The principle of inclusion-exclusion can be generalized to a finite number of sets as described in the next theorem. This theorem can be proved by using induction, so we leave the proof as an exercise.

**Theorem 7.1.16:** Let $A_1, A_2, \ldots, A_n$ be finite sets. Let

$$n_0 = |A_1| + |A_2| + \ldots + |A_n| = \sum_{i=1}^{n} |A_i|,$$

$$n_1 = |A_1 \cap A_2| + |A_1 \cap A_3| + \ldots + |A_1 \cap A_n| + |A_2 \cap A_3| + \ldots + |A_2 \cap A_n| + \ldots + |A_{n-1} \cap A_n|,$$

$$n_2 = \sum_{1 \leq i < j \leq n} |A_i \cap A_j|,$$

$$n_3 = \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| + \ldots + |A_k \cap A_l|,$$

$$n_m = |A_1 \cap A_2 \cap \ldots \cap A_m|.$$

Then

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = n_0 - n_1 + n_2 - n_3 + \ldots + (-1)^{n-1} n_n.$$

### Worked-Out Exercises

**Exercise 1:** Find the number of integers between 4 and 100 that end with 3 or 5 or 7.

**Solution:** We divide the task into the following tasks.

- $T_1$: Find all integers between 4 and 100 that end with 3.
- $T_2$: Find all integers between 4 and 100 that end with 5.
- $T_3$: Find all integers between 4 and 100 that end with 7.

Now 13, 23, 33, 43, 53, 63, 73, 83, 93, and 98 are the 9 integers between 4 and 100 that end with 3. There are 10 integers, 5, 15, 25, 35, 45, 55, 65, 75, 85, and 95, between 4 and 100 that end with 5. Also, there are 10 integers, 7, 17, 27, 37, 47, 57, 67, 77, 87, and 97, between 4 and 100 that end with 7.

Hence, the tasks $T_1$, $T_2$, and $T_3$ can be completed in 9, 10, and 10 ways, respectively. The tasks are all independent of each other. Therefore, the number of ways to do one of these tasks is $9 + 10 + 10 = 29$. Hence, the number of integers between 4 and 100 that end with 3 or 5 or 7 is 29.

**Exercise 2:** Find the number of words of length 3 using the letters $A$, $B$, $C$, $D$, and $E$ that start with the letter $C$ such that no word contains a repetition of letters.

**Solution:** A word of length 3 can be constructed in three successive steps. Choose the first letter, then choose the second letter, and then choose the third letter. In the first step, we must choose the letter $C$. Therefore, the first letter can be chosen in 1 way. Because no word contains a repetition of letters, once the first letter is chosen, the number of remaining letters is 4. The second letter can be any one of the remaining 4 letters. Therefore, the second letter can be chosen in 4 ways. After choosing the first and the second letters, the number of remaining letters is 3. The third letter can be any one of these 3 letters. Therefore, the third letter can be chosen in 3 ways.

Consequently, by the multiplication principle, there are $1 \cdot 4 \cdot 3 = 12$ different words of length 3 that start with the letter $C$ and do not contain a repetition of letters.

**Exercise 3:** Find the number of bit strings of length 8 that begin with 1011.

**Solution:** A bit string of length 8 that begins with 1101 is of the form $1101a_5a_6a_7a_8$, where each $a_i$ is 0 or 1, $i = 5, 6, 7, 8$. These strings can be constructed in four successive steps. In
the first step, we choose \( a_0 \), which is either 0 or 1. Therefore, \( a_0 \) can be selected in 2 different ways. Next we choose \( a_0 \), which can be chosen in 2 different ways. Similarly, each of \( a_i \) and \( a_0 \) can be chosen in 2 different ways. Therefore, by the multiplication principle, the number of bit strings of length 8 that begin with 1 is 2 · 2 · 2 · 2 = 16.

**Exercise 4:** Let \( A \) be a set with 8 elements and \( B \) be a set with 6 elements. Find the number of functions from \( A \) into \( B \).

**Solution:** Let \( a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8 \) be the elements of \( A \). Let \( T \) be the task of constructing a function, say \( f : A \rightarrow B \). The function \( f \) is completely determined if we know the images \( f(a_i) \) for \( i = 1, 2, \ldots, 8 \). Hence, to find the number of ways \( T \) can be completed, i.e., the number of functions from \( A \) into \( B \), we divide \( T \) into the following eight successive steps \( T_1, T_2, \ldots, T_8 \).

\[ T_i : \text{Choose the image of } a_i \text{ for } i = 1, 2, \ldots, 8. \]

Because the image of \( a_1 \) can be any member of \( B \), step \( T_1 \) can be completed in 6 different ways. Similarly, each of steps \( T_i \) for \( i = 2, \ldots, 8 \) can be completed in 6 different ways. Hence, by the multiplication principle \( T \) can be completed in \( 6 \cdot 6 \cdot 6 \cdot 6 \cdot 6 \cdot 6 \cdot 6 = 6^8 \) ways. Hence, the number functions from \( A \) into set \( B \) is \( 6^8 \).

**Exercise 5:** Let \( A \) be a set with 5 elements and \( B \) be a set with 6 elements. Find the number of one-one functions from \( A \) into \( B \).

