Let \( Y = \{y_1, y_2, \ldots, y_k\} \) be a set with \( k \) distinct elements. Let \( f : X \to Y \) be a function. Suppose \( n > k \). Let \( m = \left\lceil \frac{n}{k} \right\rceil \). Then there exist \( m \) distinct elements, say \( a_1, a_2, \ldots, a_m \) in \( X \), such that

\[
f(a_1) = f(a_2) = \cdots = f(a_m).
\]

The proof of this result is almost the same as the proof of Theorem 7.2.4. The only difference is that we have to replace the line "Suppose \( n \geq m \) \( \Rightarrow \) \( \{x \in X \mid f(x) = y_i\} \) contains at least two elements" with "Suppose \( n \geq m \) \( \Rightarrow \) \( \{x \in X \mid f(x) = y_i\} \) contains at least \( m \) elements."

**Worked-Out Exercises**

**Exercise 1:** A box that contains eight green balls and six red balls is kept in a completely dark room. What is the least number of balls one must take out from the box so that at least two balls will be the same color?

**Solution:** Let \( X \) be the set of all balls in the box and \( Y = \{G, R\} \), where \( G \) indicates green and \( R \) indicates red. Define a function \( f : X \to Y \) by \( f(b) = G \) if the color of the ball is green and \( f(b) = R \) if the color of the ball is red. (See Figure 7.7.)

![Figure 7.7 Function f from the set of balls to the set of colors](image)

If we take a subset \( A \) of three balls of \( X \), then we find that \( |A| > |Y| \), so by the pigeonhole principle, at least two elements of \( A \) must be assigned the same color in \( Y \). Therefore, at least two of the balls of \( A \) must have the same color. On the other hand, there may exist a subset \( B = \{b, c\} \) with two balls only such that \( f(b) = G \) and \( f(c) = R \). Hence, the least number of balls that one must take out of the box so that at least two balls are the same color is three.

**Exercise 2:** Let \( X = \{x_1, x_2, \ldots, x_{100}\} \) be a set of 100 distinct positive integers. If these positive integers are divided by 75, then show that at least two of the remainders must be the same.

**Solution:** Let \( r_i \) be the remainder when \( x_i \) is divided by 75, \( i = 1, 2, \ldots, 100 \). Let

\[ R = \{r_1, r_2, \ldots, r_{100}\}, \]

i.e., \( R \) is the set of remainders. Then \( |R| = 100 \). If a positive integer \( n \) is divided by 75, then the remainder \( r \) is such that \( 0 \leq r \leq 74 \). Let

\[ S = \{0, 1, \ldots, 74\}. \]

For each element of \( R \) there is a corresponding element in \( S \); i.e., each element of \( R \) is assigned a value from the set \( S \).

We can think of the elements of \( R \) as pigeons and the elements of \( S \) as pigeonholes. Then \( n = 100 \) and \( k = 75 \). Because \( n > k \), by the pigeonhole principle, at least two elements of \( R \) must be assigned the same value in \( S \). Therefore, at least two of the \( r_i \)'s must be the same. Hence, at least two of the remainders must be the same.

**Exercise 3:** From the integers in the set \( \{1, 2, \ldots, 30\} \), what is the least number of integers that must be chosen so that at least one of them is divisible by 3 or 5?

**Solution:** The numbers in the given range that are divisible by 3 or 5 are 3, 5, 6, 9, 10, 12, 15, 18, 20, 21, 24, 25, 27, and 30. Thus, there are 14 numbers in the given range that are divisible by 3 or 5. This implies that there are 16 numbers that are not divisible by 3 or 5. It follows that we must choose at least 17 numbers from the given set to ensure that at least one of them is divisible by 3 or 5.

**Exercise 4:** Let \( \{a_i\}_{i=1}^n \) be a finite sequence of length \( n \); i.e., it has \( n \) elements.

(i) \( \{a_i\}_{i=1}^n \) is called strictly increasing if \( a_1 < a_2 < \cdots < a_n \), i.e., \( a_i < a_{i+1} \) for all \( i = 1, 2, \ldots, n-1 \).

(ii) \( \{a_i\}_{i=1}^n \) is called strictly decreasing if \( a_1 > a_2 > \cdots > a_n \), i.e., \( a_i > a_{i+1} \) for all \( i = 1, 2, \ldots, n-1 \).

(iii) A subsequence of \( \{a_i\} \) is a sequence of the form \( a_{i_1}, a_{i_2}, \ldots, a_{i_k} \), where \( 1 \leq i_1 < i_2 < \cdots < i_k \leq n \).

For example, 2, 5, 8, 7, 10 is a subsequence of 1, 2, 9, 6, 3, 8, 12, 7, 10, 15, 18.

Let \( a_1, a_2, a_3, \ldots, a_{n+1} \) be a sequence of distinct \( n^2 + 1 \) real numbers. Show that this sequence has a strictly increasing or strictly decreasing subsequence of \( n + 1 \) elements.

