \infty$.

(d) Not causal because $h(t) \neq 0$ for $t < 0$. Stable because $\int_{-\infty}^{\infty} |h(t)|dt = e^{-2}/2 < \infty$.

(e) Not causal because $h(t) \neq 0$ for $t < 0$. Stable because $\int_{-\infty}^{\infty} |h(t)|dt = 1/3 < \infty$.

(f) Causal because $h(t) = 0$ for $t < 0$. Stable because $\int_{-\infty}^{\infty} |h(t)|dt = 1 < \infty$.

(g) Causal because $h(t) = 0$ for $t < 0$. Unstable because $\int_{-\infty}^{\infty} |h(t)|dt = \infty$.

2.2 Advanced Problems

2.2.1 Problem (2.40)

(a) Note that

$$y(t) = \int_{-\infty}^{t} e^{-(t-\tau)} x(\tau - 2) d\tau = \int_{-\infty}^{t-2} e^{-(t-2-\tau')} x(\tau') d\tau'.$$

Therefore,

$$h(t) = e^{-(t-2)} u(t-2).$$

(b) We have

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau \\ &= \int_{2}^{\infty} e^{-(\tau-2)} [u(t-\tau+1) - u(t-\tau-2) \end{aligned}$$

$h(\tau)$ and $x(t-\tau)$ are as shown in the figure below.

Using this figure, we may write

$$y(t) = \begin{cases} 0, & t < 1 \\ \int_{2}^{t+1} e^{-(\tau-2)} d\tau = 1 - e^{-(t-1)}, & 1 < t < 4 \\ \int_{t-2}^{t+1} e^{-(\tau-2)} d\tau = e^{-(t-4)} [1 - e^{-3}], & t > 4 \end{cases}$$

2.2.2 Problem (2.43)

(a) We first have

$$\begin{aligned}[x(t)*h(t)]*g(t) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)h(\sigma'-\tau)g(t-\sigma')d\tau d\sigma' \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)h(\sigma)g(t-\sigma-\tau)d\tau d\sigma\end{aligned}$$

Also,

$$\begin{aligned}x(t)*[h(t)*g(t)] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t-\sigma')h(\tau)g(\sigma'-\tau)d\sigma' d\tau \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\sigma)h(\tau)g(t-\tau-\sigma)d\tau d\sigma \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)h(\sigma)g(t-\sigma-\tau)d\tau d\sigma\end{aligned}$$

The equality is proved.

(b) (i) We first have

$$w[n] = u[n]*h_1[n] = \sum_{k=0}^{n}\left(-\frac{1}{2}\right)^k = \frac{2}{3}\left[1-(-\frac{1}{2})^{n+1}\right]u[n].$$

Now,

$$y[n] = w[n]*h_2[n] = (n+1)u[n].$$

(ii) We first have

$$g[n] = h_1[n]*h_2[n] = \sum_{k=0}^{n}\left(-\frac{1}{2}\right)^k + \frac{1}{2}\sum_{k=0}^{n-1}(-\frac{1}{2})^k = u[n]$$

Now,

$$y[n] = u[n]*g[n] = u[n]*u[n] = (n+1)u[n].$$

The same result was obtained in both parts (i) and (ii).

(c) Note that

$$x[n]*(h_2[n]*h_1[n]) = (x[n]*h_2[n])*h_1[n].$$

Also note that

$$x[n]*h_2[n] = \alpha^n u[n] - \alpha^n u[n-1] = \delta[n].$$

Therefore,

$$x[n]*h_1[n]*h_2[n] = \delta[n]*\sin 8n = \sin 8n.$$

2.3 Extension Problems

2.3.1 Problem (2.61)

(a) (i) From Kirchoff's voltage law, we know that the input voltage must equal the sum of the voltages across the inductor and capacitor. Therefore,

$$x(t) = LC\frac{d^2y(t)}{dt^2} + y(t).$$

Using the values of L and C we get

$$\frac{d^2y(t)}{dt^2} + y(t) = x(t).$$

(ii) Using the results of Problem 2.53, we know that the homogeneous solution of the differential equation

$$\frac{d^2y(t)}{dt^2} + a_1\frac{dy(t)}{dt} + a_2y(t) = bx(t).$$

will have terms of the form $K_1e^{s_0t}+K_2e^{s_1t}$ where s_0 and s_1 are roots of the equation

$$s^2 + a_1s + a_2 = 0.$$

(It is assumed here that $s_0 \neq s_1$.) In this problem, $a_1 = 0$ and $a_2 = 1$. Therefore, the root of the equation are $s_0 = j$ and $s_1 = -j$.. The homogeneous solution is

$$y_h(t) = K_1e^{jt} + K_2e^{-jt}.$$

And, $\omega_1 = 1 = \omega_2$.

(iii) If the voltage and current are restricted to be real, then $K_1 = K_2 = K$. Therefore,

$$y_h(t) = 2K\cos(t) = 2K\sin(t + \pi/2).$$

(b) (i) From Kirchoff's voltage law, we know that the input voltage must equal the sum of the voltages across the resistor and capacitor. Therefore,

$$x(t) = RC\frac{dy(t)}{dt} + y(t).$$

Using the values of R, L, and C we get

$$\frac{dy(t)}{dt} + y(t) = x(t).$$

(ii) The natural response of the system is the homogeneous solution of the above differential equation. Using the results of Problem 2.53, we know that the homogeneous solution of the differential equation

$$\frac{dy(t)}{dt} + a_1 y(t) = bx(t).$$

will have terms of the form $Ae^{s_0 t}$ where s_0 is the root of the equation

$$s + a_1 = 0.$$

In this problem, $a_1 = 1$. Therefore, the root of the equation are $s_0 = -1$. The homogeneous solution is

$$y_h(t) = Ke^{-t}.$$

And, $a = 1$.

