$f^n : A \rightarrow A$ is a one-one function for all integers $n \geq 1$ (see Worked-Out Exercise 7, p. 295).

\[ \text{Theorem 5.1.3: Let } A \text{ be an infinite set and } f : A \rightarrow A \text{ be a function. If } n \text{ is a positive integer, then } f^n \text{ is one-one and hence a function on } A \text{ as follows.} \]

\[ \text{Proof: Let } a \in A. \text{ Now } f^n : A \rightarrow A, \text{ so } f^n(a) \in A \text{ for all } n \geq 1. \text{ This implies that } \}
\{a, f(a), f^2(a), \ldots \} \subseteq A.

Because $A$ is finite, it follows that $\{a, f(a), f^2(a), \ldots \}$ is finite. Therefore, there must exist positive integers $r$ and $s$ such that $r > s$ and

\[ f^r(a) = f^s(a). \]

Now

\[ f^r(a) = f^s(a) \]
\[ \Rightarrow (f^s \circ f^{-1})(a) = f^s(a) \]
\[ \Rightarrow f^s(f^{-1}(a)) = f^s(a) \]
\[ \Rightarrow f^{-1}(a) = a, \quad \text{because } f \text{ is one-one.} \]

Let

\[ a' = f^{-1}(a) \in A. \]

Then

\[ f(a') = f(f^{-1}(a)) = f^{-1}(a) = a. \]

We can now conclude that $f$ is onto $A$. Consequently, $f$ is a one-to-one correspondence.

**Worked-Out Exercises**

**Exercise 1:** Determine which of the relations $f$ are functions from the set $X$ to the set $Y$.

(a) $X = \{-2, -1, 0, 1, 2\}$, $Y = \{-3, 4, 5\}$, and $f = \{(-2, -3), (-1, -3), (0, 4), (1, 5), (2, -3)\}$.

(b) $X = \{-2, -1, 0, 1, 2\}$, $Y = \{-3, 4, 5\}$, and $f = \{(-2, -3), (1, 4), (2, 5)\}$.

(c) $X = Y = \{-3, -1, 0, 2\}$, and $f = \{(-3, -1), (-3, 0), (-1, 2), (0, 2), (2, -1)\}$.

(d) $X = Y = \text{the set of all integers}$, and $f = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid b = a + 1\}$.

(e) $X = Y = \text{the set of all integers}$, and $f = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid b = \sqrt{a}\}$.

In case any of these relations are functions, determine if they are one-one, onto $Y$, and/or one-to-one correspondences.

**Solution:**

(a) The domain of $f$, $D(f) = \{-2, -1, 0, 1, 2\} = X$. Moreover, for any $a \in X$ there are no two distinct elements $b$ and $c$ in $Y$ such that $(a, b) \in f$ and $(a, c) \in f$. Hence, $f$ is a function. Now $\text{Im}(f) = \{-3, 4, 5\} = Y$. Thus, $f$ is onto $Y$. However, $f(-2) = -3 = f(-1)$ and $-2 \neq -3$. Hence, $f$ is not one-one.

(b) The domain of $f$, $D(f) = \{-2, 1, 2\} \neq X$. Hence, $f$ is not a function on $X$.

(c) The domain of $f$, $D(f) = \{-3, -1, 0, 2\} = X$. Now $(-3, -1) \in f$ and $(-3, 0) \in f$, so $f(-3) = -1$ and simultaneously $f(-3) = 0$, so $f$ is not well defined. Hence, $f$ is not a function.

(d) The domain of $f$, $D(f) = \mathbb{Z}$. Let $b, c \in \mathbb{Z}$ be such that $(a, b) \in f$ and $(a, c) \in f$. Then $b = a + 1$ and $c = a + 1$. Thus, $b = a + 1 = c$. Hence, $f$ is well defined. Consequently, $f$ is a function.