**Solution:** Let \( a_1, a_2, a_3, a_4, a_5 \) be the elements of \( A \). Let \( T \) be the task of constructing a one-one function, say \( f : A \rightarrow B \). The function \( f \) is completely determined if we know the images \( f(a_i) \) for \( i = 1, 2, \ldots, 5 \). Hence, to find the number of ways \( T \) can be completed, i.e., the number of functions from \( A \) into \( B \), we divide \( T \) into following five successive steps \( T_1, T_2, \ldots, T_5 \).

\[ T_i : \text{Choose the image of } a_i \text{ for } i = 1, 2, \ldots, 5. \]

Because the image of \( a_1 \) can be any member of \( B \), the step \( T_1 \) can be completed in 6 different ways. Now the function \( f \) must be one-one. Hence, \( f(a_2) \), the image of \( a_2 \), must be different from \( f(a_1) \), the image of \( a_1 \). Hence, after making a choice for \( f(a_1) \), only 5 elements are left in \( B \) to make a choice for \( f(a_2) \). It follows that there are 5 choices for \( f(a_2) \); that is, the image of \( a_2 \) has 5 choices, so \( T_2 \) can be completed in 5 different ways. Similarly, the image of \( a_3 \) can be chosen in 4 different ways, the image of \( a_4 \) can be chosen in 3 different ways, and the image of \( a_5 \) can be chosen in 2 different ways. Hence, by the multiplication principle task \( T \) can be completed in \( 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = 720 \) ways. Hence, the number of one-one functions from \( A \) into set \( B \) is \( 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = 720 \).

**Exercise 6:** How many license plates of 4 letters from the English alphabet, \( A \) to \( Z \), followed by 3 digits from 0 to 9 can be made, if repetition of letters is not allowed?

**Solution:** A license plate consisting of 4 letters followed by 3 digits is of the form

\[ a_1a_2a_3a_4d_1d_2d_3, \]

where \( a_i \in \{A, B, C, \ldots, Z\} \), \( i = 1, 2, 3, 4 \) and \( d_j \in \{0, 1, 2, \ldots, 9\} \), \( j = 1, 2, 3 \).

The license plates can be built in seven successive steps. In each of the first four steps, we choose a letter, and in each of the remaining three steps, we choose a digit.

Now \( a_1 \) can be chosen in 26 different ways. Because the repetition of letters is not allowed, after choosing \( a_1 \), only 25 letters are left. Therefore, in the second step, \( a_2 \) can be chosen in 25 different ways. Similarly, \( a_3 \) can be chosen in 24 ways and \( a_4 \) can be chosen in 23 ways.

Now consider the digits. Because the repetition of digits is allowed, in the remaining three steps, each of \( d_1, d_2, d_3 \) can be selected in 10 different ways.

Hence, by the multiplication principle, the number of license plates is

\[ 26 \cdot 25 \cdot 24 \cdot 23 \cdot 10 \cdot 10 \cdot 10 = 358,800,000. \]

**Exercise 7:** Suppose there are eight different books on algebra, four different books on discrete structures, five different books on computer science, and two different books on history. Determine the number of ways these different books can be selected from three different categories.

**Solution:** Let \( A \) denote an algebra book, \( D \) denote a discrete structures book, \( C \) denote a computer science book, and \( H \) denote a history book. Our task, \( T \), is to select three different books from different categories. We divide the task \( T \) into tasks \( T_1, T_2, T_3, \) and \( T_4 \).

\[ T_1 : \text{Choose an algebra book, a discrete structures book, and a computer science book.} \]
\[ T_2 : \text{Choose an algebra book, a discrete structures book, and a history book.} \]
\[ T_3 : \text{Choose an algebra book, a computer science book, and a history book.} \]
\[ T_4 : \text{Choose a discrete structures book, a computer science book, and a history book.} \]

All these tasks are independent of each other. Hence, if the tasks \( T_1, T_2, T_3, T_4 \) can be done in \( n_1, n_2, n_3, n_4 \) ways respectively, then by the addition principle task \( T \) can be done in \( n_1 + n_2 + n_3 + n_4 \) ways.

Now consider task \( T_1 \): to choose an algebra book, a discrete structures book, and a computer science book. Consider the triple \( (A, D, C) \), \( A \) for algebra book, \( D \) for discrete structures book, and \( C \) for computer science book. So task \( T_1 \) is to choose the triple \( (A, D, C) \).

There are eight ways an algebra book can be chosen, four ways a discrete structures book can be chosen, and five ways a computer science book can be chosen. Therefore, by the multiplication principle, task \( T_1 \) can be done in \( 8 \cdot 4 \cdot 5 = 160 \) different ways. Hence, \( n_1 = 160 \).

In a similar manner, we can show that \( n_2 = 8 \cdot 4 \cdot 2 = 64 \), \( n_3 = 8 \cdot 5 \cdot 2 = 80 \), and \( n_4 = 4 \cdot 5 \cdot 2 = 40 \). It now follows that

\[ n_1 + n_2 + n_3 + n_4 = 160 + 64 + 80 + 40 = 344. \]

Therefore, we can select three books from three different categories in 344 ways.
SECTION REVIEW

Key Terms

addition principle  inclusion-exclusion principle  multiplication principle

Some Key Results

1. Let $X_1, X_2, \ldots, X_k$ be sets such that the number of elements in $X_i$ is $n_i$, that is, $|X_i| = n_i$, $i = 1, 2, \ldots, k, k \geq 2$. Suppose that for any two sets $X_i$ and $X_j$, $X_i \cap X_j = \emptyset$, $i = 1, 2, \ldots, k, j = 1, 2, \ldots, k, i \neq j$. That is, the sets $X_1, X_2, \ldots, X_k$ are pairwise disjoint. Then $|X_1 \cup X_2 \cup \cdots \cup X_k| = n_1 + n_2 + \cdots + n_k$.