(c) (i) From Kirchoff's voltage law, we know that the input voltage must equal the sum of the voltages across the resistor, inductor, and capacitor. Therefore,

$$x(t) = LC\frac{d^2y(t)}{dt^2} + RC\frac{dy(t)}{dt} + y(t).$$

Using the values of R, L, and C we get

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 5y(t) = 5x(t).$$

(ii) Using the results of Problem 2.53, we know that the homogeneous solution of the differential equation

$$\frac{d^2y(t)}{dt^2} + a_1\frac{dy(t)}{dt} + a_2 y(t) = bx(t).$$

will have terms of the form $K_1e^{s_0 t} + K_2 e^{s_1 t}$ where s_0 and s_1 are roots of the equation

$$s^2 + a_1 s + a_2 = 0.$$

(It is assumed here that $s_0 \neq s_1$.) In this problem, $a_1 = 2$ and $a_2 = 5$. Therefore, the root of the equation are $s_0 = -1 + 2j$ and $s_1 = -1 - 2j$. The homogeneous solution is

$$y_h(t) = K_1 e^{-t}e^{2jt} + K_2 e^{-t}e^{-2jt}.$$

And, $a = 1$.

(iii) If the voltage and current are restricted to be real, then $K_1 = K_2 = K$. Therefore,

$$y_h(t) = 2Ke^{-t}\cos(2t) = 2Ke^{-t}\sin(2t + \pi/2).$$

2.3.2 Problem (2.66)

(a) The plot of $x_1(t)$ is as shown in Figure S2.66.

(b) The plots of $x_2(t)$ and $x_2(t)$ are as shown in Figure S2.66.

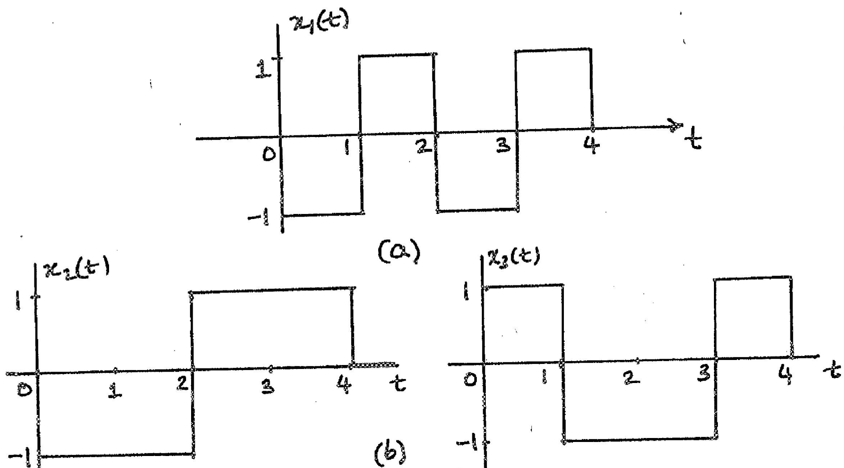


Figure S2.66

(c) $x_1(t) * h_2(t) = x_2(t) * h_3(t) = x_1(t) * h_3(t) = 0$ for $t = 4$.

Chapter 3

Fourier Series Representation of Periodic Signals

3.1 Basic Problems

3.1.1 Problem (3.1)

Using the Fourier series synthesis eq. (3.38),

$$\begin{aligned}
x(t) &= a_1 e^{j(2\pi/T)t} + a_{-1} e^{-j(2\pi/T)t} + a_3 e^{j3(2\pi/T)t} + a_{-3} e^{-j3(2\pi/T)t} \\
&= 2e^{j(2\pi/8)t} + 2e^{-j(2\pi/8)t} + 4je^{j3(2\pi/8)t} - 4je^{-j3(2\pi/8)t} \\
&= 4\cos(\frac{\pi}{4}t) - 8\sin(\frac{6\pi}{8}t) \\
&= 4\cos(\frac{\pi}{4}t) + 8\cos(\frac{3\pi}{4}t + \frac{\pi}{2})
\end{aligned}$$

3.1.2 Problem (3.3)

The given signal is

$$\begin{aligned}
x(t) &= 2 + \frac{1}{2}e^{j(2\pi/3)t} + \frac{1}{2}e^{-j(2\pi/3)t} - 2je^{j(5\pi/3)t} + 2je^{-j(5\pi/3)t} \\
&= 2 + \frac{1}{2}e^{j2(2\pi/6)t} + \frac{1}{2}e^{-j2(2\pi/6)t} - 2je^{j5(2\pi/6)t} + 2je^{-j5(2\pi/6)t}
\end{aligned}$$

From this, we may conclude that the fundamental frequency of $x(t)$ is $2\pi/6 = \pi/3$. The non-zero Fourier series coeffcients of $x(t)$ are:

$$a_0 = 2, \qquad a_2 = a_{-2} = \frac{1}{2}, \qquad a_5 = a_{-5}^* = -2j$$

3.1.3 Problem (3.4)

Since $\omega_0 = \pi$, $T = 2\pi/\omega_0 = 2$. Therefore,

$$a_k = \frac{1}{2}\int_0^2 x(t)e^{-jk\pi t}dt$$

Now,

$$a_0 = \frac{1}{2}\int_0^1 1.5dt - \frac{1}{2}\int_1^2 1.5dt = 0$$

and for $k \neq 0$

$$\begin{aligned} a_k &= \frac{1}{2}\int_0^1 1.5e^{-jk\pi t}dt - \frac{1}{2}\int_1^2 1.5e^{-jk\pi t}dt \\ &= \frac{3}{2k\pi j}[1 - e{-}jk\pi] \\ &= \frac{3}{k\pi}e^{-jk(\pi/2)}\sin(\frac{k\pi}{2}) \end{aligned}$$

3.1.4 Problem (3.5)

Both $x_1(1-t)$ and $x_1(t-1)$ are periodic with fundemental period $T_1 = \frac{2\pi}{\omega_1}$. Since $y(t)$ is a linear combination of $x_1(1-t)$ and $x_1(t-1)$, it is also periodic with fundemental period $T_2 = \frac{2\pi}{\omega_1}$. Therefore, $\omega_2 = \omega_1$.