To show $f$ is one-one, let $a, b \in \mathbb{Z}$ and $a \neq b$. Then $a + 1 \neq b + 1$, so $f(a) \neq f(b)$. Thus, $f$ is one-one. (To show $f$ is one-one, we can also argue as follows: Let $a, b \in \mathbb{Z}$ and $f(a) = f(b)$. This implies that $a + 1 = b + 1$, so $a = b$. Hence, $f$ is one-one.)
To show $f$ is onto $Z$, let $a \in Z$. Then $a - 1 \in Z$ and $f(a - 1) = (a - 1) + 1 = a$; i.e., $a - 1$ is the preimage of $a$. Hence, every element of $Z$ is a preimage and so $\text{Im}(f) = Z$. This shows that $f$ is onto $Z$. Consequently, $f$ is a one-to-one correspondence.

(c) $\exists Z \therefore \sqrt{2} \not\in Z$. Thus $(2, \sqrt{2}) \not\in f$. Hence, the domain of $f$, $D(f) \not= Z$. This implies that $f$ is not a function.

**Exercise 2:** Let $f$ be the function from the set $X = \{2, 3, 4, 5, 6, 7\}$ into the set $Y = \{0, 1, 2, 3, 4\}$ defined by $f(x) = 2x \text{mod} 5$. Write $f$ as a set of ordered pairs. Is $f$ one-one or onto $Y$?

**Solution:** (Recall that $m \text{mod} n$ is the remainder when $m$ is divided by $n$. $2 \cdot 2 \text{mod} 5 = 4, 2 \cdot 3 \text{mod} 5 = 1, 2 \cdot 4 \text{mod} 5 = 3, 2 \cdot 5 \text{mod} 5 = 0, 2 \cdot 6 \text{mod} 5 = 2, 2 \cdot 7 \text{mod} 5 = 4$. Hence, $f = (2, 4), (3, 1), (4, 3), (5, 0), (6, 2), (7, 4))$. Now $f(2) \not= f(7)$ and $f(2) = 4 = f(7)$.

Thus, $f$ is not one-one. Again, the range of $f$, $\text{Im}(f) = \{1, 3, 0, 2\} = Y$, so $f$ is onto $Y$.

**Exercise 3:** Determine which of the following functions are one-one, onto, or both one-one and onto.

(a) $f : N \to Z - \{0\}$ defined by $f(n) = n$ for all $n \in N$.

(b) $f : Z \to Z$ defined by $f(x) = x - 4$ for all $x \in Z$.

(c) $f : R \to R$ defined by $f(x) = |x| + x$ for all $x \in R$.

(d) $f : R \to R$ defined by $f(x) = x^2$ for all $x \in R$.

(e) $f : C \to R$ defined by $f(z) = |z|$ for all $z \in C$.

**Solution:**

(a) Let $n, m \in N$. Suppose that $f(n) = f(m)$. Then $-n = -m$, so $n = m$. Therefore, $f$ is one-one. Notice that for all $n \in N$, $f(n) = n < 0$. This indicates that positive integers are not in the range of $f$ so $f$ is not onto $Z - \{0\}$. To be specific, consider $3 \in Z - \{0\}$. Suppose that 3 has a preimage. Then there exists $n \in N$ such that $f(n) = 3$. This implies that $3 = f(n) = -n < 0$, a contradiction. Hence, $f$ is not onto $Z - \{0\}$.

(b) Let $x, y \in Z$. Suppose that $f(x) = f(y)$. Then $x - 4 = y - 4$, so $x = y$. This shows that $f$ is one-one. Now $f$ is onto $Z$ if and only if for all $y \in Z$ there exists $x \in Z$ such that $f(x) = y$.

(c) Consider $-1, -2 \in R$. Now $f(-1) = |1| + (-1) = 1 + (-1) = 0$ and $f(-2) = |2| + (-2) = 2 + (-2) = 0$. This shows that $-1 \neq -2$, but $f(-1) = f(-2)$. Therefore, $f$ is not one-one.

