Convex combinations of two points

What are these, anyway?

Start with the equation of a line in two dimensions

$$y = mx + b$$

Consider this as a parametric equation:

$$y(x) = mx + b$$

Now consider how values of x map into the 1-dimensional space where the values of y(x) reside. The quantity m acts as a displacement away from b. Let us rename it as d (for displacement) and rearrange the parametric equation.

$$y(x) = xd + b$$
$$y(0) =?$$
$$y(1) =?$$

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Let us do a little algebra.

$$y(x) = xd + b$$

$$= x [(d+b) - b] + b$$

$$= x(d+b) + (1-x)b$$

$$= (1-x)b + x(d+b)$$

Let $c = d + b$.
Then $y(x) = (1-x)b + xc$
and $d = c - b$

Consider once again how values of x map into the one dimensional space where the values of y(x) reside.

> y(x) = (1 - x)b + xc y(0) = ? y(1) = ? y(1/2) = ?y(1/4) = ?

What happens when $0 \le x \le 1$? What happens when x < 0? What happens when 1 < x? With a little experimentation it becomes apparent that in one or two dimensions the equation

$$y(x) = (1-x)b + xc$$

is the equation of the line that passes through points b and c.

When x moves between x = 0 and x = 1, y(x) traces out the part of the line between b and c.

When x moves away from x = 0 using negative values y(x) traces out the part of the line on the other side of b from c.

When x moves away from x = 1 using larger values it traces out the part of the line on the other side of c from b. Are the previous statements true for \mathbb{R}^n ?

Yes

For $b, c \in \mathsf{R}^n$

$$y(x) = (1-x)b + xc$$

is the equation of the line that passes through points b and c. Of special interest, when x moves between x = 0 and x = 1, y(x) traces out the part of the line between b and c.

When
$$0 \le x \le 1$$
, $y(x) = (1 - x)b + xc$

is a **convex combination** of points b and c.

When 0 < x < 1, y(x) is a **strict** convex combination of points b and c.

The set of all convex combinations of two distinct points $b, c \in \mathbb{R}^n$ is the line segment connecting those two points. We can formally define the line segment between $b, c \in \mathbb{R}^n$ as: $\{(1-x)b + xc : 0 \le x \le 1\}$

Do the distance math

$$d (b,y(x))^{2} = |b - [(1 - x)b + xc]|^{2}$$

= |[1 - (1 - x)]b - xc|^{2}
= |xb - xc|^{2}
= x^{2} |b - c|^{2}
= x^{2} d (b,c)^{2}

Thus
$$d(b,y(x)) = x d(b,c)$$

$$d (y(x),c)^{2} = |[(1-x)b + xc] - c|^{2}$$

$$= |(1-x)b + (x-1)c|^{2}$$

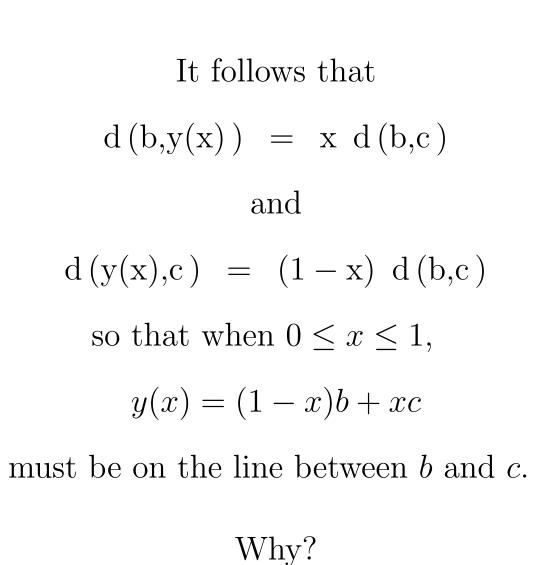
$$= |(1-x)b - (1-x)c|^{2}$$

$$= |(1-x)(b-c)|^{2}$$

$$= (1-x)^{2} |b-c|^{2}$$

$$= (1-x)^{2} d (b,c)^{2}$$

Thus
$$d(y(x),c) = (1-x) d(b,c)$$



<u>One more time</u>

We now define w = 1 - x, so that

$$y(x) = (1-x)b + xc$$

becomes

y(w,x) = wb + xc, with w + x = 1 and $w, x \ge 0$

This is a parameterized version of the usual definition of a convex combination of two points.