Since $x_1(t) \overset{FS}{\longleftrightarrow} a_k$, using the results in Table 3.1 we have

$$x_1(t+1) \overset{FS}{\longleftrightarrow} a_k e^{jk(2\pi/T_1)}$$

$$x_1(t-1) \overset{FS}{\longleftrightarrow} a_k e^{-jk(2\pi/T_1)} \Rightarrow x_1(-t+1) \overset{FS}{\longleftrightarrow} a_{-k}e^{-jk(2\pi/T_1)}$$

Therefore,

$$x_1(t+1) + x_1(1-t) \overset{FS}{\longleftrightarrow} a_k e^{jk(2\pi/T_1)} + a_{-k}e^{-jk(2\pi/T_1)} = e^{-j\omega_1 k}(a_k + a_{-k})$$

3.1.5 Problem (3.20)

(a) Current through the capacitor $= C\frac{dy(t)}{dt}$.
Voltage across resistor $= RC\frac{dy(t)}{dt}$.
Voltage across inductor $= LC\frac{d^2y(t)}{dt^2}$.
Input voltage = Voltage across resistor + Voltage across inductor + Voltage across capacitor.
Therefore,

$$x(t) = LC\frac{d^2y(t)}{dt^2} + RC\frac{dy(t)}{dt} + y(t)$$

Substituting for R, L and C, we have

$$\frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt} + y(t) = x(t)$$

(b) We will now use an approach similar to the one used in part (b) of the previous problem. If we assume that the input is of the form $e^{j\omega t}$, then the output will be of the form $H(j\omega)e^{j\omega t}$. Substituting in the above differential equation and simplifying, we obtain

$$H(j\omega) = \frac{1}{-\omega^2 + j\omega + 1}$$

(c) The signal $x(t)$ is periodic with period 2π. Since $x(t)$ can be expressed in the form

$$x(t) = \frac{1}{2j}e^{j(2\pi/2\pi)t} - \frac{1}{2j}e^{-j(2\pi/2\pi)t},$$

the non-zero Fourier series coefficients of $x(t)$ are

$$a_1 = a_{-1}^* = \frac{1}{2j}.$$

Using the results derived in Section 3.8 (see eq.(3.124)), we have

$$\begin{aligned} y(t) &= a_1 H(j)e^{jt} - a_{-1}H(-j)e^{-jt} \\ &= (1/2j)(\frac{1}{j}e^{jt} - \frac{1}{-j}e^{-jt}) \\ &= (-1/2)(e^{jt} + e^{-jt}) \\ &= -\cos(t) \end{aligned}$$

3.1.6 Problem (3.21)

Using the Fourier series synthesis eq.

$$\begin{aligned} x(t) &= a_1 e^{j(2\pi/T)t} + a_{-1}e^{-j(2\pi/T)t} + a_5 e^{j5(2\pi/T)t} + a_{-5}e^{-j5(2\pi/T)t} \\ &= je^{j(2\pi/8)t} - je^{-j(2\pi/8)t} + 2e^{j5(2\pi/8)t} + 2e^{-j5(2\pi/8)t} \\ &= -2\sin(\frac{\pi}{4}t) + 4\cos(\frac{5\pi}{4}t) \\ &= -2\cos(\frac{\pi}{4}t - \pi/2) + 4\cos(\frac{5\pi}{4}t). \end{aligned}$$

3.1.7 Problem (3.22)

(a) (i) $T = 1$, $a_0 = 0$, $a_k = \frac{j(-1)^k}{k\pi}$, $k \neq 0$.

(ii) Here,

$$x(t) = \begin{cases} t+2, & -2 < t < -1 \\ 1, & -1 < t < 1 \\ 2-t, & 1 < t < 2 \end{cases}$$

$T = 6$, $a_0 = 1/2$, and

$$a_k = \begin{cases} 0, & k \text{ even} \\ \frac{6}{\pi^2 k^2}\sin(\frac{\pi k}{2})\sin(\frac{\pi k}{6}), & k \text{ odd} \end{cases}.$$

(iii) $T = 3$, $a_0 = 1$, and

$$a_k = \frac{3j}{2\pi^2 k^2}[e^{jk2\pi/3}\sin(k2\pi/3) + 2e^{jk\pi/3}\sin(k\pi/3)], \qquad k \neq 0.$$

(iv) $T = 2$, $a_0 = -1/2$, $a_k = \frac{1}{2} - (-1)^k$, $k \neq 0$.

(v) $T = 6$, $\omega_0 = \pi/3$, and

$$a_k = \frac{\cos(2k\pi/3) - \cos(k\pi/3)}{jk\pi/3}.$$

Note that $a_0 = 0$ and $a_{k\ even} = 0$.

(vi) $T = 4$, $\omega_0 = \pi/2$, $a_0 = 3/4$ and

$$a_k = \frac{e^{-jk\pi/2}\sin(k\pi/2) + e^{-jk\pi/4}\sin(k\pi/4)}{k\pi}, \qquad \forall k.$$

(b) $T = 2$, $a_k = \frac{-1^k}{2(1+jk\pi)}[e - e^{-1}]$ for all k.

(c) $T = 3$, $\omega_0 = 2\pi/3$, $a_0 = 1$ and

$$a_k = \frac{2e^{-j\pi k/3}}{\pi k}\sin(2\pi k/3) + \frac{e^{-j\pi k}}{\pi k}\sin(\pi k).$$

3.2 Advanced Problems

3.2.1 Problem (3.40)

(a) $x(t-t_0)$ is also periodic with period T. The Fourier series coefficients b_k of $x(t - t_0)$ are

$$\begin{aligned} b_k &= \frac{1}{T}\int_T x(t-t_0)e^{-jk(2\pi/T)t}dt \\ &= \frac{e^{-jk(2\pi/T)t_0}}{T}\int_T x(\tau)e^{-jk(2\pi/T)\tau}d\tau \\ &= e^{-jk(2\pi/T)t_0}a_k \end{aligned}$$

Similarly, the Fourier series coefficients of $x(t+t_0)$ are

$$c_k = e^{jk(2\pi/T)t_0}a_k.$$

Finally, the Fourier series coefficients of $x(t-t_0)+x(t+t_0)$ are

$$d_k = b_k + c_k = e^{-jk(2\pi/T)t_0}a_k + e^{jk(2\pi/T)t_0}a_k = 2\cos(k2\pi t_0/T)a_k.$$

(b) Note that $\mathcal{E}v\{x(t)\} = [x(t)+x(-t)]/2$. The FS coefficients of $x(-t)$ are

$$\begin{aligned} b_k &= \frac{1}{T}\int_T x(-t)e^{-jk(2\pi/T)t}dt \\ &= \frac{1}{T}\int_T x(\tau)e^{jk(2\pi/T)\tau}d\tau \\ &= a_{-k} \end{aligned}$$