2. Suppose that tasks $T_1, T_2, \ldots, T_k$ can be done in $n_1, n_2, \ldots, n_k$ ways, respectively. If all these tasks are independent of each other, then the number of ways to do one of these tasks is $n_1 + n_2 + \cdots + n_k$.

3. Suppose that a task $T$ can be completed in $k$ successive steps. Suppose step 1 can be completed in $n_1$ different ways, step 2 can be completed in $n_2$ different ways, and in general, no matter how the preceding steps are completed, step $k$ can be completed in $n_k$ different ways. Then task $T$ can be completed in $n_1 n_2 \cdots n_k$ different ways.

4. Let $X_1$ and $X_2$ be finite sets. Then $|X_1 \cup X_2| = |X_1| + |X_2| - |X_1 \cap X_2|$.

EXERCISES

1. There are four routes from New York to Chicago, five routes from Chicago to Denver, and three routes from Denver to Los Angeles. Find the number of different routes from New York to Los Angeles via Chicago and Denver.

2. For a show a clown has four types of wigs, six types of dresses, and five types of shoes. How many ways can the clown dress up for the show?

3. For a birthday dinner, there are four types of soft drinks, three types of desserts, and five types of pizzas. A guest can choose one item from each group. How many ways can the dinner be served?

4. Two six-faced red and blue dice are thrown. What is the number of outcomes if the first die must show a 1?

5. Two six-faced distinct dice are thrown. What is the number of outcomes if the sum of the digits shown is 7?

6. Two six-faced distinct dice are thrown. What is the number of outcomes if the sum of the digits shown is 5?

7. Two six-faced distinct dice are thrown. What is the number of outcomes if the sum of the digits shown is 3 or 6?

8. Two six-faced distinct dice are thrown. What is the number of outcomes if the sum of the digits shown is odd?

9. How many license plates consisting of three letters followed by four digits can be prepared if repititions are allowed?

10. How many license plates consisting of three letters followed by four digits can be prepared if repititions are not allowed?

11. How many license plates consisting of three letters followed by four digits can be prepared if the licence plates contain the letter A and repetitions are allowed?

12. How many license plates consisting of three letters followed by four digits can be prepared if the license plates contain the letter A and repetitions are not allowed?

13. How many strings of 0's and 1's of length 10 are there?

14. How many strings of 0's and 1's of length 6 that begin with 1 are there?

15. How many strings of 0's and 1's of length 8 that end with 10 are there?

16. How many strings of 0's and 1's of length 12 that begin with 1 and end with 00 are there?

17. How many strings of 0's and 1's of length 6 in which the second bit is 1 and the fifth bit is 0 are there?

18. How many strings of 0's and 1's of length 7 and containing exactly one 1 are there?

19. How many strings of 0's and 1's of length 6 and containing at least one 1 are there?

20. How many strings of 0's and 1's of length less than or equal to 5 are there?

21. How many strings of 0's and 1's of length less than or equal to 7 and that start with 1 or end with 0 are there?

22. Suppose there is set of four distinct mystery novels, five distinct romance novels, and three distinct poetry books.

   a. In how many ways can these books be arranged on a shelf?
b. In how many ways can these books be arranged on a shelf if all mystery novels are arranged first?
c. In how many ways can these books be arranged on a shelf if all poetry books stay together?

23. Three coins are tossed and the outcomes are placed in a row.
   a. How many outcomes are there?
   b. How many outcomes contain at least two consecutive heads?
   c. How many outcomes do not contain at least two consecutive heads?
   d. How many outcomes do not contain exactly two heads?

24. Find the number of three-digit even numbers.
25. Find the number of three-digit even numbers divisible by 5.
26. Find the number of four-digit numbers divisible by 3 or 7.
27. Find the number of five-digit numbers with distinct digits.
28. Find the number of integers that are greater than or equal to 2 and less than or equal to 500 and
   a. have distinct digits.
   b. have distinct digits and are divisible by 5.
   c. contain the digit 3.
   d. divisible by 3.
   e. divisible by 5.
   f. divisible by 7.
29. Find the number of positive integers less than or equal to 500 and are
   a. divisible by 3.
   b. divisible by 5.
   c. divisible by 7.
   d. divisible by 3 and 5.
   e. divisible by 3 and 7 or 5.
   f. neither divisible by 3 nor divisible by 7.

30. Let $A$ be a set with 10 elements. Find the number functions from $A$ into a set $B$ such that the number of elements in $B$ is
   a. 5.
   b. 7.
   c. 10.
   d. 15.
31. Let $A$ be a set with 8 elements. Find the number of one-one functions from $A$ into a set $B$ such that the number of elements in $B$ is
   a. 4.
   b. 8.
   c. 10.
   d. 20.
32. Let $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{x, y\}$ be sets.
   a. Find the number of functions from $A$ into $B$ if each element of $A$ is mapped to $x$.
   b. Find the number of functions from $A$ into $B$ if each element of $A$ is mapped to $x$ and each odd subscripted element of $A$ is mapped to $y$.
   c. Find the number of functions from $A$ into $B$ if each element of $A$ is mapped to $x$.  

33. Let $S$ be a set with 100 elements.
   a. Find the number of subsets of $S$ that have exactly one element.
   b. Find the number of subsets of $S$ that have at least two elements.
34. Let $S$ be a set with $n$ elements, $n > 0$.
   a. Find the number of subsets of $S$ that have exactly $n - 1$ elements.
   b. Find the number of subsets of $S$ that do not have $n - 1$ elements.
35. A palindrome is a string that reads the same forward and backward. For example, madam is a palindrome. Let $A$ be a set of lowercase English letters.
   a. Find the number of palindromes over the set $A$ of length 10.
   b. Find the number of palindromes over the set $A$ of length 11.
   c. Find the number of palindromes over the set $A$ of length $n$, $n > 0$.
36. Suppose Brad uses a string $a_1 a_2 a_3 a_4$ of four digits from the digits 1, 2, 3, 4, 5, 6 as the identification number on the books in his collection. Find the maximum number of distinct identification numbers.
37. Consider an ISBN $x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9$ for books $x_i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ for $1 \leq i \leq 9$. Find the maximum number of expressions $x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9$ if not all $x'$s are 0 at the same time.