(d) Let $x$ and $y$ be two elements of $R$. Suppose that $f(x) = f(y)$. Now

$$
\begin{align*}
f(x) &= f(y) \\
x^2 &= y^2 \\
x^2 - y^2 &= 0 \\
(x - y)(x^2 + xy + y^2) &= 0 \\
(x - y)(x + y)^2 + \frac{3}{2}y^2 &= 0 \\
x - y &= 0 \text{ or } (x + \frac{1}{2}y)^2 + \frac{3}{2}y^2 &= 0.
\end{align*}
$$

If $x \neq y$, then $x \neq y$ and this implies that $(x + \frac{1}{2}y)^2 + \frac{3}{2}y^2 > 0$. Thus, it follows that $x = y$, which in turn implies that $x = y$. Hence, $f$ is one-one.

Now let $a \in R$. Because the equation $x^2 = a$ has a solution $b$ in $R$, there exists an element $b$ in $R$ such that $f(b) = b^2 = a$. Hence, $f$ is onto $R$. Consequently, $f$ is a one-to-one correspondence.

**Exercise 4:** Let $f$ be the function from the set $N$ into the set $X = \{0, 1, 2, 3, 4, 5, 6, 7\}$ defined by $f(x) = x \text{mod} 7$ for all $x \in N$. Find $\text{Im}(f)$. Is $f$ onto $X$? Is $f$ one-one?

**Solution:** We know that for any positive integer $n$, $n \text{mod} 7$ is the remainder when $n$ is divided by 7. Now by the division algorithm, $n = 7t + r$, where $0 \leq r < 7$. Then $n \text{mod} 7 = r$. Hence, $\text{Im}(f) = \{0, 1, 2, 3, 4, 5, 6\}$. Because $[0, 1, 2, 3, 4, 5, 6] \not= [0, 1, 2, 3, 4, 5, 6, 7, 8]$, it follows that $f$ is not onto $X$.

Again, $10 \text{mod} 7 = 3 = 17 \text{mod} 7$. Hence, $f(10) = f(17)$. However, $10 \neq 17$. Therefore, $f$ is not one-one.

**Exercise 5:** Let $f : R \to R$ defined by $f(x) = x^2 - 4x$. Find $\text{Im}(f)$. Is $f$ onto $R$? Is $f$ one-one?

**Solution:** Let $y \in \text{Im}(f)$. Then $f(x) = y$ for some $x \in R$, i.e., $y = f(x) = x^2 - 4x$. Now

$$
\begin{align*}
y &= x^2 - 4x \\
y + 4 &= x^2 - 4x + 4 \\
y + 4 &= (x - 2)^2 \\
y + 4 &\geq 0 \\
y &\geq -4
\end{align*}
$$

This implies that $\text{Im}(f) = \{y \in R \mid y \geq -4\}$. From this it also follows that $\text{Im}(f) \not= R$, so $f$ is not onto $R$.

We can also show that $f$ is not onto $R$ by finding an element that has no preimage. For example, consider $-5 \in R$. Suppose that $f(x) = -5$ for some $x \in R$. Then

$$
\begin{align*}
-5 &= x^2 - 4x \\
-5 + 5 &= x^2 - 4x + 4 \\
0 &= (x - 2)^2
\end{align*}
$$

which is impossible because $(x - 2)^2 \geq 0$. Thus, $f$ is not one-one.
Moreover, \( f \) is not one-one as \( 0 \neq 4 \) and \( f(0) = f(4) \).

**Exercise 6:** Suppose that \( f : A \to B \) and \( g : B \to C \). Then prove that

(a) if \( g \circ f \) is one-one, then \( f \) is one-one;
(b) if \( g \circ f \) is onto \( C \), then \( g \) is onto \( C \);
(c) if \( g \circ f \) is a one-one onto \( C \), then \( f \) is one-one and \( g \) is onto \( C \).