Therefore, the FS coefficients of $\mathcal{E}v\{x(t)\}$ are

$$c_k = \frac{a_k+b_k}{2} = \frac{a_k+a_{-k}}{2}.$$

(c) Note that $\mathcal{R}e\{x(t)\} = [x(t)+x^*(t)]/2$. The FS coefficients of $x^*(t)$ are

$$b_k = \frac{1}{T}\int_T x^*(t)e^{-jk(2\pi/T)t}dt.$$

Conjugating both sides, we get

$$b_k^* = \frac{1}{T}\int_T x(t)e^{jk(2\pi/T)t}dt = a_{-k}.$$

Therefore, the FS coefficients of $\mathcal{R}e\{x(t)\}$ are

$$c_k = \frac{a_k+b_k}{2} = \frac{a_k+a^*_{-k}}{2}.$$

(d) The Fourier series synthesis equation gives

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T)kt}.$$

Differentiating both sides wrt t twice, we get

$$\frac{d^2x(t)}{dt^2} = \sum_{k=-\infty}^{\infty} -k^2\frac{4\pi^2}{T^2}a_k e^{j(2\pi/T)kt}.$$

By inspection, we know that the Fourier series coefficients of $d^2x(t)/dt^2$ are $-k\frac{4\pi^2}{T^2}a_k$.

(e) The period of $x(3t)$ is a third of the period of $x(t)$. Therefore, the signal $x(3t-1)$ is periodic with period $T/3$. The Fourier series coefficents of $x(3t)$ are still a_k. Using the analysis of part (a), we know that the Fourier series coefficients of $x(3t-1)$ is $e^{-jk(6\pi/T)}a_k$.

3.2.2 Problem (3.46)

(a) The Fourier series coefficients of $z(t)$ are

$$\begin{aligned}
c_k &= \frac{1}{T}\int_T \sum_n \sum_l a_n b_l e^{j(n+l)\omega_0 t} e^{-jk\omega_0 t} dt \\
&= \frac{1}{T}\sum_n \sum_l a_n b_l \delta(k-(n+l)) \\
&= \sum_n a_n b_{k-n}
\end{aligned}$$

(b) (i) Here, $T_0 = 3$ and $\omega_0 = 2\pi/3$. Therefore,

$$c_k = [\frac{1}{2}\delta(k-30) + \frac{1}{2}\delta(k+30)] * \frac{2\sin(k2\pi/3)}{3k2\pi/3}.$$

Simplifying,

$$c_k = \frac{\sin\{(k-30)2\pi/3\}}{3(k-30)2\pi/3} + \frac{\sin\{(k+30)2\pi/3\}}{3(k+30)2\pi/3}$$

and $c_{\pm 30} = 1/3$.

(ii) We may express $x_2(t)$ as

$$x_2(t) = \text{sum of two shifted square waves } \times \cos(20\pi t).$$

Here, $T_0 = 3$, $\omega_0 = 2\pi/3$. Therefore,

$$\begin{aligned} c_k &= \frac{1}{3}e^{-j(k-30)(2\pi/3)}\frac{\sin\{(k-30)2\pi/3\}}{(k-30)2\pi/3} + \frac{1}{3}e^{-j(k+30)(2\pi/3)}\frac{\sin\{(k+30)2\pi/3\}}{(k+30)2\pi/3} \\ &+ \frac{1}{3}e^{-j(k-30)(\pi/3)}\frac{\sin\{(k-30)\pi/3\}}{(k-30)2\pi/3} + \frac{1}{3}e^{-j(k+30)(\pi/3)}\frac{\sin\{(k+30)\pi/3\}}{(k+30)2\pi/3} \end{aligned}$$

(iii) Here, $T_0 = 4$, $\omega_0 = \pi/2$. Therefore,

$$c_k = \left[\frac{1}{2}\delta(k-40) + \frac{1}{2}\delta(k+40)\right] * \frac{j[k\omega_0 + e^{-1}\{\sin k\omega_0 - \cos k\omega_0\}]}{2[1+(k\omega_0)^2]}.$$

Simplifying,

$$\begin{aligned} c_k &= \frac{j[(k-40)\omega_0 + e^{-1}\{\sin(k-40)\omega_0 - \cos(k-40)\omega_0\}]}{4[1+\{(k-40)\omega_0\}^2]} \\ &+ \frac{j[(k+40)\omega_0 + e^{-1}\{\sin(k+40)\omega_0 - \cos(k+40)\omega_0\}]}{4[1+\{(k+40)\omega_0\}^2]}. \end{aligned}$$

(c) From Problem 3.42, we know that $b_k = a^*_{-k}$. From part (a), we know that the FS coefficients of $z(t) = x(t)y(t) = x(t)x^*(t) = |x(t)|^2$ will be

$$c_k = \sum_{n=-\infty}^{\infty} a_n b_{n-k} = \sum_{n=-\infty}^{\infty} a_n a_{n+k}.$$

From the Fourier series analysis equation, we have

$$c_k = \frac{1}{T_0}\int_0^{T_0} |x(t)|^2 e^{-j(2\pi/T_0)kt}dt = \sum_{n=-\infty}^{\infty} a_n a^*_{n+k}.$$

Putting $k = 0$ in this equation, we get

$$\frac{1}{T_0}\int_0^{T_0} |x(t)|^2 dt = \sum_{n=-\infty}^{\infty} |a_n|^2.$$

3.2.3 Problem (3.47)

Considering $x(t)$ to be periodic with period 1, the nonzero FS coefficients of $x(t)$ are $a_1 = a_{-1} = 1/2$. If we now consider $x(t)$ to be periodic with period 3, then the the nonzero FS coefficients of $x(t)$ are $b_3 = b_{-3} = 1/2$.

3.3 Extension Problems

3.3.1 Problem (3.65)

(a) Pairs (a) and (b) are orthogonal. Pairs (c) and (d) are not orthogonal.

(b) Orthogonal, but not orthonormal. $A_m = 1/\omega_0$.

(c) Orthonormal.

(d) We have

$$\int_{t_0}^{t_0+T} e^{jm\omega_0\tau}e^{-jn\omega_0\tau}d\tau = e^{j(m-n)\omega_0 t_0}\frac{[e^{j(m-n)2\pi}-1]}{(m-n)\omega_0}$$

This evaluates to 0 when $m \neq n$ and to jT when $m = n$. Therefore, the functions are orthogonal but not orthonormal.