38. Consider the following nested loops.
   for $i := 1$ to 10 do
      for $j := 1$ to 20 do
         print "Hello";
         a. How many times is the word Hello printed?
         b. How many times does the inner loop (for $j := 1$ to 20 do) execute? What is the number of iterations of this loop?
         c. How many times does the outer (for $i := 1$ to 10 do) loop execute? What is the number of iterations of this loop?
39. Consider the following nested loops.
   for $i := 1$ to 10 do
      for $j := 1$ to 10 do
         print "Hello";
         a. How many time is the word Hello printed?
         b. How many times does the inner loop (for $j := 1$ to 10 do) execute? What is the number of iterations of this loop?
         c. How many times does the outer (for $i := 1$ to 10 do) loop execute? What is the number of iterations of this loop?
40. Consider the following nested loops.
   for $i := 1$ to 10 do
      for $j := 1$ to 20 do
         for $k := 1$ to 15 do
            print "Hello";
            a. How many times is the word Hello printed?
            b. How many times does the innermost loop (for $k := 1$ to 15 do) execute? What is the number of iterations of this loop?
7.2 Pigeonhole Principle

This chapter is about counting principles and in this section we describe another counting principle.

Let us consider the following problems.

a. There are 13 people in a room. At least 2 of these 13 people must be born in the same month.

b. Sometimes airlines, hoping for cancellations, overbook flights. If 101 people are booked for a trip and the plane has only 100 seats, then at least 2 people must be assigned the same seat.

There are many other problems like these in which an object with certain properties needs to be determined. For example, in the birthday problem, we want to show that there is a month with the property that at least two people are born in the same month. Such problems can be answered by applying the principle commonly known as the pigeonhole principle.

The Pigeonhole Principle: Suppose there are \(n\) pigeons, \(k\) pigeonholes, and \(n > k\). If these \(n\) pigeons fly into these \(k\) pigeonholes, then some pigeonhole must contain at least two pigeons.

Figure 7.4 illustrates the pigeonhole principle, where each dot represents a pigeon.

![Figure 7.4 Pigeons in the pigeonholes](image)

Notice that the pigeonhole principle only tells us that an object with the desired property exists. It does not tell us which object has the desired property or how to find that object.

The pigeonhole principle is also known as the Dirichlet drawer principle, or the shoebox principle. This principle was first formally stated by Peter Gustave Lejeune Dirichlet (1805–1859).

In order to apply the pigeonhole principle, we need to specify which objects are pigeons and which objects are pigeonholes.

Example 7.2.1

In this example, we answer the first problem posed at the beginning of this section: There are 13 people in a room. At least 2 of these 13 people must be born in the same month.
Initially, all \( n \) elements of \( S \) are available. Any of these \( n \) elements can be selected as the first element of the \( r \)-permutation. Therefore, the first step can be completed in \( n \) different ways. After completing the first step, only \( n - 1 \) elements are left. Therefore, step 2 can be completed in \( n - 1 \) different ways. In general, after selecting the first \( i - 1 \) elements of the \( r \)-permutation, for the \( i \)th element \( n - (i - 1) = n - i + 1 \) elements are left, where \( 1 \leq i \leq r \). Any of these \( n - i + 1 \) elements can be selected as the \( i \)th element of the \( r \)-permutation. Therefore, the \( i \)th step can be completed in \( n - i + 1 \) different ways.

It now follows by the multiplication principle that the number of ways an \( r \)-permutation can be constructed is:

\[
n(n-1)(n-2)\cdots(n-r+1)
\]

ways. Hence,

\[
P(n, r) = n(n-1)(n-2)\cdots(n-r+1). \quad \blacksquare
\]

**Remark 7.3.5** Let \( n \) and \( r \) be two integers such that \( 1 \leq r \leq n \). Recall that for any positive integer \( n \), the product \( n(n-1)(n-2)\cdots3 \cdot 2 \cdot 1 \) is denoted by \( n! \). Moreover, \( 0! = 1 \). Now notice that

\[
\frac{n!}{(n-r)!} = \frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)!}{(n-r)!}
\]

\[
= n(n-1)(n-2)\cdots(n-r+1).
\]

Hence,

\[
P(n, r) = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}.
\]

**Corollary 7.3.6** Let \( S \) be a set with \( n \) distinct elements, \( n > 0 \). Then \( P(n, n) \)—the number of \( n \)-permutations of \( S \)—is given by:

\[
P(n, n) = n!
\]

**Proof:** By Remark 7.3.5,

\[
P(n, n) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1} = n!. \quad \blacksquare
\]

**Worked-out Exercises**

**Exercise 1:** Let \( S \) be a set with 50 elements. Find the number of \( 3 \)-permutations of \( S \) and find the number of permutations of \( S \).

**Solution:** The number of \( 3 \)-permutations of \( S \) is:

\[
P(50, 3) = 50 \cdot 49 \cdot 48 = 117,600.
\]

The number of permutations of \( S \) is:

\[
P(50, 50) = 50!.
\]

**Exercise 2:** How many dance pairs, (dance pairs means a pair \((W, M)\), where \( W \) stands for a woman and \( M \) for man), can be formed from a group of 6 women and 10 men?