**Solution:** Consider the element \( a_k \) of the sequence, where \( 1 \leq k \leq n^2 + 1 \). Now starting at \( a_k \) we can construct a strictly increasing or strictly decreasing sequence.
For example, in the sequence 1, 2, 9, 6, 3, 8, 12, 7, 10, 15, starting at 2 we can construct the strictly increasing sequences 2, 9, 12 and 2, 8, 12, 15. Similarly, starting at 9, we can construct the strictly decreasing sequence 9, 8, 7.

Because \( a_1, a_2, a_3, \ldots, a_{n+1} \) is a finite sequence, the number of strictly increasing and strictly decreasing subsequences is finite. Among the strictly increasing subsequences, we can choose the increasing subsequence of the greatest length. Similarly, among the decreasing subsequences, we can choose the decreasing subsequence of the greatest length. Let \( i_k \) be the length of the largest strictly increasing subsequence starting at \( a_k \) and \( d_k \) be the length of the largest decreasing subsequence starting at \( a_k \). We associate the pair \((i_k, d_k)\) with the element \( a_k \). Consider the set \( A = \{(i_k, d_k) : k = 1, 2, \ldots, n^2 + 1\} \).

If possible, suppose there is no strictly increasing or strictly decreasing subsequence of length \( n + 1 \) in \( a_1, a_2, a_3, \ldots, a_{n^2+1} \). It follows that \( 1 \leq i_k \leq n \) and \( 1 \leq d_k \leq n \) for all \( k = 1, 2, \ldots, n^2 + 1 \).

For each \( k \), \( i_k \) has \( n \) possible choices and \( d_k \) has \( n \) possible choices. Hence, for each \( k \), the pair \((i_k, d_k)\) has \( n^2 \) possible choices.

The set \( A \) has \( n^2 + 1 \) elements and each pair has \( n^2 \) choices. It follows by the pigeonhole principle that at least two elements in the set \( A \) must be the same. That is, there exist integers \( u \) and \( v \), \( 1 \leq u < v \leq n^2 + 1 \) such that \((i_u, d_u) = (i_v, d_v)\), i.e., \( i_u = i_v \) and \( d_u = d_v \).

Consider \( a_u \) and \( a_v \). Because all elements of the sequence \( a_1, a_2, a_3, \ldots, a_{n^2+1} \) are distinct, it follows that \( a_u < a_v \) or \( a_u > a_v \).

Suppose \( a_u < a_v \). Now because \( i_u = i_v \), there is a strictly increasing subsequence, say \( a_u, a_{u_2}, \ldots, a_{u_k} \), of length \( i_u \) starting at \( a_u \). Then \( a_u, a_{u_2}, \ldots, a_{u_k} \) is a strictly increasing subsequence of length \( i_u + 1 \) starting at \( a_u \). This is a contradiction to the fact that the largest strictly increasing subsequence starting at \( a_u \) is of the length \( i_u \).

Similarly, if \( a_u > a_v \), then using the fact that \( d_u = d_v \), we can show that \( a_v \) has a strictly decreasing subsequence of length \( d_v + 1 \).

Thus, our assumption is false. Consequently, the sequence \( a_1, a_2, a_3, \ldots, a_{n^2+1} \) has a strictly increasing or strictly decreasing subsequence of \( n + 1 \) elements.

**Exercise 5:** Let \( A \) be a square such that each side is of length 2 inches. Show that if 5 points are placed inside \( A \), then the distance between at least two of the points is \( \leq \sqrt{2} \).

**Solution:** Let \( A \) be as shown in Figure 7.8.

![Figure 7.8](image)

We have decomposed the square \( A \) into four smaller squares, each side of length 1 inch, as shown in Figure 7.8.

Let \( a \) and \( b \) be two points in a smaller square (see Figure 7.8(a)). The greatest distance between points \( a \) and \( b \) would occur only when they are on opposite corners of the square, in which case the distance between them is \( \sqrt{2} \).

Now consider Figure 7.8(b). There are four squares, each side of length 1 inch. We need to place 5 points in the square \( A \). By the pigeonhole principle, at least two of these points must be either inside a smaller square or on the boundaries of the smaller square. As observed in Figure 7.8(a), the greatest distance between these two points is \( \sqrt{2} \). Thus, the distance between at least two of these points is \( \leq \sqrt{2} \).

**Section Review**

**Key Terms**

- pigeonhole principle
- generalized pigeonhole principle

**Some Key Definitions**

1. Suppose that there are \( n \) pigeons, \( k \) pigeonholes, and \( n > k \). If these \( n \) pigeons fly into these \( k \) pigeonholes, then some pigeonhole must contain at least two pigeons.