(e) We have

$$\begin{aligned}
\int_{-T}^{T} x_e(t)x_o(t)dt &= \frac{1}{4}\int_{-T}^{T}[x(t)+x(-t)][x(t)-x(-t)]dt \\
&= \frac{1}{4}\int_{-T}^{T}x^2(t)dt - \frac{1}{4}\int_{-T}^{T}x^2(-t)dt \\
&= 0.
\end{aligned}$$

(f) Consider

$$\int_a^b \frac{1}{\sqrt{A_k}}\phi_k(t)\frac{1}{\sqrt{A_l}}\phi_l^*(t)dt = \frac{1}{\sqrt{A_kA_l}}\int_a^b\int_a^b \phi_k(t)\phi_l^*(t)dt.$$

This valuates to zero for $k \neq l$. For $k = l$, it evaluates to $A_k/A_k = 1$. Therefore, the functions are orthonormal.

(g) We have

$$\begin{aligned}
\int_a^b |x(t)|^2dt &= \int_a^b x(t)x^*(t)dt \\
&= \int_a^b \sum_i a_i\phi_i(t)\sum_j a_j\phi_j^*(t)dt \\
&= \sum_i\sum_j a_ia_j^*\int_a^b \phi_i(t)\phi_j^*(t)dt \\
&= \sum_i |a_i|^2.
\end{aligned}$$

(h) We have

$$\begin{aligned}
y(T) &= \int_{-\infty}^{\infty} h_i(T-\tau)\phi_j(\tau)d\tau \\
&= \int_{-\infty}^{\infty} \phi_i(\tau)\phi_j(\tau)d\tau \\
&= \delta_{ij} = 1 \text{ for } i=j \text{ and } 0 \text{ for } i \neq j.
\end{aligned}$$

Chapter 4

The Continuous-Time Fourier Transform

4.1 Basic Problems

4.1.1 Problem (4.1)

(a) Let $x(t) = e^{-2(t-1)}u(t-1)$. Then the Fourier transform $X(j\omega)$ of $x(t)$ is:

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} e^{-2(t-1)}u(t-1)e^{-j\omega t}dt \\ &= \int_{1}^{\infty} e^{-2(t-1)}e^{-j\omega t}dt \\ &= e^{-j\omega}/(2+j\omega) \end{aligned}$$

$|X(j\omega)|$ is as shown in Figure S4.1.

(b) Let $x(t) = e^{-2|t-1|}$. Then the Fourier transform $X(j\omega)$ of $x(t)$ is:

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} e^{-2|t-1|}e^{-j\omega t}dt \\ &= \int_{1}^{\infty} e^{-2(t-1)}e^{-j\omega t}dt + \int_{-\infty}^{1} e^{2(t-1)}e^{-j\omega t}dt \\ &= e^{-j\omega}/(2+j\omega) + e^{-j\omega}/(2-j\omega) \\ &= 4e^{-j\omega}/(4+\omega^2) \end{aligned}$$

$|X(j\omega)|$ is as shown in Figure S4.1.

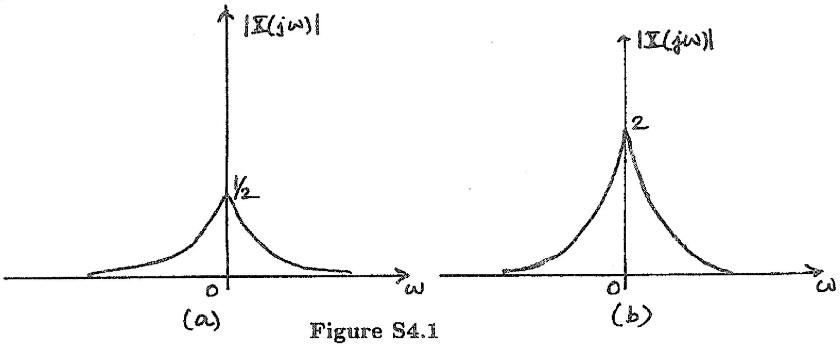


Figure S4.1

4.1.2 Problem (4.3)

(a) The signal $x_1(t) = \sin(2\pi t + \pi/4)$ is periodic with a fundamental period of $T = 1$. This translates to a fundamental frequency of $\omega_0 = 2\pi$. The nonzero Fourier series coefficients of this signal may be found by writing it in the form

$$\begin{aligned} x_1(t) &= \frac{1}{2j}\left(e^{j(2\pi t+\pi/4)} - e^{-j(2\pi t+\pi/4)}\right) \\ &= \frac{1}{2j}e^{j\pi/4}e^{j2\pi t} - \frac{1}{2j}e^{-j\pi/4}e^{-j2\pi t} \end{aligned}$$

Therefore, the nonzero Fourier series coefficients of $x_1(t)$ are

$$a_1 = \frac{1}{2j}e^{j\pi/4}e^{j2\pi t}, \qquad a_{-1} = -\frac{1}{2j}e^{-j\pi/4}e^{-j2\pi t}$$

From Section 4.2, we know that for periodic signals, the Fourier transform consists of a train of impulses occurring at $k\omega_0$. Furthermore, the area under each impulse is 2π times the Fourier series coefficient a_k. Therefore, for $x_1(t)$, the corresponding Fourier transform $X_1(j\omega)$ is given by

$$\begin{aligned} X_1(j\omega) &= 2\pi a_1\delta(\omega - \omega_0) + 2\pi a_{-1}\delta(\omega + \omega_0) \\ &= (\pi/j)e^{j\pi/4}\delta(\omega - 2\pi) - (\pi/j)e^{-j\pi/4}\delta(\omega + 2\pi) \end{aligned}$$

(b) The signal $x_2(t) = 1 + \cos(6\pi t + \pi/8)$ is periodic with a fundamental period of $T = 1/3$. This translates to a fundamental frequency of $\omega_0 = 6\pi$. The nonzero Fourier series coefficients of this signal may be found by writing it in the form

$$\begin{aligned} x_2(t) &= 1 + \frac{1}{2}\left(e^{j(6\pi t+\pi/8)} - e^{-j(6\pi t+\pi/8)}\right) \\ &= 1 + \frac{1}{2}e^{j\pi/8}e^{j6\pi t} + \frac{1}{2}e^{-j\pi/8}e^{-j6\pi t} \end{aligned}$$