**Solution:** The problem is equivalent to finding all one-one functions from the set of 6 women to the set of 10 men, which is the same as finding all 6-permutations of a set of 10 elements. Now the number of 6-permutations of a set of 10 elements is \( P(10, 6) = \frac{10!}{(10-6)!} = \frac{10!}{4!} = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = 151200 \).
Exercise 3: How many four-letter words can be formed from the letters G, R, O, U, P, S if no letter is to be used more than once in any word?

Solution: We will arrange four distinct letters in a row to form a four-letter word. If we change the ordering of letters we will get a different word.

Therefore, the number of four-letter words where no letter is to be used more than once in any word
= the number of ordered arrangements of the letters G, R, O, U, P, S taken four at a time
= the number of permutations of {G, R, O, U, P, S}
= the number of permutations of six distinct elements taken four at a time
= \( P(6, 4) = \frac{6!}{(6-4)!} = \frac{6!}{2!} = 6 \cdot 5 \cdot 4 \cdot 3 = 360 \).

Exercise 4: How many numbers between 20000 and 50000 can be formed with the digits 1, 2, 3, 4, 5, 6 such that no digits are repeated in any of the numbers so formed?

Solution: Any number between 20000 and 50000 is a five-digit number of the form \( \alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5 \) such that \( \alpha_i \neq 1, 5, 6 \). If all the five digits are distinct and there are no restrictions for \( \alpha_i \), then the number of five-digit numbers is the same as the number of permutations of five elements of the set \{1, 2, 3, 4, 5, 6\}, which equals \( P(6, 5) = \frac{6!}{(6-5)!} = \frac{6!}{1!} = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = 720 \).

Let us now find in how many of the numbers \( \alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5 \), \( \alpha_5 \) is 1, 5, or 6.

If \( \alpha_5 \) is 1, then we will find the numbers \( 1 \alpha_2 \alpha_3 \alpha_4 \alpha_5 \), where \( \alpha_i \in \{2, 3, 4, 5, 6\} \) for \( i = 1, 2, 3, 4 \) and the number of such numbers is the same as the number of permutations of four elements of the set \{2, 3, 4, 5, 6\}, which equals \( P(5, 4) = \frac{5!}{(5-4)!} = \frac{5!}{1!} = 5 \cdot 4 \cdot 3 \cdot 2 = 120 \).

Similarly, there are 120 numbers of the form \( 5 \alpha_2 \alpha_3 \alpha_4 \alpha_5 \) and also there are 120 numbers of the form \( 6 \alpha_2 \alpha_3 \alpha_4 \alpha_5 \). Because we will not count the numbers \( 1 \alpha_2 \alpha_3 \alpha_4 \alpha_5 \), \( 5 \alpha_2 \alpha_3 \alpha_4 \alpha_5 \), and \( 6 \alpha_2 \alpha_3 \alpha_4 \alpha_5 \), we find that there are \( 720 - 120 - 120 = 360 \) numbers lying between 20000 and 50000 that can be formed with the digits 1, 2, 3, 4, 5, 6 such that digits are not repeated in any of the numbers so formed.

Exercise 5: In how many ways can six boys and five girls stand in a line so that no two girls are next to each other?

Solution: According to the given conditions, between two girls there must be a boy. Suppose the six boys are \( B_1, B_2, B_3, B_4, B_5, B_6 \) and the girls are \( G_1, G_2, G_3, G_4, G_5, G_6 \) where \( G_i \) denote the positions for girls. For girls there are seven positions. In these seven positions, five different girls can stand in \( P(7, 5) \) different ways (because it is a 5-permutation of seven elements). After each arrangement of the girls, the six boys can stand in six different places in \( P(6, 6) \) different ways (because it is a 6-permutation of six elements).

Hence, by the multiplication principle, the number of ways six boys and five girls may stand in a line so that no two girls are next to each other is
\[
P(7, 5) \cdot P(6, 6) = \frac{7!}{(7-5)!} \cdot 6! = \frac{7!}{2!} \cdot 6! = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2!}{2!} \cdot 6! = 1814400.
\]

Exercise 6: In how many ways can six boys and five girls stand in a line so that all the boys stand side by side and all the girls stand side by side?

Solution: An arrangement of the required type looks as follows:
\[
G \ G \ G \ G \ G \ B \ B \ B \ B \ B \ B
\]
or
\[
B \ B \ B \ B \ B \ G \ G \ G \ G \ G
\]
Here \( G \) denotes the position of a girl and \( B \) denotes the position of a boy. Now five girls can be arranged in a row in \( P(5, 5) \) ways (because it is a 5-permutation of five elements). After each arrangement of the girls, the six boys can stand in six different places in \( P(6, 6) \) different ways (because it is a six-permutation of six elements). Hence, by the multiplication principle, the number of ways five girls and six boys may stand in a line so that all the boys stand first and then all the girls is
\[
P(6, 6) \cdot P(5, 5) = 6! \cdot 5!.
\]
Similarly, the number of ways five girls and six boys can stand in a line so that all the boys stand first and then all the girls stand first and then all the girls is
\[
P(6, 6) \cdot P(5, 5) = 5! \cdot 6!.
\]
Hence the number of ways six boys and five girls stand in a line so that all the boys stand side by side and all the girls stand side by side is
\[
5! \cdot 6! + 6! \cdot 5! = 2 \cdot 5! \cdot 6!.
\]
Some Key Results

1. Let \( S \) be a set with \( n \) distinct elements, \( n > 0 \). Let \( 0 < r \leq n \). Then \( P(n, r) \) is given by the following formula: \( P(n, r) = n(n-1)(n-2)\cdots(n-r+1) \).

