Convex combinations of more than two points

We first consider the extension to three points in R^n.

Given $a, b, c \in \mathsf{R}^n$ and $v, w, x \in \mathsf{R}$,

$$y(v, w, x) = va + wb + xc \ ,$$

$$\text{with } \ v + w + x = 1 \ \text{ and } \ v, w, x \geq 0$$

is a (parameterized) **convex combination** of the given points.

Let us do a little algebra, assuming that $v + w \neq 0$.

$$\begin{aligned} y(v,w,x) &= va + wb + xc \\ &= (v+w)\left[\left(\frac{v}{v+w}\right)a + \left(\frac{w}{v+w}\right)b\right] + xc \\ &= (1-x)\left[\left(\frac{v}{v+w}\right)a + \left(\frac{w}{v+w}\right)b\right] + xc \end{aligned}$$

Define $b' = \left(\frac{v}{v+w}\right)a + \left(\frac{w}{v+w}\right)b$.

Then b' is a convex combination of a and b (Why?) and $y(v,w,x)$ is a convex combination of b' and c.

So, assuming that the points are distinct, what is

$$\{\ va+wb+xc\ :\ v+w+x=1 \text{ and } 0 \le v,w,x \le 1\ \}\ ,$$

the set of all convex combinations of $a, b, c \in \mathsf{R}^n$?

We have that

$$b' = \left(\frac{v}{v+w}\right) a + \left(\frac{w}{v+w}\right) b \ \ (\text{with } \ v+w \neq 0)$$

and

$$va + wb + xc = (1-x)b' + xc \ \ (\text{with } \ x = 1 - v - w)$$

$$\{ \ va + wb + xc \ \ : \ \ v + w + x = 1 \ \text{ and } \ 0 \leq v, w, x \leq 1 \ \}$$

consists of all the points on the edges and inside the triangle with corner points $a, b,$ and c.

We now consider the extension to many points in R^n.

Let $\mathcal{A} = \{a^1, \ldots, a^m\}$ be a given finite set of m points in R^n. The set $\mathcal{A}$ can be used to generate different *polyhedral* objects in R^n by combining its elements using various linear operations. The elements of $\mathcal{A}$ are the **generators** of the objects they define.

One of these fundamental objects is the **convex hull** of the points in $\mathcal{A}$ (a *polytope*) defined by

$$H(\mathcal{A}) = \left\{ \sum_{i=1}^{m} \lambda_i a^i \;:\; \sum_{i=1}^{m} \lambda_i = 1 \text{ and } \lambda_1, \ldots, \lambda_m \geq 0 \right\}$$

The convex hull consists of all convex combinations of the generators.

In R^3 one can visualize the convex hull of many points as a multi-faceted diamond.

Any set is said to be **convex** if it contains all convex combinations of any finite set of points from that set.

$H(\mathcal{A})$ is convex since it can be shown that *a convex combination of convex combinations of given points is a convex combination of those points.*

Example:

$$\frac{1}{2}\left[\frac{1}{3}a^1 + \frac{2}{3}a^2\right] + \frac{1}{2}\left[\frac{2}{3}a^2 + \frac{1}{3}a^3\right] \;=\; \frac{1}{6}a^1 + \frac{2}{3}a^2 + \frac{1}{6}a^3$$

The minimum cardinality subset $\mathcal{F} \subset \mathcal{A}$ which generates $H(\mathcal{A})$ is called the **frame** of $H(\mathcal{A})$.

A frame is to a convex hull what a *basis* is to a linear combination.

In R^3 when visualizing the convex hull of many points as a multi-faceted diamond, the corner points are the generators.

It can be shown that *a point is a member of $\mathcal{F}$ if and only if it cannot be written as a strict convex combination of two distinct points of $H(\mathcal{A})$.*

The fundamental tool for determining the frame $\mathcal{F}$ from $\mathcal{A}$ is the generic linear program:

$$(LP)$$

$$z = \min \sum_{j \in J} \lambda_j$$

s.t.

$$\sum_{j \in J} \lambda_j a^j = a^k$$

$$\lambda_j \geq 0 \quad , \quad j \in J$$

where $\mathcal{A}' \subset \mathcal{A}$ and $J = \{ \ j \ : \ a^j \in \mathcal{A}' \backslash a^k \ \}$.

Fundamental Results:

If $\mathcal{F} \subset \mathcal{A}'$, $a^k \neq 0$, and the linear program (LP) is feasible, then $a^k \in \mathcal{F}$ if and only if at optimality $z > 1$.

If $\mathcal{F} \subset \mathcal{A}'$, $a^k \neq 0$, and the linear program (LP) is infeasible, then $a^k \in \mathcal{F}$.

Naive approaches to finding $\mathcal{F}$ based on iterative solution of problems of type (LP) have long been used. Computationally they suffer by starting with large size sets $\mathcal{A}'$ (and J) and only slowly decreasing their sizes.

Dula, Helgason, and Hickman have shown how to more efficiently compute $\mathcal{F}$, in part making use of iterative solution of problems of type (LP) starting with small size sets $\mathcal{A}'$ (and J) and slowly increasing their sizes.

References:

J.H. Dulá and R.V. Helgason (1996), A new procedure for identifying the frame of the convex hull of a finite collection of points in multidimensional space, *European Journal of Operational Research 92*, 352-367.

J.H. Dulá, R.V. Helgason, and B.L. Hickman (1992), Preprocessing schemes and a solution method for the convex hull problem in multidimensional space, *Computer Science and Operations Research: New Developments in Their Interfaces*, O. Balci (ed.), 59-70, Pergamon Press, U.K.