2. Let \( X = \{x_1, x_2, \ldots, x_n\} \) be a set with \( n \) distinct elements and \( Y = \{y_1, y_2, \ldots, y_k\} \) be a set with \( k \) distinct elements. Let \( f : X \rightarrow Y \) be a function. Suppose \( n > k \). Then there exist \( x_i, x_j \in X \), \( i \neq j \) and \( y_k \in Y \) such that \( f(x_i) = y_k \) and
3. Suppose that there are \( n \) pigeons, \( k \) pigeonholes, \( n > k \), and \( m = \binom{n}{k} \). If these \( n \) pigeons fly into these \( k \) pigeonholes, then some pigeonhole must contain at least \( m \) pigeons.

**Exercises**

1. There are 400 students in a programming class. Show that at least two of them were born on the same day of the month.

2. Let \( A = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7\} \) be a set of seven integers. Show that if these numbers are divided by 6, then at least two of them must have the same remainder.

3. Let \( A = \{1, 2, 3, 4, 5, 6, 7, 8\} \) be a set of integers. Show that if you choose any five distinct members of \( A \), then there will be two integers such that their sum is 9.

4. Let \( A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \). Is it true that if we choose any five distinct members of \( A \), then the sum of two of the numbers chosen is 11? Justify your answer.

5. Let \( A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \). Is it true that if we choose any six distinct members of \( A \), then the sum of two of the numbers chosen is 11? Justify your answer.

6. Let \( n \) be a positive integer. Show that if a set of \( n + 1 \) distinct integers are divided by \( n \), then at least two of the remainders must be the same.

7. Suppose there is a group of 10 senators. Each senator must serve in one of eight committees. Show that there is at least one committee with more than one senator.

8. Let \( A \) be the set of integers \( 1, 2, 3, 4, 5 \). Show that if 4 numbers are selected from \( A \), then at least two must add up to 6.

9. Let \( A \) be a set of \( 2n \) integers, \( 1, 2, 3, \ldots, 2n - 1, 2n \).
   a. Show that if \( n + 1 \) numbers are chosen from \( A \), then at least one of them is divisible by 2.
   b. Show that if \( n + 1 \) are chosen from \( A \), then the multiplication of at least two of them must be even.

10. Suppose that we have a deck of 52 cards.
   a. How many cards must be picked up so that at least one of them is red?
   b. How many cards must be picked up so that at least one of them is a heart?
   c. How many cards must be picked up so that at least one of them is an ace?

11. From the integers in the set \( \{1, 2, \ldots, 20\} \), what is the least number of integers that must be chosen so that at least one of them is divisible by 4?

12. From a set of 50 integers, how many must be chosen so that at least two of them have the same remainder when divided by 9?

13. Let \( f : A \rightarrow B \) be a function from set \( A \) into set \( B \) and \( |A| = 10 \) and \( |B| = 8 \). Show that \( f \) is not one-one.

14. Let \( a \) and \( b \) be integers such that \( b \neq 0 \). Show that when \( \frac{a}{b} \) is written as a decimal number, then either the expansion stops or a certain set of digits repeats.

15. In a group of 38 people, at least how many must have been born in the same month?

16. A group of 40 students in a class must share a set of 15 computers. To avoid conflict, each student is assigned only 1 computer. Moreover, no computer is assigned to more than 4 students. Prove that at least 2 computers are assigned to 3 or more students.

17. In a quiz taken by 70 students, the scores range from 60 to 88. At least how many students must have the same score?

18. Let \( a_1, a_2, a_3, \ldots, a_{10} \) be a sequence of 10 elements. Show that this sequence has a strictly increasing or decreasing subsequence of 4 elements.

19. Let \( A = \{1, 2, \ldots, 2n - 1, 2n\} \) be a set of integers. Let \( S \subseteq A \) such that \( S \) has \( n + 1 \) elements. Show that \( S \) contains two elements \( a \) and \( b \) such that \( a \mid b \).

20. Let \( A = \{a_1, a_2, a_3, a_4, a_5\} \) such that for all \( i \), \( 1 \leq a_i \leq 7 \). Show that \( A \) has subsets \( S_1 \) and \( S_2 \) such that the sum of the elements of \( S_1 \) and the sum of the elements of \( S_2 \) are the same.

21. Suppose that we have the list of integers 100 to 500.
   a. What is the maximum number of integers that can be chosen from this list so that the digits in each number are distinct?
   b. What is the least number of integers that must be chosen so that at least one number has distinct digits?
   c. What is the least number of integers that must be chosen so that at least two numbers have a digit in common?

22. Let \( A \) be a set of 101 integers such that none is divisible by 100. Show that there exists \( a, b \in A \) such that \( a - b \) is divisible by 100.

23. A set of \( n + 1 \) integers, \( n \geq 1 \), such that none is divisible by \( n \). Show that there exists \( a, b \in A \) such that \( a - b \) is divisible by \( n \).

24. Let \( A \) be a square such that each side is of length 4 inches.
   a. Show that if 5 points are placed inside \( A \), then the distance between at least two of the points is \( \leq \sqrt{2} \).
   b. Show that if 17 points are placed inside \( A \), then the distance between at least two of the points is \( \leq \sqrt{2} \).