2. Let \( S \) be a set with \( n \) distinct elements, \( n > 0 \). Then \( P(n, n) \) the number of \( n \)-permutations of \( S \) is given by \( P(n, n) = n! \).

**Exercises**

1. Find \( P(10, 3) \), \( P(15, 10) \), \( P(6, 0) \), \( P(6, 6) \).
2. Find the positive integer \( n \) such that \( P(n+1, 3) = 10 \cdot P(n-1, 2) \).
3. Show that for any positive integer \( n \), \( P(n, n-1) = n! \).
4. Find the number of different one-one functions from the set \( \{1, 2, 3, 4\} \) into \( \{E, F, G, H, I, J\} \).
5. Find the number of different one-one correspondences from a set of four distinct elements into itself.
6. Find the number of different ways a grade of A, B, C, or D can be assigned to three students of a class so that no two students receive the same grade.
7. Three friends go to a movie where they find seven vacant seats in a row. In how many different ways they can seat themselves?
8. There are 12 hospitals in a town. How many different ways can seven patients be sent to the hospitals so that no two patients may be in the same hospital?
9. How many three-letter words can be formed from the letters \( A, N, G, R, Y \) if no letter is to be used more than once in a word?
10. How many distinct five-letter words can be formed from the letters \( A, N, G, R, Y \) if no letter is to be used more than once in a word?
11. How many different ways can the letters of the word \( MONDAY \) be arranged? How many of them start with \( M \) and do not end with \( Y \)?
12. Find the different numbers of three digits that can be formed with the digits 1, 2, 3, 4, 5, 6 such that no digit is repeated in any of the numbers so formed.
13. How many numbers between 5000 and 4000, with distinct digits, can be formed using the digits 1, 3, 4, 5, 6?
14. How many numbers greater than 3000, with distinct digits, can be formed using the digits 1, 3, 4, 5, 6?
15. How many even numbers greater than 500, with distinct digits, can be formed using the digits 3, 4, 5, 6, 7?
16. Determine the four-digit numbers, with distinct digits, that can be formed using the digits 0, 1, 2, 3, 4, 5, 6, 7 such that none of the numbers have a leading 0.
17. In how many ways can six boys and eight girls be arranged in a line so that no two boys may occupy consecutive positions?
18. How many ways can six boys and six girls be seated in a row if the boys and girls are to have alternate seats?
19. Seven rooms are available in a motel. Four visitors come to the motel and ask for separate rooms. In how many ways can the manager assign the rooms?
20. Find the number of seating arrangements in a row of eight students so that two particular students will not sit side by side.
21. Find the number of six-letter words that can be formed from the letters of the word \( HISTORY \) if no letter is used more than once in any word subject to the conditions given below.
   a. The first letter of each word is \( H \).
   b. The first letter of each word is either \( H \) or \( Y \).
   c. The word starts with \( HIS \).
   d. The word contains \( HIS \) as a substring.
22. How many dance pairs \( (G, B) \), where \( G \) stands for girl and \( B \) for boy, can be formed from a group of 10 girls and 15 boys?
23. How many different four-digit numbers with distinct digits can be formed using the digits 1, 2, 3, 4, 5, 6?

**7.4 Combinations**

In the previous section, we were interested not only in selecting certain elements, but also in arranging them in a row. However, there are many situations in which we are only interested in selecting certain elements. For example, consider the following problem.

Suppose that there are 4 students, \( s_1, s_2, s_3, \) and \( s_4 \), interested in serving on a committee that handles dorm-room assignments. How many ways can such a committee of 2 students be formed? Notice that here we are only interested in selecting 2 students out of 4. Let \( S = \{s_1, s_2, s_3, s_4\} \) be the set of these students. A committee of 2 students out of these 4 students is a 2-element subset of \( S \). Thus, the number of ways to form a committee of 2 students reduces to finding the number
REMARK 7.4.7 The proof of Corollary 7.4.6 can also be given as follows: Let \( 0 \leq r \leq n \) in a selection of \( r \) elements, we do not select the other \( n - r \) elements. Therefore, we have two subsets of \( S \), one with \( r \) elements and the other with \( n - r \) elements. For each selection of \( r \) elements, we have a selection of \( n - r \) elements. And conversely, for each selection of \( n - r \) elements, we have a selection of \( r \) elements. Hence, \( C(n, r) = C(n, n - r) \).

COROLLARY 7.4.8: Let \( n \) be a nonnegative integer. Then
\[
C(n, n) = 1.
\]

WORKED-OUT EXERCISES

Exercise 1: In how many ways can a soccer team of 11 players be selected from a group of 20 players?

Solution: We are to select 11 players out of 20 players. For this we find all 11-combinations of a set of 20 elements. The number of such combinations is
\[
C(20, 11) = \frac{20!}{11!(20 - 11)!} = \frac{20!}{11!9!} = \frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12}{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} = 167960.
\]

Exercise 2: If \( C(16, r) = C(16, r + 2) \), then find \( r \).

Solution: \( C(16, r) = C(16, r + 2) \) implies either \( r = r + 2 \) or \( r = (r + 2) = 16 \) or \( r + (r + 2) = 16 \). Now \( r \neq r + 2 \). Therefore, \( r + (r + 2) = 16 \), which implies \( r = 7 \).

