Slide 1

- Mathematical Induction
- Weak Induction
- Strong Induction
- Inductive Definition

Slide 2

Mathematical Induction

- Common method of proof based on a "strong" property of the integers.
- Formal approach uses a predicate whose variable will take on integral values.
- There are two general versions:
  - weak induction
  - strong induction

Slide 3

Example:

$$P(n) \quad : \quad 1 + 2 + \cdots + n = \frac{n \cdot (n+1)}{2}$$

Mathematical Induction can be used to establish that

$P(1), P(2), P(3), \ldots$ (an infinite sequence of statements)

are all true.

Slide 4

Weak Induction Proof Format

1. Establish the predicate $P(n)$.
2. Show that a "base case", $P(b)$, is true – typically $P(1)$.
3. Make the **inductive assumption**: for some fixed arbitrary integer $k \geq b$, $P(k)$ is true.
4. Using the inductive assumption, show it then follows that $P(k+1)$ is true.

Slide 5

Why are they all true?

You showed:

- $P(b)$ is true.
- When $k \geq b$, $P(k) \vdash P(k+1)$

Slide 6

Truth Has Been Established Sequentially

(When $k \geq b$, $P(k) \vdash P(k+1)$)

$P(b)$ is true.

Since $b \geq b$ and $P(b)$ is true , $P(b+1)$ is true.

Since $b+1 \geq b$ and $P(b+1)$ is true , $P(b+2)$ is true.

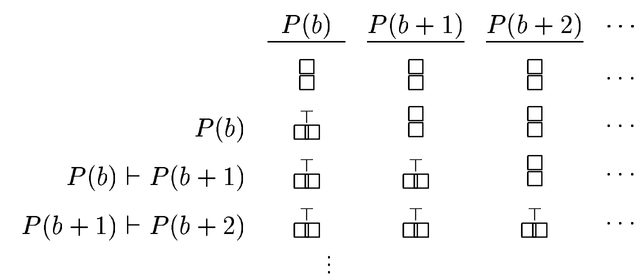
Since $b+2 \geq b$ and $P(b+2)$ is true , $P(b+3)$ is true.

$\vdots$

Slide 7

It's Like Dominoes

(When $k \geq b$, $P(k) \vdash P(k+1)$)



Slide 8

Induction Example:

Show for all n, $1 + 2 + \cdots + n = \frac{n \cdot (n+1)}{2}$

Proof (by induction)

Define

$$P(n) \ : \ 1 + 2 + \cdots + n = \frac{n \cdot (n+1)}{2}$$

Consider

$$P(1) \ : \ 1 = \frac{1 \cdot (1+1)}{2}.$$

Since $\frac{1 \cdot (1+1)}{2} = \frac{1 \cdot 2}{2} = 1$, $P(1)$ is true.

Slide 9

Let k be an arbitrary fixed integer such that $k \geq 1$.

Assume $P(k)$ is true. Thus, for this k,

$$1+2+\cdots+k = \frac{k \cdot (k+1)}{2}.$$

Consider

$$P(k+1) \ : \ 1+2+\cdots+(k+1) = \frac{(k+1)\cdot(k+2)}{2}.$$

$$1+2+\cdots+(k+1) = (\ 1+2+\cdots+k\ ) + (k+1).$$

And, by the inductive assumption,

$$(\ 1+2+\cdots+k\ ) + (k+1) = \frac{k \cdot (k+1)}{2} + (k+1).$$

Slide 10

So,

$$1+2+\cdots+(k+1) = \frac{k \cdot (k+1)}{2} + (k+1) =$$

$$\frac{k \cdot (k+1)}{2} + \frac{2 \cdot (k+1)}{2} = \frac{(k+1)\cdot(k+2)}{2}.$$

Thus, $P(k)$ being true implies $P(k+1)$ is true.

Hence by induction, $P(n)$ is true for all $n \geq 1$.

∎

Slide 11

Strong Mathematical Induction

Sometimes it is extremely difficult (perhaps impossible) to simply establish that

$$P(k) \vdash P(k+1)$$

It may be that the (assumed) truth of all of the statements

$$P(b), P(b+1), \ldots, P(k)\ ,$$

not just the (assumed) truth of $P(k)$, is necessary to show that $P(k+1)$ is true.

Slide 12

This expanded need is reflected in a change in the inductive assumption for **strong** mathematical induction.

Slide 13

Strong Induction Proof Format

1. Establish the predicate $P(n)$.
2. Show that a "base case", $P(b)$, is true – typically $P(1)$.
3. Make the **inductive assumption**: for some fixed arbitrary integer $k \geq b$, $P(b), P(b+1), \ldots, P(k)$ are all true.
4. Using the inductive assumption, show it then follows that $P(k+1)$ is true.

Slide 14

Truth Has Been Established Sequentially

(When $k \geq b$, $P(b) \wedge \ldots \wedge P(k) \vdash P(k+1)$)

$P(b)$ is true.

Since $b \geq b$ and $P(b)$ is true, $P(b+1)$ is true.

Since $b+1 \geq b$ and $P(b)$ and $P(b+1)$ are true, $P(b+2)$ is true.

Since $b+2 \geq b$ and $P(b)$, $P(b+1)$, and $P(b+2)$ are true, $P(b+3)$ is true.

$\vdots$

Slide 15

Inductive Definition

Sequences can be defined by first specifying values for "base cases" or starting values and giving an **inductive formula** by means of which the values can be extended further in the sequence.

Example - Fibonacci numbers

$$f_1 = 1$$
$$f_2 = 1$$
$$\text{For } n \geq 3, \; f_n = f_{n-1} + f_{n-2}$$

Slide 16

Fibonacci numbers

$$f_1 = f_2 = 1 \, , \text{ for } n \geq 3, \; f_n = f_{n-1} + f_{n-2}$$

f_1	f_2	f_3	f_4	f_5	f_6	$\ldots$
1	1					
1	1	1+1=2				
1	1	2	1+2=3			
1	1	2	3	2+3=5		
1	1	2	3	5	3+5=8	
						$\ddots$

Slide 17

Example - $2^n \cdot 10 + 1$

$$A_1 = 6$$
$$A_2 = 11$$
$$\text{For } n \geq 3, \; A_n = 3A_{n-1} - 2A_{n-2}$$

Slide 18

Example - $2^n \cdot 10 + 1$

$A_1 = 6 \;\; A_2 = 11$, for $n \geq 3$, $A_n = 3A_{n-1} - 2A_{n-2}$

A_1	A_2	A_3	A_4	A_5	$\ldots$
6	11				
6	11	33-12=21			
6	11	21	63-22=41		
6	11	21	41	123-42=81	
					$\ddots$

Slide 19

Example

Prove that for any n greater than or equal to 2, n is a prime number or a product of prime numbers.

Proof by strong induction is appropriate together with proof by cases.