Then in symbols, the given sentence takes the form $\forall x (P(x) \to Q(x))$.

(e) Let

$$P(x): x \text{ is even.}$$
$$Q(x): x \text{ is prime.}$$

Then in symbols, the given sentence takes the form $\exists x (P(x) \wedge Q(x))$. The domain of discourse is the set of integers.

(f) Let $P(x, y)$ denote the sentence: $x + y = 0$. Hence, in the domain of integers the given sentence may be symbolized as $\forall x \exists y P(x, y)$.

Exercise 2: In the following, use $P(x)$: x is an odd integer; $Q(x)$: x is a prime integer; and $R(x)$: x^2 is an odd integer. Write a statement in English corresponding to each symbolic statement.

(a) $\forall x (P(x) \to R(x))$ b. $\forall x (P(x) \wedge Q(x))$

(c) $\exists x (P(x) \wedge Q(x))$

Solution:

(a) The squares of all odd integers are odd.

(b) All integers are odd and prime.

(c) Some odd integers are prime.

Exercise 3: Let $P(x, y)$ denote the sentence: $xy = x$. What is the truth value of $\forall x \exists y P(x, y)$, where the domain of x, y is the set of all integers?

Solution: Since $P(x, 1) : x1 = x$ and it is true for all integers x, we find that the truth value of $\forall x \exists y P(x, y)$ is T.

Exercise 4: Let $P(x, y)$ denote the sentence: $xy = 1$. What is the truth value of $\forall x \exists y P(x, y)$, where the domain of x, y is the set of all integers?

Solution: For $x = 2, P(2, y) : 2y = 1$. There are no integers y such that $2y = 1$. Hence, the statement $\forall x \exists y P(x, y)$ is false.

Exercise 5: Let $P(x, y)$ denote the sentence: $x + y = 1$. What are the truth values of $\forall x \exists y P(x, y)$, $\forall x \forall y P(x, y)$, and $\exists x \exists y P(x, y)$, where the domain of x and y is the set of all integers?

Solution: Let x be an integer. Then $y = 1 - x$ is an integer such that $x + (1 - x) = 1$. Thus, for any integer x there exists an integer $y = 1 - x$ such that $x + y = 1$. Therefore, we find that the truth value of $\forall x \exists y P(x, y)$ is T.

To find the truth value of $\forall x \forall y P(x, y)$, we consider the integers 2 and 3. Because $2 + 3 \neq 1$, the truth value of $\forall x \forall y P(x, y)$ is F.

We now consider the statement $\exists x \exists y P(x, y)$. Because $0 + 1 = 1$, we find that there are integers x and y such that $x + y = 1$. Hence, the truth value of the statement $\exists x \exists y P(x, y)$ is T.

Exercise 6: Test the validity of the following argument: Some rational numbers are powers of 5. All integers are rational numbers. Therefore, some integers are powers of 5.

Solution: We first translate the given argument in the following form.

There exists x, if x is a rational number, then x is a power of 5.

For all x, if x is an integer, then x is a rational number.
5 is an integer.

Therefore, there exists x such that if x is a integer, then x is a power of 5.

We now symbolize the above arguments: Let

$$P(x): x \text{ is an integer.}$$
$$Q(x): x \text{ is a rational number.}$$
$$R(x): x \text{ is a power of 5.}$$

We can write the above argument in the following form:

$$\exists x (Q(x) \to R(x))$$
$$\forall x (P(x) \to Q(x))$$
$$P(5)$$
$$\text{Therefore, } \exists x (P(x) \to R(x)).$$

To verify the validity we now consider the following sequence of formulas.

$B_1 : \exists x (Q(x) \to R(x))$	hypothesis
$B_2 : Q(a) \to R(a)$	for some member of the domain the set of rational numbers, by the rule of inference ES
$B_3 : \forall x (P(x) \to Q(x))$	hypothesis
$B_4 : P(a) \to Q(a)$	by the rule of inference US
$B_5 : P(a) \to R(a)$	B_4, B_2, B_5 is a logically valid argument, by hypothetical syllogism.

Therefore,

$B_6 : \exists x (P(x) \to R(x))$	by the rule of inference EG

Hence, the consequence is valid.

Section Review

Key Terms

statement logic
propositional logic
predicate
propositional function
domain
free variable

n-place predicate	bound variable	first-order logic
universal quantifier	counterexample	
existential quantifier	disproof	

Some Key Definitions

1. Let x be a variable and let D be a set. $P(x)$ is a sentence. Then $P(x)$ is called a predicate or propositional function with respect to the set D if each value of x in D, $P(x)$ is a statement; i.e., $P(x)$ is true or false. Moreover, D is called the domain of the discourse and x is called the free variable.
2. Let $P(x)$ be a predicate and let D be the domain of the discourse. The universal quantification of $P(x)$ is the statement for all x, $P(x)$ or for every x, $P(x)$. In symbols, the universal quantification of the predicate $P(x)$ is written as $\forall x\, P(x)$.
3. Let $P(x)$ be a predicate and let D be the domain of the discourse. The existential quantification of $P(x)$ is the statement there exists $x\, P(x)$. In symbols, the existential quantification of the predicate $P(x)$ is written as $\exists x\, P(x)$.

Some Key Results

1. Let $P(x)$ be a predicate with domain of discourse D. Then

   (i) $\sim\forall x\, P(x) \equiv \exists x \sim P(x)$.
   (ii) $\sim\exists x\, P(x) \equiv \forall x \sim P(x)$.