**Solution:**

(a) Suppose that \( g \circ f \) is one-one. To show \( f \) is one-one, let \( a_1, a_2 \in A \) and \( f(a_1) = f(a_2) \). Now \( f(a_1) = f(a_2) \) and \( g \) is a function from \( B \) to \( C \). Therefore,

\[
g(f(a_1)) = g(f(a_2)),
\]

i.e.,

\[
(g \circ f)(a_1) = (g \circ f)(a_2).
\]

This implies that \( a_1 = a_2 \), because \( g \circ f \) is one-one. Hence, \( f \) is one-one.

(b) Suppose that \( g \circ f \) is onto \( C \). To show \( g \) is onto \( C \), let \( c \in C \). Because \( g \circ f \) is onto \( C \) and \( c \in C \), there exists \( a \in A \) such that

\[
(g \circ f)(a) = c.
\]

This implies that

\[
g(f(a)) = c.
\]

Let \( b = f(a) \in B \). Then we have

\[
c = (g \circ f)(a) = g(f(a)) = g(b).
\]

That is, \( b = f(a) \) is the preimage of \( c \). Because \( c \) is an arbitrary element of \( C \), we can conclude that \( g \) is onto \( C \).

(c) This follows from parts (i) and (ii).

**Exercise 7:** Let \( A \) be any set and \( f : A \to A \) be a one-one function. Then \( f^n : A \to A \) is a one-one function for all integers \( n \geq 1 \).

**Solution:** If possible, suppose there exists an integer \( n > 1 \) such that \( f^n \) is not one-one. Let \( k \) be the smallest such integer. That is, \( f, f^2, \ldots, f^{k-1} \) are one-one, but \( f^k \) is not one-one, \( k > 1 \). Because \( f^k \) is not one-one, there exist \( a, b \in A \) such that \( a \neq b \) and \( f^k(a) = f^k(b) \). Now,

\[
f^k(a) = f^k(b)
\]

\[
\Rightarrow (f \circ f^{k-1})(a) = (f \circ f^{k-1})(b)
\]

\[
\Rightarrow f(f^{k-1}(a)) = f(f^{k-1}(b))
\]

\[
\Rightarrow f^{k-1}(a) = f^{k-1}(b),
\]

because \( f \) is one-one

\[
\Rightarrow a = b,
\]

because \( f^{k-1} \) is one-one.

This is a contradiction as \( a \neq b \). Consequently, \( f^n \) is one-one for all integers \( n \geq 1 \).

**Exercise 8:** Let \( S = \{ x \in \mathbb{R} | -1 < x < 1 \} \). Show that the function \( f : \mathbb{R} \to S \) defined by

\[
f(x) = \frac{x}{1 + |x|}
\]

is a one-one and onto function.

**Solution:** Let \( x \in \mathbb{R} \). Then

\[
-|x| < x < |x|,
\]

\[
-1 - |x| < -|x|,
\]

and

\[
|1 + |x|| > |x|.
\]

Hence, \( -1 < x < 1 \) and so \( -1 < f(x) < 1 \). This shows that \( f(x) \in S \).

Let \( x, y \in \mathbb{R} \) and \( f(x) = f(y) \). Then \( \frac{x}{1 + |x|} = \frac{y}{1 + |y|} \). Thus,

\[
\frac{|x|}{1 + |x|} = \frac{|y|}{1 + |y|}.
\]

This implies that \( |x| + |x||y| = |y| + |x||y| \) and so \( |x| = |y| \). Now \( \frac{|x|}{1 + |x|} = \frac{1}{1 + |x|} \) implies that \( x \geq 0 \) if and only if \( y \geq 0 \). Therefore, because \( |x| = |y|, x = y \). Thus, \( f \) is one-one.