We can also determine the value of \( r \) using a direct computation as follows:
\[
C(16, r) = C(16, r + 2)
\]
\[
\Rightarrow \frac{16!}{r!(16 - r)!} = \frac{16!}{(r + 2)!(16 - r - 2)!}
\]
\[
\Rightarrow (r + 2)!(16 - r - 2)! = r!(16 - r - 2)!
\]
\[
\Rightarrow (r + 2)(r + 1) = (16 - r)(16 - r - 2)
\]
\[
\Rightarrow r^2 + 3r + 2 = 16r - 32r + r^2 - 16 + r
\]
\[
\Rightarrow 3r + 2 = 240 - 31r
\]
\[
\Rightarrow 34r = 238
\]
\[
\Rightarrow r = 7.
\]

Exercise 3: Let \( S \) be a set containing \( n \) elements, where \( n \) is a positive integer. If \( r \) is an integer such that \( 0 \leq r \leq n \), then show that the number of subsets of \( S \) containing exactly \( r \) elements is
\[
\frac{n!}{r!(n - r)!}.
\]

Solution: We prove this result by induction on \( n \).

Let \( P(n) \) be the statement: If \( S \) is a set containing \( n \) \((\geq 0)\) elements, then the number of subsets of \( S \) containing exactly \( r \) elements \((0 \leq r \leq n)\) is \( \frac{n!}{r!(n - r)!} \).

**Basis step:** Suppose \( n = 1 \). Then \( S \) has only one element, say \( a \). Then \( \emptyset \) and \( \{a\} \) are the only subsets of \( S \). Thus, if \( r = 0 \), then \( \emptyset \) is the only subset with \( 0 \) element. Hence,
\[
1 = \frac{1!}{0!(1 - 0)!}.
\]

Again, for \( r = 1 \), \( \{a\} \) is the only subset with \( 1 \) element. So,
\[
1 = \frac{1!}{1!(1 - 1)!}.
\]

Hence, we see that the statement is true for \( n = 1 \).

**Inductive hypothesis:** Suppose \( k \) is a positive integer. Assume that \( P(k) \) holds for any set with \( k \) elements.

**Inductive step:** Let \( S \) be a set with \( k + 1 \), \( k \geq 1 \), elements. Let us write \( S = \{a_1, a_2, \ldots, a_n, a_{n+1}\} \). We now determine the number of subsets of \( S \) containing exactly \( r \) elements where \( 0 \leq r \leq k + 1 \).

If \( r = 0 \), then the empty set, \( \emptyset \), is the only subset with zero elements. If \( r = k + 1 \), then the set \( S \) is the only subset with \( k + 1 \) elements. In both of these cases, \( P(k + 1) \) holds because
\[
1 = \frac{(k + 1)!}{0!(k + 1 - 0)!} \quad \text{and} \quad 1 = \frac{(k + 1)!}{(k + 1)(k + 1 - k - 1)!}.
\]

Now, let \( A \) be any subset with exactly \( r \) elements where \( 0 < r < k + 1 \). There are two cases to be considered.
Case 1: \( a_{i1} \not\in A \). In this case, \( A \) is a subset of the set \( \{a_1, a_2, \ldots, a_r\} \). By the inductive hypothesis, the number of such subsets is
\[
\frac{k!}{(r-1)!(k-r+1)!}.
\]

Case 2: \( a_{i1} \in A \). In this case, if we remove \( a_{i1} \) from \( A \), then \( A \setminus \{a_{i1}\} \) is a subset of \( \{a_1, a_2, \ldots, a_r\} \) and the number of elements in \( A \setminus \{a_{i1}\} \) is \( r - 1 \). By the inductive hypothesis, the number of such subsets is
\[
\frac{k!}{(k-r+1)!}. \]

Now from Case 1 and Case 2, we find that the total number of subsets \( A \) of \( S \) with \( r \) elements is
\[
\frac{k!}{r!(k-r)!} + \frac{k!}{(r-1)!(k-r+1)!} = \frac{k!(k-r+1) + k!(k-r)}{r!(k-r)!} = \frac{(k+1)!}{r!(k+1-r)!}.
\]

Hence, \( P(k+1) \) is true. The result now follows by induction.

Exercise 4: Let \( X = \{0, 1, 2, 3, 4\} \). Find the number of subsets of \( X \) that contain three elements of \( X \).

Solution: The number of subsets of \( X \) that contain three elements of \( X \) is the number of 3-combinations of \( X \). This number is \( C(5, 3) = \binom{5}{3} = \frac{5!}{3!2!} = 10 \).

Exercise 5: A committee of six is to be made from four students and eight teachers. In how many ways can this be done:

a. If the committee contains exactly three students?

b. If the committee contains at least three students?

Solution:

a. We are to select a committee of six from four students and eight teachers such that the committee contains exactly three students. Hence, the other three members are the teachers. Now, we can select three students out of four in \( C(4, 3) \) ways. For each of these selections, the remaining three members are selected from eight teachers and this can be done in \( C(8, 3) \) ways. Hence, the number of ways a committee of six from four students and eight teachers such that the committee contains exactly three students is
\[
C(4, 3) \cdot C(8, 3) = \frac{4!}{3!(4-3)!} \cdot \frac{8!}{3!(8-3)!} = \frac{4! \cdot 8!}{3! \cdot 3!} = 4 \cdot \frac{8 \cdot 7 \cdot 6}{3 \cdot 2} = 224.
\]

b. We have to consider two cases.

Case 1: 3 students and 3 teachers.

Case 2: 4 students and 2 teachers.

In Case 1, from part (a) we find that the number of ways the committee can be formed is 224.