Therefore, the nonzero Fourier series coefficients of $x_2(t)$ are

$$a_0 = 1, \qquad a_1 = \frac{1}{2}e^{j\pi/8}e^{j6\pi t}, \qquad a_{-1} = \frac{1}{2}e^{-j\pi/8}e^{-j6\pi t}$$

From Section 4.2, we know that for periodic signals, the Fourier transform consists of a train of impulses occurring at $k\omega_0$. Furthermore, the area under each impulse is 2π times the Fourier series coefficient a_k. Therefore, for $x_2(t)$, the corresponding Fourier transform $X_2(j\omega)$ is given by

$$\begin{aligned} X_2(j\omega) &= 2\pi a_0\delta(\omega) + 2\pi a_1\delta(\omega - \omega_0) + 2\pi a_{-1}\delta(\omega + \omega_0) \\ &= 2\pi\delta(\omega) + \pi e^{j\pi/8}\delta(\omega - 6\pi) + \pi e^{-j\pi/8}\delta(\omega + 6\pi) \end{aligned}$$

4.1.3 Problem (4.5)

From the given information,

$$\begin{aligned} x(t) &= (1/2\pi)\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \\ &= (1/2\pi)\int_{-\infty}^{\infty} |X(j\omega)|e^{j\sphericalangle\{X(j\omega)\}}e^{j\omega t}d\omega \\ &= (1/2\pi)\int_{-3}^{3} 2e^{-\frac{3}{2}\omega+\pi}e^{j\omega t}d\omega \\ &= \frac{-2}{\pi(t-3/2)}\sin[3(t-3/2)] \end{aligned}$$

The signal $x(t)$ is zero when $3(t-3/2)$ is a nonzero integer multiple of π. This gives

$$t = \frac{k\pi}{2} + \frac{3}{2}, \qquad \text{for } k \in \mathcal{I}, \text{ and } k \neq 0.$$

4.1.4 Problem (4.8)

(a) The signal $x(t)$ is as shown in the Figure S4.8.

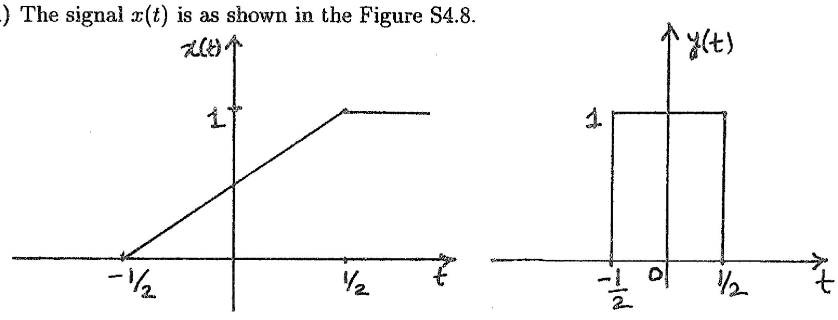


Figure S4.8

We may express this signal as

$$x(t) = \int_{-\infty}^{t} y(t)dt,$$

where $y(t)$ is the rectangular pulse shown in Figure S4.8. Using the integration property of the Fourier transform, we have

$$x(t) \overset{FT}{\longleftrightarrow} X(j\omega) = \frac{1}{j\omega}Y(j\omega) + \pi Y(j0)\delta(\omega)$$

We know from Table 4.2 that

$$Y(j\omega) = \frac{2\sin(\omega/2)}{\omega}$$

Therefore,

$$X(j\omega) = \frac{2\sin(\omega/2)}{j\omega^2} + \pi\delta(\omega)$$

(b) If $g(t) = x(t) - 1/2$, then the Fourier transform $G(j\omega)$ of $g(t)$ is given by

$$G(j\omega) = X(j\omega) - (1/2)2\pi\delta(\omega) = \frac{2\sin(\omega/2)}{j\omega^2}.$$

4.1.5 Problem (4.12)

(a) From Example 4.2 we know that

$$e^{-|t|} \stackrel{FT}{\longleftrightarrow} \frac{2}{1+\omega^2}.$$

Using the differentiation in frequency property, we have

$$te^{-|t|} \stackrel{FT}{\longleftrightarrow} j\frac{d}{d\omega}\left\{\frac{2}{1+\omega^2}\right\} = -\frac{4j\omega}{(1+\omega^2)^2}.$$

(b) The duality property states that if

$$g(t) \stackrel{FT}{\longleftrightarrow} G(j\omega)$$

then

$$G(t) \stackrel{FT}{\longleftrightarrow} 2\pi g(j\omega).$$

Now, since

$$te^{-|t|} \stackrel{FT}{\longleftrightarrow} -\frac{4j\omega}{(1+\omega^2)^2}$$

we may use duality to write

$$-\frac{4jt}{(1+t^2)^2} \stackrel{FT}{\longleftrightarrow} 2\pi\omega e^{-|\omega|}$$

Multiplying both sides by j, we obtain

$$\frac{4t}{(1+t^2)^2} \stackrel{FT}{\longleftrightarrow} j2\pi\omega e^{-|\omega|}.$$

4.1.6 Problem (4.13)

(a) Taking the inverse Fourier transform of $X(j\omega)$, we obtain

$$x(t) = \frac{1}{2\pi} + \frac{1}{2\pi}e^{j\pi t} + \frac{1}{2\pi}e^{j5t}$$

The signal $x(t)$ is therefore a constant summed with two complex exponentials whose fundamental frequencies are $2\pi/5$ rad/sec and 2 rad/sec. These two complex exponentials are not harmonically related. That is, the fundamental frequencies of these complex complex exponentials can never be integral multiples of a common fundamental frequency. Therefore, the signal is not periodic.

(b) Consider the signal $y(t) = x(t) * h(t)$. From the convolution property, we know that $Y(j\omega) = X(j\omega)H(j\omega)$. Also, from $h(t)$, we know that

$$H(j\omega) = e^{-j\omega}\frac{2\sin\omega}{\omega}.$$

The function $H(j\omega)$ is zero when $\omega = k\pi$, where k is a nonzero integer. Therefore,

$$Y(j\omega) = X(j\omega)H(j\omega) = \delta(\omega) + \delta(\omega - 5)$$

This gives

$$y(t) = \frac{1}{2\pi} + \frac{1}{2\pi}e^{j5t}$$

Therefore, $y(t)$ is a complex exponential summed with a constant. We know that a complex exponential is periodic. Adding a constant to a complex exponential does not affect its periodicity. Therefore, $y(t)$ will be a signal with a fundamental frequency of $2\pi/5$.