Now let \( z \in \mathbb{R} \) and \( -1 < z < 1 \). We show that there exists \( y \in S \) such that \( f(y) = z \). For this, first suppose that \( \bar{0} < z < 1 \). Let \( y \in \mathbb{R} \) be such that \( z = f(\bar{y}) \). Then,

\[
z = f(y) = \frac{y}{1 + |y|}.
\]

From this, notice that \( \bar{0} < 0 \). Thus, \( z = \frac{\bar{y}}{1 + |\bar{y}|} \). This implies that \( z(1 + y) = x \). Solve this for \( \bar{y} \) to get \( \bar{y} = \frac{z}{1 + z} \). This suggests that to find a preimage \( y \) of \( z \), we can use \( \bar{y} \) to be \( \frac{z}{1 + z} \). Let us verify this. Now

\[
f\left( \frac{z}{1 + z} \right) = \frac{\frac{z}{1 + z}}{1 + \left| \frac{z}{1 + z} \right|} = \frac{z}{1 + \frac{z}{1 + z}} = z.
\]

Now suppose \( -1 < z < 0 \). Here we can show that the preimage of \( z \) is \( \frac{z}{1 + z} \).

\[
f\left( \frac{z}{1 + z} \right) = \frac{\frac{z}{1 + z}}{1 + \left| \frac{z}{1 + z} \right|} = \frac{z}{1 + \frac{z}{1 + z}} = z.
\]

Hence, \( f \) is onto \( \mathbb{R} \). Consequently, \( f \) is a one-one and onto function.
SECTION REVIEW

Key Terms

function  target  onto
well defined  range  surjective
single valued  numeric functions  surjection
image  identity function  one-to-one correspondence
preimage  constant function  bijective
mapped  one-one  bijection
domain  injective  composition
codomain  injection

Some Key Definitions

1. Let $A$ and $B$ be nonempty sets and $f$ be a relation from $A$ into $B$. Then $f$ is called a function from $A$ into $B$, if
   
   (i) the domain of $f$ is $A$, i.e., $\mathcal{D}(f) = A$, and
   (ii) for all $(a, b), (a', b') \in f$, $a = a'$ implies $b = b'$. In this case, we say that $f$ is well defined or single valued.

2. Let $A$ and $B$ be sets and $f : A \rightarrow B$ be a function. The set $A$ is referred to as the domain of the function and the set $B$ is called the codomain, or target, of $f$. The set
   
   \[ f(A) = \{ f(x) \mid x \in A \} \]
   
   is a subset of the codomain $B$. The set $f(A)$ is called the range of the function $f$, or the image of the set $A$ under the function $f$, denoted by $\text{Im}(f)$ or $\mathcal{I}(f)$.

3. A function $f : A \rightarrow A$ is said to be the identity function if $f(x) = x$ for all $x \in A$. This function is usually denoted by $i_A$.

4. A function $f : A \rightarrow B$ is said to be a constant function if there exists $b \in B$ such that $f(x) = b$ for all $x \in A$. That is, all elements of $A$ are mapped to only one element of $B$.

5. Let $A$ and $B$ be sets and $f : A \rightarrow B$. Then,
   
   (i) $f$ is called one-one (or injective or injection) if for all $a_1, a_2 \in A$,
   \[ a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2) \]
   (i.e., images of distinct elements of the domain are distinct).
   (ii) $f$ is called onto $B$ (or surjective or surjection) if for every $b \in B$ there exists at least one $a \in A$ such that $f(a) = b$, i.e.,
   \[ \text{Im}(f) = B \]
   (iii) $f$ is called one-to-one correspondence (or bijective or bijection) if $f$ is both one-one and onto.
6. Let \( f : A \to B \) and \( g : B \to C \) be functions. The composition of \( f \) and \( g \), written \( g \circ f \), is the function from \( A \) to \( C \) defined as
\[
(g \circ f)(a) = g(f(a)), \quad \text{for all } a \in A.
\]

Some Key Results

1. Let \( f : A \to B \), \( g : B \to C \), and \( h : C \to D \). Then \( h \circ (g \circ f) = (h \circ g) \circ f \); i.e., composition of functions is associative, provided the composition is defined.

2. Suppose that \( f : A \to B \) and \( g : B \to C \). The following assertions hold.
   
   (i) If both \( f \) and \( g \) are one-one, then \( g \circ f \) is also one-one.
   