In Case 2, proceeding as in part (a), the number of ways the committee can be formed is
\[
C(4, 4) \cdot C(8, 2) = \frac{4!}{4!(4-4)!} \cdot \frac{8!}{2!(8-2)!} = \frac{4! \cdot 8!}{4! \cdot 2!} = \frac{8 \cdot 7}{2} = 28.
\]

Now combining parts (a) and (b) (using the addition principle), a committee of six from four students and eight teachers such that the committee contains at least three students is 224 + 28 = 252.

Exercise 6: A student is required to answer 7 out of 12 questions, which are divided into two groups, each containing 6 questions. The student is not permitted to answer more than 5 questions from either group. In how many different ways can the student choose the 7 questions?

Solution: The student can choose 7 questions satisfying the given restrictions in the following ways:

<table>
<thead>
<tr>
<th>Number of questions from group A</th>
<th>Number of questions from group B</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
</tr>
</tbody>
</table>

Hence, the total number of ways in which the student can choose the 7 questions is
\[
C(6, 5) \cdot C(6, 2) + C(6, 3) \cdot C(6, 4) = 90 \cdot 300 + 300 = 780.
\]

Exercise 7: Seema has six friends. In how many ways can she invite one or more friends to a dinner party?

Solution: Seema may invite one friend out of six, or two out of six, or three out of six, or four out of six, or five out of six, or six out of six. Hence, the number of ways Seema can invite her friends is
\[
C(6, 1) + C(6, 2) + C(6, 3) + C(6, 4) + C(6, 5) + C(6, 6) = \frac{6!}{1!(6-1)!} + \frac{6!}{2!(6-2)!} + \frac{6!}{3!(6-3)!} + \frac{6!}{4!(6-4)!} + \frac{6!}{5!(6-5)!} + \frac{6!}{6!(6-6)!} = 6 + 15 + 20 + 15 + 6 + 1 = 63.
\]

Another Solution: Seema may invite one friend out of six, or two out of six, or three out of six, or four out of six, or five out of six, or six out of six. Thus, we need to find the number of nonempty subsets of a set of six elements. Now the number of subsets of a set of six elements is \( 2^6 = 64 \). One of these subsets is empty. Therefore, the number of nonempty subsets of a set of six elements is \( 64 - 1 = 63 \). Hence, there are 63 ways Seema can invite one or more friends to a dinner party.
SECTION REVIEW

Key Terms

combination

\[ C(n, r) \]

r-combination

Some Key Results

1. Let \( S \) be a set with \( n > 0 \) elements. Let \( r \) be an integer such that \( 0 \leq r \leq n \). The number of subsets that contain \( r \) elements of \( S \) is \( \frac{n!}{r!(n-r)!} \).

2. Let \( S \) be a set with \( n > 0 \) elements. Let \( r \) be an integer such that \( 0 \leq r \leq n \). Then \( C(n, r) = \frac{n!}{r!(n-r)!} \).

3. Let \( n \) and \( r \) be nonnegative integers such that \( 0 \leq r \leq n \). Then \( C(n, r) = C(n, n-r) \).

EXERCISES

1. Find \( C(10, 3) \), \( C(15, 10) \), \( C(6, 0) \), \( C(6, 6) \).
2. Find the positive integer \( n \) such that \( C(20, 2n) = C(20, 2n + 4) \).
3. Let \( X = \{2, 3, 4, 5, 6, 7\} \). Find the number of subsets of \( X \) that contain four elements.
4. Let \( X = \{1, 2, 3, 4, 5, 6, 7, 8\} \). Find the number of subsets of \( X \) that contain four odd integers.
5. Find the number of subsets of \( \{A, B, C, D, E\} \) that contain no vowels.
6. How many four-element subsets of \( \{a, b, c, d, e, i, o, u, x\} \) contain no vowels?
7. Find the number of committees consisting of four different members from a group of 16 people.
8. A box contains 15 apples. How many different selections of 3 apples can be made so as to include a particular apple?
9. How many ways can a soccer team of 11 players be selected from 18 players?
10. How many different triangles can be formed by joining the vertices of a square?
11. In an athletic club, there are 16 male members and 10 female members. How many ways can we form a committee of 7 members subject to the following conditions:
   a. there must be 3 males and 4 females.
   b. the committee must contain at least 2 females.
   c. the committee must contain at least 2 males.
12. Find the number of ways we can form a committee of four Republicans and three Democrats from a group of ten Republicans and eight Democrats.
13. From a group of seven Democrats and four Republicans, a committee of five with at least one Republican is to be formed. In how many ways can this be done?
14. A test consists of 12 questions that are divided into three sections. There are 5 questions in the first section, 4 in the second section, and 3 in the third section. A student is required to answer 6 out of these 12 questions. In how many ways can the student answer 6 questions if the student selects 3 from the first section, 2 from the second section, and 1 from the third section?
15. An exam consists of 10 questions that are divided into two sections. Each section contains 5 questions. A student is required to answer 6 out of 10 questions. The student is not permitted to answer more than 4 questions from any group. In how many ways can the student select the questions?
16. How many different words consisting of four consonants and three vowels can be formed from an alphabet of ten consonants and four vowels? (Repetition of a letter in a word is not allowed.)
17. In a group of 20 students, 8 students are girls. In how many ways can 12 students be selected so as to include (i) exactly 7 girls (ii) at least 7 girls?
18. A club has 12 members. A president, a vice president, a secretary, and a treasurer are to be chosen from these members. A member cannot serve in more than one position. In how many ways can these officers be chosen?
19. A club has two groups of players labeled group A and group B. The number of players in group A is 6 and the number of players in group B is 8. Find the number of ways a football team of 11 players can be formed from the two groups so that the team contains at least 4 players from group A.
20. Hamid has seven friends. In how many ways can he invite one or more of them to a dinner?