(c) From the results of parts (a) and (b), we see that the answer is yes.

4.1.7 Problem (4.21)

(a) The given signal is

$$e^{-\alpha t}\cos(\omega_0 t)u(t) = \frac{1}{2}e^{-\alpha t}e^{j\omega_0 t}u(t) + \frac{1}{2}e^{-\alpha t}e^{-j\omega_0 t}u(t).$$

Therefore,

$$X(j\omega) = \frac{1}{2(\alpha - j\omega_0 + j\omega)} - \frac{1}{2(\alpha - j\omega_0 + j\omega)}.$$

(b) The given signal is

$$x(t) = e^{-3t}\sin(2t)u(t) + e^{3t}\sin(2t)u(-t).$$

We have

$$x_1(t) = e^{-3t}\sin(2t)u(t) \overset{FT}{\longleftrightarrow} X_1(j\omega) = \frac{1/2j}{3 - j2 + j\omega} - \frac{1/2j}{3 + j2 + j\omega}.$$

Also,

$$x_2(t) = e^{3t}\sin(2t)u(-t) = -x_1(-t) \overset{FT}{\longleftrightarrow} X_2(j\omega) = -X_1(-j\omega) = \frac{1/2j}{3 - j2 - j\omega} - \frac{1/2j}{3 + j2 - j\omega}.$$

Therefore,

$$X(j\omega) = X_1(j\omega + X_2(j\omega) = \frac{3j}{9 + (\omega + 2)^2} - \frac{3j}{9 + (\omega - 2)^2}.$$

(c) Using the Fourier transform analysis equation (4.9) we have

$$X(j\omega) = \frac{2\sin\omega}{\omega} + \frac{\sin\omega}{\pi - \omega} - \frac{\sin\omega}{\pi + \omega}.$$

(d) Using the Fourier transform analysis equation (4.9) we have

$$X(j\omega) = \frac{1}{1 - \alpha e^{-j\omega T}}.$$

(e) We have

$$x(t) = (1/2j)te^{-2t}e^{j4t}u(t) - (1/2j)te^{-2t}e^{-j4t}u(t).$$

Therefore,

$$X(j\omega) = \frac{1/2j}{(2 - j4 + j\omega)^2} - \frac{1/2j}{(2 + j4 - j\omega)^2}.$$

(f) We have

$$x_1(t) = \frac{\sin\pi t}{\pi t} \stackrel{FT}{\longleftrightarrow} X_1(j\omega) = \begin{cases} 1, & |\omega| < \pi \\ 0, & \text{otherwise} \end{cases}.$$

Also

$$x_2(t) = \frac{\sin 2\pi(t-1)}{\pi(t-1)} \stackrel{FT}{\longleftrightarrow} X_2(j\omega) = \begin{cases} e^{-2\omega}, & |\omega| < 2\pi \\ 0, & \text{otherwise} \end{cases}.$$

$$x(t) = x_1(t)x_2(t) \stackrel{FT}{\longleftrightarrow} X(j\omega) = \frac{1}{2\pi}\{X_1(j\omega) * X_2(j\omega)\}.$$

Therefore,

$$X(j\omega) = \begin{cases} e^{-j\omega}, & |\omega| < \pi \\ (1/2\pi)(3\pi + \omega)e^{-j\omega}, & -3\pi < \omega < -\pi \\ (1/2\pi)(3\pi - \omega)e^{-j\omega}, & \pi < \omega < 3\pi \\ 0, & \text{otherwise} \end{cases}.$$

(g) Using the Fourier transform analysis eq. (4.9) we obtain

$$X(j\omega) = \frac{2j}{\omega}\left[\cos 2\omega - \frac{\sin\omega}{\omega}\right].$$

(h) If

$$x_1(t) = \sum_{k=-\infty}^{\infty} \delta(t - 2k),$$

then

$$x(t) = 2x_1(t) + x_1(t-1).$$

Therefore,

$$X(j\omega) = X_1(j\omega)[2 + e^{-\omega}] = \pi \sum_{k=-\infty}^{\infty} \delta(\omega - k\pi)[2 + (-1)^k].$$

(i) Using the Fourier transform analysis eq. (4.9) we obtain

$$X(j\omega) = \frac{1}{j\omega} + \frac{2e^{-j\omega}}{-\omega^2} - \frac{2e^{-j\omega} - 2}{j\omega^2}.$$

(j) $x(t)$ is periodic with period 2. Therefore,

$$X(j\omega) = \pi \sum_{k=-\infty}^{\infty} \tilde{X}(jk\pi)\delta(\omega - k\pi),$$

where $\tilde{X}(j\omega)$ is the Fourier transform of one period of $x(t)$. That is,

$$\tilde{X}(j\omega) = \frac{1}{1 - e^{-2}} \left[\frac{1 - e^{-2(1+j\omega)}}{1 + j\omega} - \frac{e^{-2}[1 - e^{-2(1+j\omega)}]}{1 - j\omega} \right].$$

4.1.8 Problem (4.24)

(a) (i) For $\mathcal{R}e\{X(j\omega)\}$ to be 0, the signal $x(t)$ must be real and odd. Therefore, signals in figures (a) and (c) have this property.

(ii) For $\mathcal{I}m\{X(j\omega)\}$ to be 0, the signal $x(t)$ must be real and even. Therefore, signals in figures (e) and (f) have this property.

(iii) For there to exist a real α such that $e^{j\alpha\omega}X(j\omega)$ is real, we require that $x(t + \alpha)$ be a real and even signal. Therefore, signals in figures (a), (b), (e), and (f) have this property.

(iv) For this condition to be true, $x(0) = 0$. Therefore, signals in figures (a), (b), (c), (d), and (f) have this property.

(v) For this condition to be true the derivative of $x(t)$ has to be zero at $t = 0$. Therefore, signals in figures (b), (c), (e), and (f) have this property.

(vi) For this to be true, the signal $x(t)$ has to be periodic. Only the signal in figure (a) has this property.