   (ii) If \( f \) is onto \( B \) and \( g \) are onto \( C \), then \( g \circ f \) is also onto \( C \).
   
   (iii) If both \( f \) and \( g \) are one-to-one correspondences, then \( g \circ f \) is also a one-to-one correspondence.

EXERCISES

1. Determine which of the relations \( f \) are functions from the set \( X \) to the set \( Y \).
   
   a. \( X = \{ -3, -2, -1, 0, 1, 2 \} \), \( Y = \{ 3, 4, 5, 6, 7 \} \), and \( f = \{ (-2, 3), (-1, 6), (0, 4), (1, 5), (2, 7) \} \).
   
   b. \( X = \{ -3, -2, -1, 0, 1, 2 \} \), \( Y = \{ 3, 4, 5, 6, 7 \} \), and \( f = \{ (-3, 3), (-2, 5), (0, 4), (-2, 6), (1, 5), (2, 7) \} \).
   
   c. \( X = \{ -3, -2, -1, 0, 1, 2 \} \), \( Y = \{ 3, 4, 5, 6, 7 \} \), and \( f = \{ (-2, 3), (0, 4), (-3, 6), (-1, 7), (1, 5), (2, 7) \} \).
   
   d. \( X = \{ -3, -1, 0, 2 \} \), and \( f = \{ (-3, -1), (-1, 2), (0, 2), (2, -1) \} \).
   
   e. \( X = Y \) is the set of all integers, \( f = \{ (a, b) \in \mathbb{Z} \times \mathbb{Z} \mid b = 2a - 1 \} \).
   
   f. \( X = Y \) is the set of all integers, \( f = \{ (a, b) \in \mathbb{Z} \times \mathbb{Z} \mid a^2 = b \} \).
   
   g. \( X = \mathbb{Q} \), \( Y = \mathbb{Q} \), defined by \( f(\frac{a}{b}) = a + m \) for all \( \frac{a}{b} \in \mathbb{Q} \).

2. Let \( A = \{ -3, -2, -1, 0, 1, 2 \} \). Find the range of the function \( f : A \to \mathbb{R} \), defined by \( f(x) = x^2 + 1 \) for all \( x \in A \).

3. Find the range of the function \( f : \mathbb{R} \to \mathbb{R} \), defined by \( f(x) = x^3 + x + 1 \) for all \( x \in \mathbb{R} \).

4. Consider the function \( f = \{(x, x^2) \mid x \in S\} \) from the set \( S = \{ -3, -2, -1, 0, 1, 2, 3 \} \) into \( \mathbb{Z} \). Is \( f \) one-one? Is \( f \) onto \( \mathbb{Z} \)?

5. Let \( f \) be the function from the set \( X = \{ 2, 3, 4, 5, 6, 7, 8 \} \) into the set \( Y = \{ 0, 1, 2, 3, 4, 5, 6, 7 \} \), defined by \( f(x) = 3x \pmod{7} \) for all \( x \in X \). Write \( f \) as a set of ordered pairs. Is \( f \) one-one, or onto \( Y \)?

6. Let \( f \) be the function from the set \( X = \{ 1, 2, 3, 4, 5, 6, 7, 8 \} \) into the set \( Y = \{ 0, 1, 2, 3, 4, 5, 6, 7, 8 \} \), defined by \( f(x) = (2x + 5) \pmod{5} \) for all \( x \in X \). Write \( f \) as a set of ordered pairs. Is \( f \) one-one or onto \( Y \)?

8. Show that the following functions are neither one-one nor onto (\( Z \) in (a); (b); and \( \mathbb{R} \) in (c), (d), and (e)).

   a. \( f : \mathbb{Z} \to \mathbb{Z} \), defined by \( f(x) = 4x^2 + 3 \) for all \( x \in \mathbb{Z} \).
   
   b. \( f : \mathbb{Z} \to \mathbb{Z} \), defined by \( f(0) = 0 \).
   
   c. \( f : \mathbb{R} \to \mathbb{R} \), defined by \( f(x) = \frac{x + 1}{x - 1} \) for all \( x \in \mathbb{R} \).
   
   d. \( f : \mathbb{R} \to \mathbb{R} \), defined by \( f(x) = \frac{x}{x - 1} \) for all \( x \in \mathbb{R} \).
   
   e. \( f : \mathbb{R} \to \mathbb{R} \), defined by \( f(x) = \cos x \) for all \( x \in \mathbb{R} \).

