Convex combinations of two points

What are these, anyway?

Start with the equation of a line in two dimensions

$$y = mx + b$$

Consider this as a parametric equation:

$$y(x) = mx + b$$

Now consider how values of x map into the 1-dimensional space where the values of $y(x)$ reside.

The quantity m acts as a displacement away from b. Let us rename it as d (for displacement) and rearrange the parametric equation.

$$y(x) = xd + b$$

$$y(0) = ?$$

$$y(1) = ?$$

Let us do a little algebra.

$$\begin{aligned} y(x) &= xd + b \\ &= x\left[(d+b) - b\right] + b \\ &= x(d+b) + (1-x)b \\ &= (1-x)b + x(d+b) \end{aligned}$$

Let $c = d + b$.

Then $\quad y(x) = (1-x)b + xc$

and $\quad d = c - b$

Consider once again how values of x map into the one dimensional space where the values of $y(x)$ reside.

$$y(x) = (1 - x)b + xc$$

$$y(0) = ?$$

$$y(1) = ?$$

$$y(1/2) = ?$$

$$y(1/4) = ?$$

What happens when $0 \leq x \leq 1$?

What happens when $x < 0$?

What happens when $1 < x$?

With a little experimentation it becomes apparent that in one or two dimensions the equation

$$y(x) = (1 - x)b + xc$$

is the equation of the line that passes through points b and c.

When x moves between $x = 0$ and $x = 1$, $y(x)$ traces out the part of the line between b and c.

When x moves away from $x = 0$ using negative values $y(x)$ traces out the part of the line on the other side of b from c.

When x moves away from $x = 1$ using larger values it traces out the part of the line on the other side of c from b.

Are the previous statements true for R^n?

<u>Yes</u>

For $b, c \in \mathsf{R}^n$

$$y(x) = (1 - x)b + xc$$

is the equation of the line that passes through points b and c. Of special interest, when x moves between $x = 0$ and $x = 1$, $y(x)$ traces out the part of the line between b and c.

When $0 \leq x \leq 1$, $y(x) = (1-x)b + xc$

is a **convex combination** of points b and c.

When $0 < x < 1$, $y(x)$ is a **strict** convex combination of points b and c.

The set of all convex combinations of two distinct points $b, c \in \mathsf{R}^n$ is the line segment connecting those two points. We can formally define the line segment between $b, c \in \mathsf{R}^n$ as: $\{ \ (1-x)b + xc \ : \ 0 \leq x \leq 1 \ \}$

Do the distance math

$$
\begin{aligned}
\mathrm{d}\,(\mathrm{b},\mathrm{y}(\mathrm{x})\,)^2 &= \mid b - [(1-x)b + xc] \mid^2 \\
&= \mid [1-(1-x)]\, b - xc \mid^2 \\
&= \mid xb - xc \mid^2 \\
&= x^2 \;\; \mid b - c \mid^2 \\
&= x^2 \; \mathrm{d}\,(\mathrm{b},\mathrm{c}\,)^2
\end{aligned}
$$

Thus $\mathrm{d}\,(\mathrm{b},\mathrm{y}(\mathrm{x})\,) \;=\; \mathrm{x}\;\mathrm{d}\,(\mathrm{b},\mathrm{c}\,)$

$$
\begin{aligned}
\mathrm{d}\,(\mathrm{y(x),c}\,)^2 &= |\, [(1-x)b + xc] - c\, |^2 \\
&= |\, (1-x)b + (x-1)c\, |^2 \\
&= |\, (1-x)b - (1-x)c\, |^2 \\
&= |\, (1-x)(b-c)\, |^2 \\
&= (1-x)^2 \;\; |\, b - c\, |^2 \\
&= (1-x)^2 \; \mathrm{d}\,(\mathrm{b,c}\,)^2
\end{aligned}
$$

Thus $\mathrm{d}\,(\mathrm{y(x),c}\,) \;=\; (1-\mathrm{x})\;\mathrm{d}\,(\mathrm{b,c}\,)$

It follows that

$$\mathrm{d}\,(\mathrm{b},\mathrm{y}(\mathrm{x})\,) \quad = \quad \mathrm{x}\ \ \mathrm{d}\,(\mathrm{b},\mathrm{c}\,)$$

and

$$\mathrm{d}\,(\mathrm{y}(\mathrm{x}),\mathrm{c}\,) \quad = \quad (1-\mathrm{x})\ \ \mathrm{d}\,(\mathrm{b},\mathrm{c}\,)$$

so that when $0 \leq x \leq 1$,

$$y(x) = (1-x)b + xc$$

must be on the line between b and c.

Why?

One more time

We now define $w = 1 - x$, so that

$$y(x) = (1 - x)b + xc$$

becomes

$$y(w, x) = wb + xc \ , \ \text{ with } \ w + x = 1 \ \text{ and } \ w, x \geq 0$$

This is a parameterized version of the usual definition of a convex combination of two points.