(b) For a signal to satisfy only properties (i), (iv), and (v), it must be real and odd, and

$$x(t) = 0, \qquad x'(0) = 0.$$

The signal shown below is an example of that.

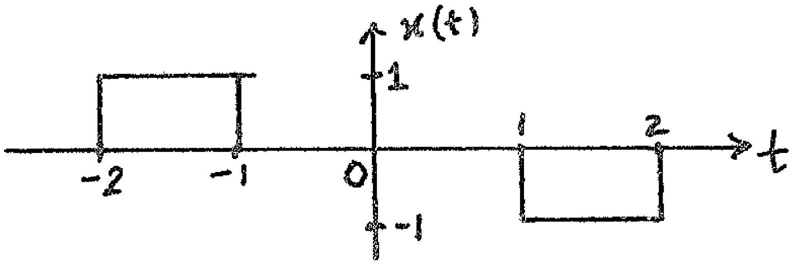


Figure S4.24

4.1.9 Problem (4.26)

(a) (i) We have

$$\begin{aligned} Y(j\omega) &= X(j\omega)H(j\omega) = \left[\frac{1}{(2+j\omega)^2}\right]\left[\frac{1}{4+j\omega}\right] \\ &= \frac{(1/4)}{4+j\omega} - \frac{(1/4)}{2+j\omega} + \frac{(1/2)}{(2+j\omega)^2} \end{aligned}$$

Taking the inverse Fourier transform we obtain

$$y(t) = \frac{1}{4}e^{-4t}u(t) - \frac{1}{4}e^{-2t}u(t) + \frac{1}{2}te^{-2t}u(t).$$

(ii) We have

$$\begin{aligned} Y(j\omega) &= X(j\omega)H(j\omega) = \left[\frac{1}{(2+j\omega)^2}\right]\left[\frac{1}{(4+j\omega)^2}\right] \\ &= \frac{(1/4)}{2+j\omega} + \frac{(1/4)}{(2+j\omega)^2} - \frac{(1/4)}{4+j\omega} + \frac{(1/4)}{(4+j\omega)^2} \end{aligned}$$

Taking the inverse Fourier transform we obtain

$$y(t) = \frac{1}{4}e^{-2t}u(t) + \frac{1}{4}te^{-2t}u(t) - \frac{1}{4}e^{-4t}u(t) + \frac{1}{4}te^{-4t}u(t).$$

(iii) We have

$$\begin{aligned} Y(j\omega) &= X(j\omega)H(j\omega) \\ &= \left[\frac{1}{1+j\omega}\right]\left[\frac{1}{1-j\omega}\right] \\ &= \frac{1/2}{1+j\omega} + \frac{1/2}{1-j\omega} \end{aligned}$$

Taking the inverse Fourier transform, we obtain

$$y(t) = \frac{1}{2}e^{-|t|}.$$

(b) By direct convolution of $x(t)$ with $h(t)$ we obtain

$$y(t) = \begin{cases} 0, & t < 1 \\ 1 - e^{-(t-1)}, & 1 < t \leq 5 \\ e^{-(t-5)} - e^{-(t-1)}, & t > 5 \end{cases}.$$

Taking the Fourier transform of $y(t)$,

$$\begin{aligned} Y(j\omega) &= \frac{2e^{-j3\omega}\sin(2\omega)}{\omega(1+j\omega)} \\ &= \left[\frac{e^{-j2\omega}}{1+j\omega}\right]\frac{e^{-j\omega}2\sin(2\omega)}{\omega} \\ &= X(j\omega)H(j\omega) \end{aligned}$$

4.1.10 Problem (4.34)

(a) We have
$$\frac{Y(j\omega}{X(j\omega)} = \frac{j\omega + 4}{6 - \omega^2 + 5j\omega}.$$
Cross-multiplying and taking the inverse Fourier transform, we obtain
$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = \frac{dx(t)}{dt} + 4x(t).$$

(b) We have
$$H(j\omega) = \frac{2}{2 + j\omega} - \frac{1}{3 + j\omega}.$$
Taking the inverse Fourier transform we obtain,
$$h(t) = 2e^{-2t}u(t) - e^{-3t}u(t).$$

(c) We have
$$X(j\omega) = \frac{1}{4 + j\omega} - \frac{1}{(4 + j\omega)^2}.$$
Therefore,
$$Y(j\omega) = X(j\omega)H(j\omega) = \frac{1}{(4 + j\omega)(2 + j\omega)}.$$
Finding the partial fraction expansion of $Y(j\omega)$ and taking the inverse Fourier transform,
$$y(t) = \frac{1}{2}e^{-2t}u(t) - \frac{1}{2}e^{-4t}u(t).$$

4.2 Advanced Problems

4.2.1 Problem (4.38)

(a) Applying a frequency shift to the analysis equation, we have

$$X(j(\omega-\omega_0)) = \int_{-\infty}^{\infty} x(t)e^{-j(\omega-\omega_0)t}dt = \int_{-\infty}^{\infty} x(t)e^{j\omega_0 t}e^{-j\omega t}dt = \mathcal{FT}\{x(t)e^{j\omega_0 t}\}.$$

(b) We have

$$w(t) = e^{j\omega_0 t} \overset{FT}{\longleftrightarrow} W(j\omega) = 2\pi\delta(\omega-\omega_0).$$

Also,

$$\begin{aligned} x(t)w(t) \overset{FT}{\longleftrightarrow} & \quad \frac{1}{2\pi}\left[X(j\omega) * W(j\omega)\right] \\ = & \quad X(j\omega) * \delta(\omega-\omega_0) \\ = & \quad X(j(\omega-\omega_0)) \end{aligned}$$

4.2.2 Problem (4.44)

(a) Taking the Fourier transform of both sides of the given differential equation, we have

$$Y(j\omega)[10+j\omega] = X(j\omega)[Z(j\omega)-1].$$

Since, $Z(j\omega) = \frac{1}{1+j\omega} + 3$, we obtain from the above equation

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{3+2j\omega}{(1+j\omega)(10+j\omega)}.$$

(b) Finding the partial fraction expansion of $H(j\omega)$ and then taking its inverse Fourier transform we obtain

$$h(t) = \frac{1}{9}e^{-t}u(t) + \frac{17}{9}e^{-10t}u(t).$$