9. Show that the following functions are onto \( \mathbb{Z} \), but not one-one.

   a. \( f : \mathbb{Z} \to \mathbb{Z} \), defined by \( f(0) = 0 \).
   
   b. \( f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \), defined by \( f(m, n) = m + n \) for all \( m, n \in \mathbb{Z} \).

10. Show that the following functions are one-one, but not onto \( \mathbb{Z} \).

    a. \( f : \mathbb{Z} \to \mathbb{Z} \), defined by \( f(n) = 9n + 1 \) for all \( n \in \mathbb{Z} \).
    
    b. \( f : \mathbb{Z} \to \mathbb{Z} \), defined by \( f(n) = 3^n \) for all \( n \in \mathbb{Z} \).

11. Show that the following functions are one-to-one correspondences.

    a. \( f : \mathbb{R} \to \mathbb{R} \), defined by \( f(a) = a^2 \) for all \( a \in \mathbb{R} \).
    
    b. \( f : \mathbb{R} \to \mathbb{R} \), defined by \( f(x) = \frac{x + 1}{x} \) for all \( x \in \mathbb{R} \).
    
    c. \( f : \mathbb{R} \to \mathbb{R} \), defined by \( f(x) = x^2 - 1 \) for all \( x \in \mathbb{R} \).

12. Determine which of the following functions are one-one, onto, or both one-one and onto.
a. $f : \mathbb{Z} \to \mathbb{Z}$, defined by $f(n) = 4n - 3$ for all $n \in \mathbb{Z}$.

b. $f : \mathbb{Z} \times \mathbb{N} \to \mathbb{Z}$, defined by $f(n, m) = \frac{m}{n}$ for all $n \in \mathbb{Z}$ and for all $m \in \mathbb{N}$.

c. $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = x^2 - x$ for all $x \in \mathbb{R}$.

d. $f : \mathbb{Z} \to \mathbb{Q}$, defined by $f(n) = \frac{n}{2}$ for all $n \in \mathbb{Z}$.

e. $f : \mathbb{R}^+ \to \mathbb{R}^+$, defined by $f(x) = \frac{1}{x}$ for all $x \in \mathbb{R}^+$.

f. $f : \mathbb{R} \to \mathbb{R}^+$, defined by $f(x) = e^x$ for all $x \in \mathbb{R}$.

13. Let $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = x^2 - 6x$ for all $x \in \mathbb{R}$. Find $\text{Im}(f)$. Is $f$ onto $\mathbb{R}$? Is $f$ one-one?

14. Let $f : X \to Y$ and $g : X \to Y$ be functions. Show that $f = g$ if and only if $f(x) = g(x)$ for all $x \in X$.

15. Let $M_2(\mathbb{R})$ denote the set of all 2x2 matrices over real numbers. Define $f : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ by $f(A) = A^T$ (the transpose of a matrix $A$) for all $A \in M_2(\mathbb{R})$. Find $f(A)$, when $A = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$. Is $f$ one-one? Is it onto $M_2(\mathbb{R})$?

16. Define $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ by $f((x, y)) = (u, v)$, where $\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$. Find $f((1, 0))$, $f((0, 2))$. Is $f$ one-one? Is it onto $\mathbb{R} \times \mathbb{R}$?

17. Let $M_2(\mathbb{R})$ denote the set of all 2x2 matrices over real numbers. Define $f : M_2(\mathbb{R}) \to \mathbb{R}$ by $f(A) = a - d$ for all $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{R})$. Find $f(A)$, when $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Is $f$ one-one? Is it onto $\mathbb{R}$?