EXERCISES

1. Symbolize the following by using quantifiers, predicates, and logical connectives.
   a. All integers are rational numbers.
   b. Some rational numbers are integers.
   c. All positive integers are multiples of 5.
   d. Some rectangles are square.
   e. For all integers n, $2n + 1$ is an odd integer.
   f. Every integer is either odd or even.
   g. Every integer is a multiple of 6 if and only if it is a multiple of both 3 and 2.
   h. There is no integer n such that n^2 is 5.
2. In parts (a)–(d), use $P(x)$: x is an integer, $Q(x)$: x is a rational number, and $R(x)$: x is a prime integer. Write a statement in English corresponding to the following symbolic statements.
   a. $P(5)$ b. $\forall x \sim P(x)$
   c. $\exists x\, R(x)$ d. $\exists x \sim Q(x)$
3. What is the universal quantification of the sentence: $x^2 + x$ is an even integer, where x is an odd integer? Is the universal quantification a true statement?
4. What is the universal quantification of the sentence: $x^2 + x$ is an even integer, where x is an even integer? Is the universal quantification a true statement?
5. What is the existential quantification of the sentence: x is a prime integer, where x is an odd integer? Is the existential quantification a true statement?
6. What is the existential quantification of the sentence: $x < 0$, where x is an integer? Is the existential quantification a true statement?
7. What is the truth value of the quantification $\forall x\, P(x)$? The domain of the discourse is the set of all positive integers.
   a. $P(x) : (x + 1)(x + 2)$ is an even integer.
   b. $P(x) : x + 1 > x$
   c. $P(x) : x + 2 > 5$
   d. $P(x) : x^2 + 2 = 3$
8. What is the truth value of the quantification $\exists x\, P(x)$? The domain of the discourse is the set of all real numbers.
   a. $P(x) : x + 1 = 1$ b. $P(x) : x^3 + 1 < x$
   c. $P(x) : x \cdot \frac{1}{2} = 1$ d. $P(x) : x^2 + 2x + 1 < 0$

9. Let $P(x,y)$ denote the sentence: $x+y=7$. What are the truth values of $\forall x \exists y P(x,y)$, $\forall x \forall y P(x,y)$, and $\exists x \exists y P(x,y)$, where the domain of x,y is the set of all integers?
10. Let $P(x,y)$ denote the sentence: $2x+y=1$. What are the truth values of $\forall x \exists y P(x,y)$, $\forall x \forall y P(x,y)$, and $\exists x \exists y P(x,y)$, where the domain of x,y is the set of all integers?
11. Let $P(x,y)$ denote the sentence: x divides y. What are the truth values of $\forall x \exists y P(x,y)$, $\forall x \forall y P(x,y)$, and $\exists x \exists y P(x,y)$, where the domain of x,y is the set $\{1,2,4,6,12\}$?
12. Symbolize the following sentences by using predicates, quantifiers, and logical connectives.
    a. Any finite set with n elements has 2^n subsets.
    b. Not all real numbers are rational numbers.
13. Symbolize the following sentences by using predicates, quantifiers, and logical connectives.
    a. Every computer science major takes a programming course.
    b. If you buy a car, then you must pay a sales tax.
    c. Some people are vegetarians.
14. Express the following using predicates, quantifiers, and logical connectives. Also verify the validity of the consequence.

    Everyone who graduates gets a job.

    Jennifer graduated.

    Therefore, Jennifer got a job.

In Exercises 15–23, test the validity of the logical consequences.

15. All men are mortal.

    Randy is a man.

    Therefore, Randy is mortal.
16. All birds can fly.

    A crow is a bird.

    Therefore, a crow can fly.
17. All polynomials with real coefficients are differentiable functions.

    All differentiable functions are continuous.

    Therefore, all polynomials with real coefficients are continuous.
18. Everyone who studies logic is good in reasoning.

    Lance is good in reasoning.

    Therefore, Lance studies logic.
19. All employers pay their employees.

    Juan is an employer.

    Therefore, Juan pays his employees.
20. All athletes exercise.

    Emily is an athlete.

    Therefore, Emily exercises.
21. All drivers take a driving test.

    Tom did not take the driving test.

    Therefore, Tom is not a driver.
22. All athletes are healthy.

    All healthy people take vitamins.

    Grant is an athlete.

    Therefore, Grant takes vitamins.
23. All dogs fetch.

    Kitty does not fetch.

    Therefore, Kitty is not a dog.
24. Show that $\forall x\,(P(x) \wedge Q(x))$ and $\forall x\, P(x) \wedge \forall x\, Q(x)$ are equivalent.
25. Show that $\exists x\,(P(x) \vee Q(x))$ and $\exists x\, P(x) \vee \exists x\, Q(x)$ are equivalent.
26. Show that $\forall x\,(P(x) \to Q(x))$ is not equivalent to $\forall x\, P(x) \to \forall x\, Q(x)$.
27. Find a counterexample to show that the following propositions are false.
    a. $\forall x \in \mathbb{R},\ x < x^2$
    b. $\forall m, n \in \mathbb{Z},\ m \cdot n > m + n$
28. Find a counterexample to show that the following propositions are false.
    a. $\forall a, b \in \mathbb{R},\ \sqrt{ab} = \sqrt{a}\sqrt{b}$
    b. $\forall a, b \in \mathbb{R},\ \sqrt{a+b} = \sqrt{a} + \sqrt{b}$
29. Find a counterexample to show that the proposition $\forall a, b, c \in \mathbb{R},\ c \neq 0,\ \frac{ac+b}{c} = a + b$ is false.

1.5 PROOF TECHNIQUES

In the preceding sections, we presented various ways of using logical arguments and deriving conclusions. As stated earlier, in mathematics and computer science, mathematical logic is used to prove theorems and the correctness of programs. In this section, after formally defining the term *theorem*, we describe some general