18. Let $A = \{1, 2, 3\}$. List all one-one functions from $A$ onto $A$.

19. Let $A = \{1, 2, \ldots, n\}$. Show that the number of one-one and onto functions from $A$ into $A$ is $n!$.

20. Let $f : A \to B$ be a function. Define a relation $R$ on $A$ by for all $a, b \in A$, $a R b$ if and only if $f(a) = f(b)$. Show that $R$ is an equivalence relation.

21. Let $A = \{x \in \mathbb{Z} \mid -5 < x \leq 0\}$, $B = \{x \in \mathbb{Z} \mid 0 < x \leq 8\}$, and $C = \{x \in \mathbb{Z} \mid -8 < x \leq 2\}$. Consider the functions $f : A \to B$ defined by $f(x) = x + 3$ for all $x \in A$ and $g : B \to C$ defined by $g(x) = -x + 1$ for all $x \in B$. Draw the arrow diagrams of the functions $f : A \to B$ and $g : B \to C$. Then draw the arrow diagram of $g \circ f$.

22. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be functions defined by $f(x) = x^2 - 2x + 4$ and $g(x) = 7x - 2$ for all $x \in \mathbb{R}$. Find $f \circ g$, $g \circ f$, and $f \circ g(-2)$, $g \circ f(-2)$.

23. Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ and $g : \mathbb{R}^+ \to \mathbb{R}^+$ be functions defined by $f(x) = \sqrt{x}$ and $g(x) = \frac{x}{2} + 1$ for all $x \in \mathbb{R}^+$, where $\mathbb{R}^+$ is the set of all positive real numbers. Find $f \circ g$ and $g \circ f$. Is $f \circ g = g \circ f$?

24. Let $f : \mathbb{Q}^+ \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x - 1$ for all $x \in \mathbb{Q}^+$ and $g(x) = x + 1$ for all $x \in \mathbb{R}$, where $\mathbb{Q}^+$ is the set of all positive rational numbers. Find $g \circ f$.

25. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 3x - 2$ and $g(x) = x^3 + 2$ for all $x \in \mathbb{R}$. Find $g \circ f$, $f \circ f$, and $g \circ g$.

26. Prove Theorem 5.1.28.

27. For the following statement, write the proof if the statement is true, otherwise give a counter example.

A function $f : A \to B$ is one-one if and only if $g \circ f = h \circ f$ for all functions $g, h : B \to A$.

5.2 SPECIAL FUNCTIONS AND CARDINALITY OF A SET

This section continues the discussion of functions. Here we discuss inverse, restriction, and composition of a function. We also discuss the floor and ceiling functions, which are often encountered in computer science, especially in algorithm analysis. We conclude with a discussion of the cardinality of a set.

Inverse of a Function

Let $f : A \to B$ be a function from a set $A$ into a set $B$. Then $f \subseteq A \times B$ is a relation from $A$ into $B$. In Chapter 3, we defined the inverse relation $f^{-1} \subseteq B \times A$. Now the natural question is: Is $f^{-1}$ a function from $B$ into $A$? Before giving the answer, let us consider the following examples of functions.

Example 5.2.1

(i) Let $A = \{1, 2, 3, 4, 5\}$, $B = \{a, b, c, d\}$, and $f : A \to B$ be defined by $f(1) = a$, $f(2) = a$, $f(3) = b$, $f(4) = c$, $f(5) = d$.

The arrow diagrams of $f$ and $f^{-1}$ are shown in Figure 5.16.

We see that the distinct elements 1 and 2 of $A$ are both mapped to $a$. Therefore, it follows that function $f$ is not one-one. Because every element of $B$ has a preimage, $f$ is onto $B$. Hence, $f$ is onto $B$ but not one-one.

The inverse relation $f^{-1} \subseteq B \times A$ is given by $f^{-1} = \{(a, 1), (a, 2), (b, 3), (c, 4), (d, 5)\}$. 